Arithmetic Correlations

Mark Goresky* and Andrew Klapper†

*partially supported by DARPA grant HR0011-04-1-0031

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Their cross-correlation with shift τ is

$$\mathcal{C}_\tau(\mathbf{a}, \mathbf{b}) = \sum_{k=0}^{T-1} \zeta^{a_k} \zeta^{-b_{k+\tau}} = \sum_{k=0}^{T-1} \zeta^{a_k - b_{k+\tau}}$$

$$= Z(\mathbf{a} - \mathbf{b}^\tau)$$

where $\mathbf{b}^\tau = b_\tau, b_{\tau+1}, \cdots$ is the τ-shift of $\mathbf{b}$.

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There is another way to say this:

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \in \mathbb{Z}(p)[[x]]$$
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Then $\mathbf{a} - \mathbf{b}^\tau$ is the coefficient sequence of
$c(x) = a(x) - b^\tau(x) \qquad \in \mathbb{Z}/(p)[[x]]$

so: $\mathcal{C}_\tau(\mathbf{a}, \mathbf{b}) = Z(c) = \sum_{k=0}^{T-1} \zeta^{c_k} = \sum_{k=N}^{N+T-1} \zeta^{c_k}.$

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Arithmetic analog:

$$\alpha = a_0 + a_1 p + a_2 p^2 + \cdots \qquad \in \mathbb{Z}_p.$$
$$\beta^\tau = b_\tau + b_{\tau+1} p + b_{\tau+2} p^2 + \cdots \qquad \in \mathbb{Z}_p.$$

$$\gamma = \alpha - \beta^\tau = \gamma_0 + \gamma_1 p + \gamma_2 p^2 + \cdots \qquad \in \mathbb{Z}_p.$$

Note: $\gamma = \alpha - \beta^\tau$ is *eventually* periodic.

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Set $\mathcal{C}_\tau^{arith}(\mathbf{a}, \mathbf{b}) = Z(\gamma) = \sum_{k=N}^{N+T-1} \zeta^{\gamma_k}.$
(sum over periodic part)

- Averaged over all pairs of sequences,

$$E(\mathcal{C}_\tau(\mathbf{a},\mathbf{b})) = \begin{cases} T & (\mathbf{a}=\mathbf{b} \text{ and } \tau = 0) \\ 0 & (\mathbf{a}\neq\mathbf{b} \text{ or } \tau \neq 0) \end{cases}$$

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- RMS cross-correlation is

$$\sqrt{E(\mathcal{C}_\tau(\mathbf{a},\mathbf{b})^2)} = \begin{cases} \mathrm{T} & (\mathbf{a}=\mathbf{b} \text{ and } \tau = 0) \\ \sqrt{2T} & (\mathbf{a}=\mathbf{b}, \tau \neq 0, p = 2) \\ \sqrt{T} & (\text{otherwise}) \end{cases}$$

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- Average Arithmetic cross-correlation is:

$$E(\mathcal{C}_\tau^{arith}(\mathbf{a},\mathbf{b})) = \begin{cases} T & (\mathbf{a}=\mathbf{b}, \tau = 0) \\ T/p^{T-\mathsf{gcd}(\tau,T)} & (\mathbf{a}=\mathbf{b}, \tau \neq 0) \\ T/p^T & (\text{otherwise}) \end{cases}$$

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$(e_1, e_2, e_3 \sim 1)$

m-sequences

Let $\mathbf{a} = a_0, a_1, a_2, \cdots$ be T-periodic, $\quad a_i \in \mathbb{Z}/(p)$.

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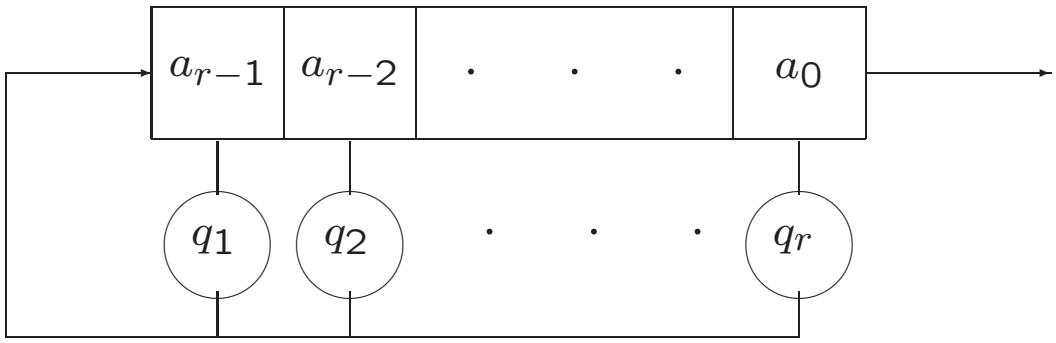
- $q(x) = -1 + q_1 x + \cdots + q_r x^r$
  is the connection polynomial of a LFSR
  that generates the sequence $\mathbf{a}$.



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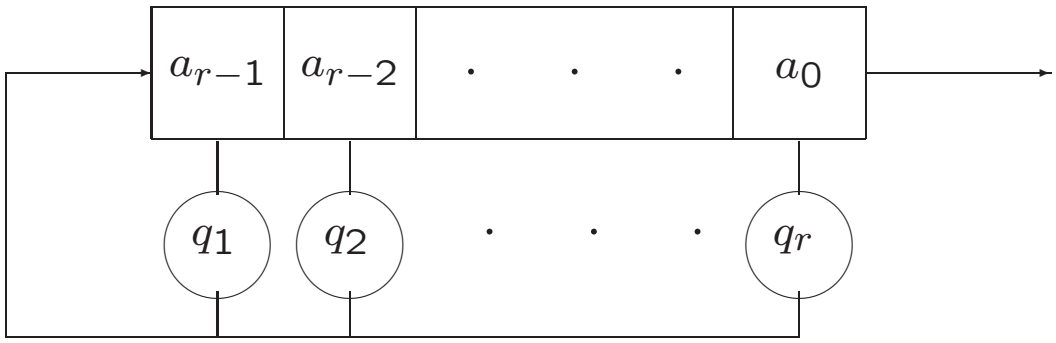

- **m-sequence** $\Leftrightarrow$ $q(x)$ is a primitive polynomial

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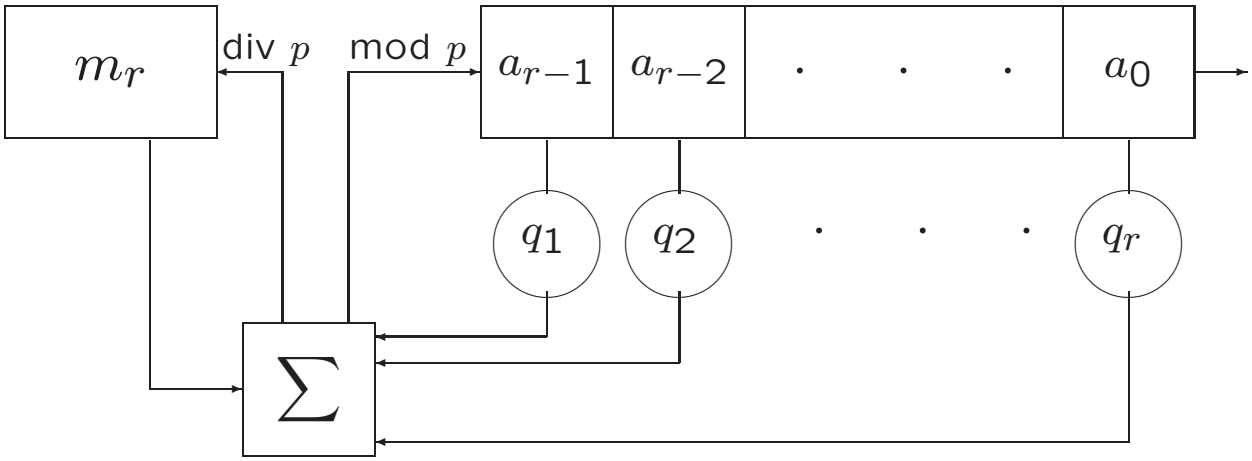

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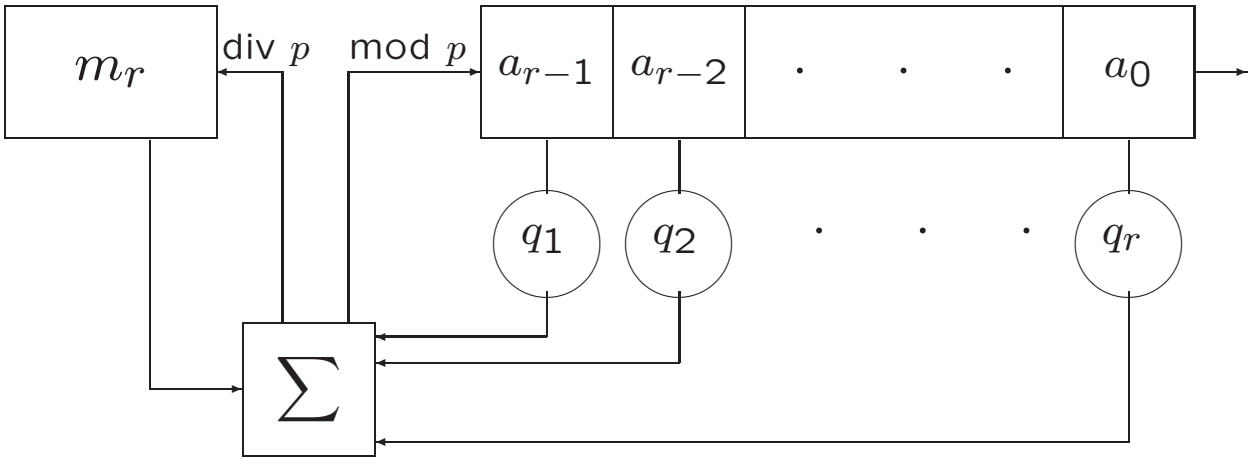

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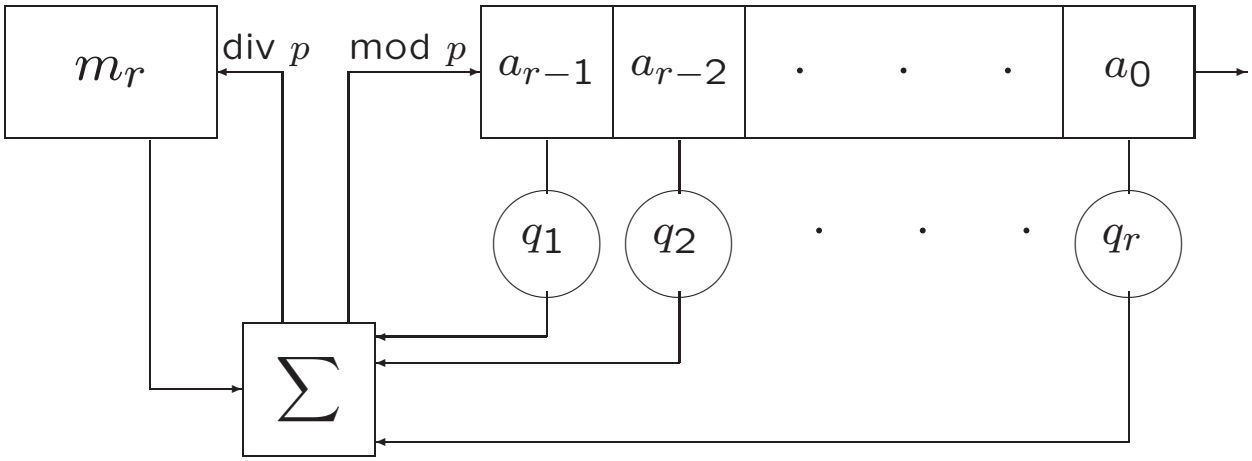

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- ℓ-**sequence** $\Leftrightarrow$ p is a primitive root modulo q.

Shift and add sequences

Let $\mathbf{a} = a_0, a_1, \cdots$ be T-periodic. $(a_i \in \mathbb{Z}/(p))$

It is a **shift and add sequence** if for any shift τ,

$$\mathbf{a} + \mathbf{a}^{\tau} = \begin{cases} \mathbf{a}^{\tau'} & \text{some shift } \tau', \text{ or} \\ 0 \end{cases}$$

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Let $\mathbf{a} = a_0, a_1, \cdots$ be T-periodic $(a_i \in \mathbb{Z}/(p))$
It is an **arithmetic shift and add** sequence if, $\forall \tau$,

$$\text{periodic part of } (\alpha + \alpha^\tau) = \text{shift of } \alpha$$

where $\alpha = a_0 + a_1 p + a_2 p^2 + \cdots \in \mathbb{Z}_p$.

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Theorem. *The sequence* $\mathbf{a}$ *is an arithmetic shift and add sequence if and only if it is an* ℓ*-sequence.*

Summary of properties

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Decimations:	
$\mathbf{a} \neq \mathbf{a}^\tau$ for $(\tau, T) = 1$	$\mathbf{a} \neq \mathbf{a}^\tau$ for $(\tau, T) = 1$ and $q > 13$ (conj.)

Generalizations

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