

7. Lecture 7: Stratifications

7.1. For some historical comments see:
http://www.math.ias.edu/~goresky/math2710/ThomMather.pdf

7.2. The plan is to decompose a reasonable space into a locally finite union of smooth manifolds (called strata) which satisfy the axiom of the frontier: the closure of each stratum should be a union of lower dimensional strata. If \( Y \subseteq X \) are strata we write \( Y < X \). But this should be done in a locally trivial way. In Whitney’s example below, it is not enough to divide this figure

![Figure 1. Three strata needed](image)

into 1- and 2-dimensional strata, even though this gives a decomposition into smooth manifolds. If the origin is not treated as another stratum then the stratification fails to be “locally trivial”. Whitney proposed a condition that identifies the origin as a separate stratum in this example. Let us say that a stratification of a closed subset \( W \) of some smooth manifold \( M \) is a locally finite decomposition \( W = \bigsqcup S_\alpha \) into locally closed smooth submanifolds \( S_\alpha \subseteq M \) (called strata) so as to satisfy the axiom of the frontier. A stratified homeomorphism \( h : W_1 \to W_2 \) between two stratified sets is a homeomorphism that takes strata to strata and is smooth on each stratum.

7.3. Definition. Let \( Y \subseteq X \) be strata in a stratification of a closed set \( W \subseteq M \). The pair \((X,Y)\) satisfies Whitney’s condition B at a point \( y \in Y \) if the following holds. Suppose that \( x_1, x_2, \ldots \in X \) is a sequence that converges to \( y \), and suppose that \( y_1, y_2, \ldots \in Y \) is a sequence that also converges to \( y \). Suppose that (in some local coordinate system near \( y \)) the secant lines \( \ell_i = \overline{x_i, y_i} \) converge to some limiting line \( \ell \). Suppose that the tangent planes \( T_{x_i}X \) converge to some limiting plane \( \tau \). Then \( \ell \subseteq \tau \).

We say the pair \((X,Y)\) satisfies condition B if it does so at every point \( y \in Y \). The decomposition into strata is a Whitney stratification if every pair of strata \( Y < X \) satisfies condition B at every point in the smaller stratum \( Y \).

(If condition B is satisfied, and if the tangent planes \( T_{y_i} Y \) also converge to some limiting plane \( \eta \) then \( \eta \subseteq \tau \) as well, which Whitney had originally proposed as an additional condition, which he called Condition A.) It turned out that Whitney’s condition B was just the right condition to
guarantee that a stratification is locally trivial, but the verification involved the full development of stratification theory by René Thom and John Mather. The problem is that stratifications satisfying condition B may still exhibit certain pathologies, such as infinite spirals, so there is a very delicate balance between proving that local triviality holds while avoiding a host of counterexamples to similar sounding statements.

Suppose \( W \subset M \) has a stratification that satisfies condition B. Let \( Y \) be a stratum and let \( y \in Y \). Let \( N_y \subset M \) be a normal slice, that is, a smooth submanifold of dimension \( \dim(N_y) = \dim(M) - \dim(Y) \) that intersects \( Y \) transversally in the single point \( \{y\} \). Define the link of the stratum \( Y \),

\[
L_Y = L_Y(y, \epsilon) = (\partial B_\epsilon(y)) \cap N_y \cap W
\]

where \( B_\epsilon(y) \) is a ball of radius \( \epsilon \) (measured in some Riemannian metric on \( M \)) centered at the point \( y \).

7.4. **Theorem.** (R. Thom, J. Mather) If \( \epsilon \) is chosen sufficiently small then

1. the closed set \( L_Y \) is stratified by its intersection with the strata \( Z \) of \( W \) such that \( Z > Y \)
2. this stratification satisfies condition B
3. the stratified homeomorphism type of \( L_Y \) is independent of the choice of \( N_y, \epsilon, \) and the Riemannian metric
4. if the stratum \( Y \) is connected then the stratified homeomorphism type of \( L_Y \) is also independent of the point \( y \).

Moreover, the point \( y \) has a basic neighborhood \( U_y \subset W \) and a stratified homeomorphism

\[
U_y \cong c^0(L_Y) \times B
\]

where \( c^0(L_Y) \) denotes the open cone on \( L_Y \) (with its obvious stratification) and where \( B \) denotes the open ball of radius 1 in \( \mathbb{R}^{\dim(Y)} \).

This homeomorphism preserves strata in the obvious way: it takes

1. \( \{y\} \times B \to Y \cap U \) with \( \{y\} \times \{0\} \to \{y\} \)
2. \( (L_Y \cap X) \times (0, 1) \times B \to X \cap U \) for each stratum \( X > Y \)

This result says that the set \( W \) does not have infinitely many holes or infinitely much topology as we approach the singular stratum \( Y \) and it says that the normal structure near \( Y \) is locally trivial as we move around in \( Y \). In particular, the collection of links \( L_Y(y) \) form the fibers of a stratified fiber bundle over \( Y \).

It also implies that (for any \( r \geq 0 \)) the local homology \( H_r(W, W - y; \mathbb{Z}) \) forms a local coefficient system on \( Y \) with stalk

\[
H_r(W, W - y; \mathbb{Z}) \cong H_{r - \dim(Y) - 1}(L_Y; \mathbb{Z}).
\]

7.5. In fact, Thom and Mather proved that a Whitney stratified \( W \) set admits a system of control data consisting of a triple \( (T_Z, \pi_Z, \rho_Z) \) for each stratum \( Z \), where \( T_Z \) is a neighborhood of \( Z \) in \( W \), where \( \pi_Z : T_Z \to Z \) is a “tubular projection”, \( \rho_Z : T_Z \to [0, \epsilon) \) is a “tubular distance function”
so that the pair \((\pi_Z, \rho_Z)|Y \cap T_Z : Y \cap T_Z \to Z \times (0, \epsilon)\) is a smooth submersion for each stratum \(Y > Z\), and where \(\pi_Z \pi_Y = \pi_Z\) in \(T_Z \cap T_Y\) and \(\rho_Z \pi_Y = \rho_Z\). (picture).

7.6. This data was then used to construct *controlled vector fields* that trace out the local triviality of the stratification.

7.7. Whitney himself outlined a procedure for proving that any closed subset \(W\) of Euclidean space defined by analytic equations admits a Whitney stratification. The idea is to start with the open, nonsingular part \(W^0\) of \(W\) as the “top” stratum, and then to look at the set of points in the singular set \(\Sigma = W - W^0\) where condition B *fails*. He proves that this is an analytic subset of codimension two, whose complement in \(\Sigma\) is therefore the first singular stratum, \(W^1\). Now, carry both \(W^0\) and \(W^1\) along, looking at the set of points (in what remains) where condition B fails, and continue in this way inductively. Since Whitney’s early work, many advances have been made in the subject. The following statement is at best a partial summary of the work of many people.

7.8. **Theorem.** The following sets admit Whitney stratifications: real and complex algebraic varieties, real and complex analytic varieties, semi-algebraic and semi-analytic varieties, subanalytic sets, and sets with o-minimal structure. Given such an algebraic (resp. analytic etc.) variety \(W\) and a locally finite union \(Z\) of algebraic (resp. analytic etc.) subvarieties, the stratification of \(W\) can be chosen so that \(Z\) is a union of strata. Given an algebraic (resp. analytic etc.) mapping \(f : W \to W'\) of algebraic (resp. analytic etc.) varieties, it is possible to find Whitney stratifications of \(W,W'\) so that the mapping \(f\) takes strata to strata, and so that for each stratum \(X\) of \(W\) the mapping \(X \to f(X)\) is a smooth submersion onto a stratum of \(W'\). Whitney stratified sets can be triangulated by a triangulation that is smooth on each stratum, such that that the closure of each stratum is a subcomplex of the triangulation.

[A subanalytic set is the image under a projection (for example, a linear projection \(\mathbb{R}^m \to \mathbb{R}^n\)) of an analytic or a semi-analytic set. O-minimal structures allow for certain non-analytic functions to be included in the definition of the set. Whitney stratifications also make sense for algebraic varieties defined over fields of finite characteristic. Given an algebraic mapping \(f : W \to W'\) between complex algebraic varieties, it is not generally possible to choose triangulations of \(W,W'\) so that \(f\) becomes a simplicial mapping.]
7.9. Pseudomanifolds and Poincaré duality. A pseudomanifold of dimension $n$ is a purely $n$-dimensional (Whitney) stratified space that can be triangulated so that every $n - 1$ dimensional simplex is a face of exactly two $n$-dimensional simplices. This implies that the $n - 1$ dimensional simplices can be combined with the $n$-dimensional simplices to form an $n$-dimensional manifold, and that the remainder (hence, the “singularity set”) has dimension $\leq n - 2$. If this manifold is orientable then an orientation defines a fundamental class $[W] \in H_n(W; \mathbb{Z})$. Cap product with the fundamental class defines the Poincaré duality map $H^r(W; \mathbb{Z}) \to H_{n-r}(W; \mathbb{Z})$ which is an isomorphism if $W$ is a manifold (or even a homology manifold) but which, in general, is not an isomorphism.

There is a sheaf-theoretic way to say this. If $W$ is oriented and $n$-dimensional then a choice of orientation determines a sheaf map $\mathbb{Z}_W \to \mathcal{C}^n_{W}$ to the sheaf of $n$-chains. On any open set $U$, choose a triangulation of $U$ and map $m \in \mathbb{Z}$ to $m$ times the sum of all the $n$-dimensional simplices in $U$. (Recall that a PL chains are identified under subdivision.) Therefore, if $W$ is an $n$-dimensional homology manifold, that is, if $H_r(W, W - x) = 0$ for all $0 \leq r < n$ and $H_n(W, W - x; \mathbb{Z}) = \mathbb{Z}$ then the map

$$\mathbb{Z}_W[n] \to \mathcal{C}^n_{W}$$

is a quasi-isomorphism. This simple statement is the Poincaré duality theorem. For, it says that this quasi-isomorphism induces an isomorphism on cohomology, that is,

$$H^r(W; \mathbb{Z}) \cong H^{BM}_{n-r}(W; \mathbb{Z})$$

and an isomorphism on cohomology with compact supports, that is,

$$H^*_c(W; \mathbb{Z}) \cong H_{n-r}(W; \mathbb{Z}).$$

[Actually, from this point of view, the deep fact is that $H^i(W; k)$ and $H_i(W; k)$ are dual over any field $k$, but this is not a fact about manifolds. Rather, it is a fact about the sheaf of chains.]

More generally if $W$ is not necessarily orientable then the orientation sheaf $\mathcal{O}_W$ is the local system whose stalk at $x \in W$ is the top local homology $H_n(W, W - x)$ and the mapping $\mathcal{O}_W \to \mathcal{C}^*$ is a quasi-isomorphism. So, for any local coefficient system $\mathcal{L}$ on $W$ the mapping $\mathcal{L} \otimes \mathcal{O}_W \to \mathcal{C}^*(\mathcal{L})$ is a quasi-isomorphism, giving an isomorphism on cohomology,

$$H^r(W; \mathcal{L} \otimes \mathcal{O}_W) \cong H^{BM}_{n-r}(W; \mathcal{L})$$

and on cohomology with compact supports,

$$H^*_c(W; \mathcal{L} \otimes \mathcal{O}_W) \cong H_{n-r}(W; \mathcal{L}).$$

So this quasi-isomorphism statement includes the Poincaré duality theorem for orientable and non-orientable manifolds, for non-compact manifolds, and for manifolds with boundary, and with possibly nontrivial local coefficient systems.