6. Lecture 6: The derived category

There are several different ‘models’ for the derived category. The first definition we give is easy to understand and useful for proofs but the objects themselves are not very natural. The second model is less intuitive but the objects occur naturally.

6.1. The derived category: first definition. Let $X$ be a topological space. The bounded derived category $D^b(X)$ is the category whose objects are complexes of injective sheaves whose cohomology sheaves are bounded (meaning that $H^r(A^\bullet) = 0$ for sufficiently large $r$ and for sufficiently small $r$). The morphisms are homotopy classes of morphisms (of complexes of sheaves), so that $D^b(X)$ is the homotopy category of (complexes of) injective sheaves.

If $S(X)$ denotes the category of sheaves on $X$, if $C^b(X)$ denotes the category of complexes of sheaves with bounded cohomology and if $K^b(X)$ denotes the homotopy category of (complexes of) sheaves with bounded cohomology then we have a canonical functor

$$S(X) 	o C^b(X) 	o K^b(X) \xrightarrow{\text{God}} D^b(X)$$

that associates to any sheaf $S$ the corresponding complex concentrated in degree zero, and to any complex $A^\bullet$ its Godement injective resolution. [This construction makes sense if we replace the category of sheaves with any abelian category $C$ provided it has enough injectives. In this way we define the bounded derived category $D^b(C)$ with functors $C \to K^b(C) \to D^b(C)$.]

From the previous lecture on “properties of injective sheaves” we therefore conclude:

- The mapping $K^b(X) \to D^b(X)$ is a functor (that is, a morphism between complexes determines a morphism in the derived category also).
- If $A^\bullet \to B^\bullet$ is a quasi-isomorphism of complexes of sheaves then it becomes an isomorphism in $D^b(X)$.
- If $A^\bullet$ is a complex of sheaves such that $H^m(A^\bullet) = 0$ for all $m$ then $A^\bullet$ is isomorphic to the zero sheaf.

6.2. Definition. Let $T : S(X) \to \mathcal{B}$ be a covariant (and additive) functor from the category of sheaves to some other abelian category with enough injectives. For the moment, let us also assume that it takes injectives to injectives. Define the right derived functor $R T : D^b(X) \to D^b(B)$ by $R T(A^\bullet)$ to be the complex $T(I^0) \to T(I^1) \to T(I^2) \to \cdots$ where $A^\bullet \to I^\bullet$ is the canonical (or the chosen) injective resolution of $A^\bullet$. Define $R^mT(A)$ (“the $m$-th derived functor”, an older terminology) to be the cohomology object of this complex, $H^m(R T(A^\bullet))$.

Let $A^\bullet \to I^\bullet$ and $B^\bullet \to J^\bullet$ be the canonical injective resolutions of $A^\bullet, B^\bullet$. Then any morphism $f : A^\bullet \to B^\bullet$ has a lift $\tilde{f} : I^\bullet \to J^\bullet$ that is unique up to homotopy, which is to say that $\tilde{f}$ is a uniquely defined morphism in the category $D^b(X)$, and we obtain a well defined morphism

$$R T(f) = T(\tilde{f}) : R T(A^\bullet) \to T(J^\bullet) = R T(B^\bullet).$$

In other words, the right derived functor of $T$ is obtained by replacing each complex $A^\bullet$ by its injective resolution $I^\bullet$ and then applying $T$ to that complex.
6.3. Examples.

1. If \( f : X \to Y \) is a continuous map between topological spaces, then we show (below) that \( f_* : \mathcal{S}(X) \to \mathcal{S}(Y) \) takes injectives to injectives. Taking \( Y = \{\text{pt}\} \) we get that the global sections functor \( \Gamma \) takes injectives to injectives. Then, for any complex of sheaves \( A^\bullet \),

\[ R^m \Gamma(A^\bullet) = H^m(\Gamma(X, I^\bullet)) \]

where \( A^\bullet \to I^\bullet \) is the canonical injective resolution. (This is how we defined the hypercohomology of the complex of sheaves \( A \) in §XIII.)

2. Let \( f : X \to Y \) be a continuous map and let \( \mathbb{Z} \) be the constant sheaf on \( X \). If \( f \) is surjective and its fibers are connected then \( f_*(\mathbb{Z}) \) is again the constant sheaf, because as a presheaf, \( f_*(\mathbb{Z}(U)) = \mathbb{Z}(f^{-1}(U)) = \mathbb{Z} \) for any connected open set \( U \subset Y \). Although we cannot hope to understand \( Rf_*(\mathbb{Z}) \) we can understand its cohomology sheaves:

\[ R^m f_*(\mathbb{Z})(U) = H^m(\Gamma(f^{-1}(U), I^\bullet)) \]

where \( I^\bullet \) is an injective resolution (or perhaps the canonical injective resolution) of the constant sheaf. But this is exactly the definition of the hypercohomology \( H^m(f^{-1}(U), \mathbb{Z}) = H^m(f^{-1}(U, \mathbb{Z})) \).

If \( f \) is proper then the stalk cohomology (of the cohomology sheaf of \( Rf_*(\mathbb{Z}) \)) at a point \( y \in Y \) is equal to \( H^m(f^{-1}(y); \mathbb{Z}) \), the cohomology of the fiber. In other words, the sheaf \( \mathbb{H}^m(Rf_*(\mathbb{Z})) \) is a sheaf on \( Y \) which, as you move around in \( Y \), displays the cohomology of the fiber.

3. We can also determine the global cohomology of the complex \( Rf_*(\mathbb{Z}) \), for it is the cohomology of the global sections \( \Gamma(X, I^\bullet) \), that is, the cohomology of \( X \). More generally, the same argument shows that: for any complex of sheaves \( A^\bullet \) on \( X \), the complex \( Rf_*(A^\bullet) \) is a sheaf on \( Y \) whose global cohomology is

\[ H^*(Y, Rf_*(A^\bullet)) \cong H^*(X, A^\bullet). \]

This complex of sheaves therefore provides data on \( Y \) which allows us to compute the cohomology of \( X \). It is called the Leray Sheaf (although historically, Leray really considered only its cohomology sheaves \( R^m f_*(A^\bullet) \)). In particular we see that the functor \( Rf_* \) does not change the hypercohomology. For \( f : X \to \{\text{pt}\} \), if \( S \) is a sheaf on \( X \) then \( f_*(S) = \Gamma(X, S) \) is the functor of global sections (or rather, it is a sheaf on a single point whose value is the global sections), so \( R^0 f_*(A^\bullet) = H^0(X, A^\bullet) \) is the hypercohomology.

4. The \( m \)-th derived functor of Hom is called \( \text{Ext}^m \), i.e., it is the group

\[ \text{Ext}^m(A^\bullet, B^\bullet) = H^m(\text{RHom}(A^\bullet, B^\bullet)) = H^m(\text{Hom}^\bullet(A^\bullet, J^\bullet)) = H^0(\text{Hom}^\bullet(A^\bullet, J^\bullet[m])) \]

where \( B^\bullet \to J^\bullet \) is an injective resolution. (We consider \( \text{Hom}(A^\bullet, B^\bullet) \) to be a functor of the \( B^\bullet \) variable and derive it by injectively resolving. It turns out, as we will see later, that the same
result can be obtained by *projectively* resolving $A^\bullet$.) As before, there is a sheaf version of Hom, which also gives a sheaf version of Ext:

$$\text{Ext}^m(A^\bullet, B^\bullet) = H^m(\text{RHom}(A^\bullet, B^\bullet))$$

Exercise. Let $G, H$ be abelian groups, considered as complexes in degree zero. Show that $\text{Ext}^1(G, H)$ coincides with the usual definition of $\text{Ext}_{\mathbb{Z}}(G, H)$.

6.4. **The derived category: second construction.** The derived category can be constructed as a sort of quotient category of the homotopy category $K^b(X)$ of complexes, by inverting quasi-isomorphisms. Let $E^b(X)$ be the category whose objects are complexes of sheaves on $X$, and where a morphism $A^\bullet \to B^\bullet$ is an equivalence class of diagrams

$$
\begin{array}{ccc}
A^\bullet & \xleftarrow{qi} & C^\bullet \\
\downarrow & & \downarrow \\
B^\bullet
\end{array}
$$

where $C^\bullet \to A^\bullet$ is a quasi-isomorphism, and where two such morphisms $A^\bullet \leftarrow C_1^\bullet \to B^\bullet$ and $A^\bullet \leftarrow C_2^\bullet \to B^\bullet$ are considered to be equivalent if there exists a diagram

$$
\begin{array}{ccc}
A^\bullet & \xleftarrow{qi} & C_3^\bullet & \xrightarrow{qi} & B^\bullet \\
\downarrow & & \downarrow & & \downarrow \\
C_1^\bullet & \xleftarrow{qi} & C_2^\bullet
\end{array}
$$

that is commutative up to homotopy. (Exercise: figure out how to compose two morphisms and then check that the result is well defined with respect to be above equivalence relation.)

6.5. **Theorem.** The natural functor $D^b(X) \to E^b(X)$ is an equivalence of categories.

*Proof.* To show that a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories it suffices to show (a) that it is essentially surjective, meaning that every object in $\mathcal{D}$ is isomorphic to an object $F(C)$ for some object $C$ in $\mathcal{C}$, and (b) that $F$ induces an isomorphism on Hom sets. The first part (a) is clear because we have injective resolutions. Part (b) follows immediately from the fact that a quasi-isomorphism of injective complexes is a homotopy equivalence and has a homotopy inverse. □

This gives a way of referring to elements of the derived category without having to injectively resolve. Each complex of sheaves is automatically an object in the derived category $E^b(X)$. Here are some applications.
6.6. T-acyclic resolutions. A functor $T : \mathcal{C} \to \mathcal{D}$ between abelian categories is exact if it takes exact sequences to exact sequences. It is left exact if it preserves kernels, that is, if $f : X \to Y$ and if $Z = \ker(f)$ (so $0 \to Z \to X \to Y$ is exact) then $T(Z) = \ker(T(f))$ (that is, $0 \to T(Z) \to T(X) \to T(Y)$ is exact). An object $X$ is T-acyclic if $R^iT(X) = 0$ for all $i \neq 0$. This means: take an injective resolution $X \to I^\bullet$, apply $T$, take cohomology, the result should be zero except possibly in degree zero.

The great advantage of T-acyclic objects is that they may be often used in place of injective objects when computing the derived functors of $T$, that is

6.7. Lemma. Let $T$ be a left exact functor from the category of sheaves to some abelian category with enough injectives. Let $A^\bullet$ be a complex of sheaves and let $A^\bullet \to X^\bullet$ be a quasi-isomorphism, where each of the sheaves $X^r$ is T-acyclic. Then $R^iT(A^\bullet)$ is canonically isomorphic to the $r$-th cohomology object of the complex

$$T(X^0) \to T(X^1) \to T(X^2) \to \cdots.$$  

If $T$ is exact then there is no need to take a resolution at all: $RT(A^\bullet)$ is canonically isomorphic to $T(A^\bullet)$.

The proof is the standard double complex argument: Suppose $T$ is left exact. Let $I^{\bullet\bullet}$ be a double complex of injective sheaves, the $r$-th row of which is an injective resolution of $X^r$. Let $Z^\bullet$ be the total complex of this double complex. It follows that $A^\bullet \to X^\bullet \to Z^\bullet$ are quasi-isomorphisms and so the complex $Z^\bullet$ is an injective resolution of $A^\bullet$. Now augment the double complex by attaching $X^\bullet$ to the zeroth column, and apply the functor $T$ to the augmented complex. Since each $X^r$ is T-acyclic, each of the rows remains exact except possibly at the zeroth spot. Since $T$ is left exact, the rows are also exact at the zeroth spot. So our lemma says that $T(X^\bullet) \to T(Z^\bullet) = RT(A^\bullet)$ is also a quasi-isomorphism, which gives an isomorphism between their cohomology objects. If the functor $T$ is exact then every object $A^r$ is T-acyclic (exercise), so the original complex $A^\bullet$ may be used as its own T-acyclic resolution.

6.8. Key exercises. Show that injective objects are T-acyclic for any left exact functor $T$. If $f : X \to Y$ is a continuous map, show that $f^* : S(Y) \to S(X)$ is exact (and so it does not need to be derived). Using this and the adjunction formula $\text{Hom}_{S(Y)}(B, f_*^*(I)) \cong \text{Hom}_{S(X)}(f^*(B), I)$ show that $f_*$ is left exact and takes injectives to injectives. Show that fine, flabby, and soft sheaves are $\Gamma$-acyclic. In particular, the cohomology of a sheaf (or of a complex of sheaves) may be computed with respect to any injective, fine, flabby, or soft resolution.

6.9. More derived functors. This also gives us a way to define “the” derived functor $RT : D^b(X) \to D^b(\mathcal{C})$ for any left exact functor $T : Sh(X) \to \mathcal{C}$ provided the category $\mathcal{C}$ has enough injectives, namely, if $A^\bullet$ is a complex of sheaves on $X$, take an injective resolution $A^\bullet \to I^\bullet$, then apply the functor $T$ to obtain a complex $T(I^\bullet)$ of objects in the category $\mathcal{C}$, then injectively resolve this complex by the usual method of resolving each $T(I^r)$ to obtain a double complex, then forming the associated total complex. Let $RT$ denote the resulting complex. Different choices of resolutions give isomorphic complexes $RT$. 

6.10. **The sheaf of chains in the derived category.** Suppose $X$ is a finite simplicial complex. If $\sigma$ is a (closed) simplex let $Q_\sigma$ denote the constant sheaf on $\sigma$. It is injective in the category of simplicial sheaves on $X$, and every injective simplicial sheaf (of rational vector spaces) is a direct sum of such elementary injectives. The sheaf of chains can be realized as the (injective) complex $C_*$:

$$
\bigoplus_{\dim(\sigma)=0} Q_\sigma \leftarrow \bigoplus_{\dim(\sigma)=1} Q_\sigma \leftarrow \bigoplus_{\dim(\sigma)=2} Q_\sigma \leftarrow \cdots
$$

in degrees $0, -1, -2, \cdots$ respectively. (We place the chains in negative degrees so that the differentials will increase degree; it is a purely formal convention.) The global sections of this complex equals the usual complex $C_*(X)$ of simplicial chains. If we use the constant sheaf $\mathbb{Z}_\sigma$ everywhere, then the resulting sheaves are soft, rather than injective, but they may still be used to compute the homology of $X$.

Now consider the limit over all subdivisions of the simplicial complex $X$. We define a “topological” sheaf on $X$ that is, in some sense, the sheafification of the direct limit of these sheaves. To be precise, let $U \subset X$ be an open subset and let $T$ be a locally finite triangulation of $U$, and let $C^T_r(U)$ be the group of $r$-dimensional simplicial chains with respect to this triangulation. Then the sheaf of piecewise linear chains is the sheaf $C^{\bullet}_{PL}$ with sections

$$
\Gamma(U, C^{\bullet}_{PL}) = \lim_{\longrightarrow} C^T_r(U)
$$

for $r \geq 0$. It is a soft sheaf, and the resulting complex

$$
C^0_{PL} \leftarrow C^{-1}_{PL} \leftarrow C^{-2}_{PL} \leftarrow \cdots
$$

is quasi-isomorphic to the sheaf of Borel-Moore chains. If the space $X$ has a real analytic (or semi-analytic or subanalytic or $\mathcal{O}$-minimal) structure then one similarly has the sheaf of locally finite subanalytic or $\mathcal{O}$-minimal chains, which gives another quasi-isomorphic “incarnation” of the sheaf of chains.

Evidently, the derived category would be more useful if such complexes could be considered to be objects in the derived category (without having to resort to taking injective resolutions).

6.11. **The bad news.** The derived category is not an abelian category. In fact, the homotopy category of complexes is not an abelian category. Kernels and cokernels do not make sense in these categories. The saving grace is that the cone operation still makes sense and in fact, it passes to the homotopy category. So we have to replace kernels and cokernels with triangles.

6.12. **Definition.** A triangle of morphisms

$$
\begin{array}{ccc}
A^\bullet & \longrightarrow & B^\bullet \\
\downarrow & & \downarrow \\
C^\bullet & \longrightarrow & [1]
\end{array}
$$
in $K^b(X)$ or in $D^b(X)$ is said to be a distinguished triangle if it is homotopy equivalent to a triangle
\[ \begin{array}{c}
X^\bullet \\
\phi \\
[1] \\
C(\phi)
\end{array} \rightarrow \begin{array}{c}
Y^\bullet \\
[1]
\end{array} \]

where $C(\phi)$ denotes the cone on the morphism $\phi$.

This means that there are morphisms between corresponding objects in the triangles such that the resulting squares commute up to homotopy. (In the homotopy categories $K(X)$ and $D^b(X)$ homotopy equivalences are isomorphisms, so people often define a distinguished triangle to be a triangle in $K(X)$ that is isomorphic to a mapping cone.)

6.13. **Lemma.** The natural functor $K^b(X) \to D^b(X)$ takes distinguished triangles to distinguished triangles. If
\[ \begin{array}{c}
A^\bullet \\
[1] \\
C^\bullet
\end{array} \rightarrow \begin{array}{c}
B^\bullet
\end{array} \]
is a distinguished triangle and if $X^\bullet$ is a complex (bounded from below) then there are distinguished triangles
\[
\begin{array}{c}
\text{RHom}(X^\bullet, A^\bullet) \\
[1] \\
\text{RHom}(X^\bullet, C^\bullet)
\end{array} \rightarrow \begin{array}{c}
\text{RHom}(X^\bullet, B^\bullet)
\end{array} \rightarrow \begin{array}{c}
\text{RHom}(B^\bullet, X^\bullet)
\end{array} \rightarrow \begin{array}{c}
\text{RHom}(C^\bullet, X^\bullet)
\end{array}
\]

and
\[
\begin{array}{c}
\text{RHom}(A^\bullet, X^\bullet)
\end{array} \rightarrow \begin{array}{c}
\text{RHom}(B^\bullet, X^\bullet)
\end{array}
\]

The proof is the observation (from the last lecture) that Hom into a cone is equal to the cone of the Homs. We stress again that the hypercohomology of RHom is exactly the group of homomorphisms in the derived category: $H^0(X, \text{RHom}(A^\bullet, B^\bullet)) = \text{Hom}_{D^b(X)}(A^\bullet, B^\bullet)$.

6.14. **Exact sequence of a pair.** Let $Z$ be a closed subspace of a topological space $X$, and let $U = X - Z$, say $Z \xrightarrow{i} X \xleftarrow{j} U$.

If $S$ is a sheaf on $X$ then there is a short exact sequence of sheaves $0 \rightarrow j_* j^* S \rightarrow S \rightarrow i_* i^* S \rightarrow 0$. The morphisms are obtained by adjunction, and exactness can be checked stalk by stalk: If $x \in Z$ then the sequence reads $0 \rightarrow 0 \rightarrow S_x \rightarrow S_x \rightarrow 0$. If $x \in U$ then the sequence reads $0 \rightarrow S_x \rightarrow S_x \rightarrow 0 \rightarrow 0$. Consequently if $A^\bullet$ is a complex of sheaves then there is a distinguished
(recall the exercise that the cokernel of an injective morphism is quasi-isomorphic to the mapping cone). For the constant sheaf the triangle gives an exact sequence
\[ H^r(X) \to H^r(Z) \to H^{r+1}(U) \to \cdots \]
but observe that \( H^*_c(U) = H^*(X, Z) \) is the cohomology of cochains on \( X \) that vanish on \( Z \). For the sheaf of chains this gives an exact sequence \( H_r(U) \to H_r(X) \to H_r(X, U) \to \cdots \) because \( i^*(\text{chains}) \) is the limit over open sets containing \( Z \) of Borel-Moore chains on that open set.

If \( S \) is a sheaf on \( X \) define \( i^!(S) \) to be the restriction to \( Z \) of the presheaf with sections supported in \( Z \), that is
\[ i^!(S) = i^*(S^Z) \text{ where } \Gamma(V, S^Z) = \{ s \in \Gamma(V, S) \mid \text{spt}(s) \subset Z \} . \]
Thus, if \( W \subset Z \) is open then
\[ \Gamma(W, i^!(S)) = \lim_{V \supset W} \Gamma_Z(V, S) \] (the limit is over open sets \( V \subset X \) containing \( W \)). The functor \( i^! \) is a right adjoint to the push-forward with compact support \( i_! \), that is, \( \text{Hom}_X(i_!A, B) = \text{Hom}_Z(A, i^!B) \). In fact, \( \text{Hom}_Z(A, i^!B) \) consists of mappings of the leaf space \( LA \to LB|Z \) that can be extended by zero to a neighborhood of \( Z \) in \( X \), and this is the same as \( \text{Hom}_X(i_!A, B) \). (In particular we obtain a canonical morphism \( i_!i^!B \to B \).)

6.15. Proposition. Let \( A^\bullet \) be a complex of sheaves on \( X \). There is another distinguished triangle
\[ R j_i^*(A^\bullet) \to A^\bullet \]
Later we will prove this using Verdier duality. For now, observe that in the case of the constant sheaf this gives an exact sequence
\[ H^r(X) \to H^r(U) \to H^r(X, U) \to \cdots \]
For the sheaf of chains, \( i^!C_{-r} \) will give the homology of a tiny neighborhood of \( Z \) in \( X \) which, for most nice spaces, will be homotopy equivalent to \( Z \) itself. The sheaf \( j^*(C_{-r}) \) will give the Borel-Moore homology of \( U \), which is the relative homology \( H_r(X, Z) \). So this triangle gives the long exact sequence for the homology of the pair \((X, Z)\).