5. Lecture 5: Homotopy and Injectives

5.1. The sheaf of chains. Most of the complexes of sheaves that were discussed until now have the property that their cohomology sheaves live only in degree zero. The sheaf of chains, however, is a naturally occurring entity with complicated cohomology sheaves. But it is not so obvious how to construct a sheaf on \( X \) that corresponds to the singular chains. One might define a presheaf \( C_r \) with sections \((U, C_r) = C_r(U)\) to be the group of singular chains on \( U \) and with restriction mapping \( C_r(U) \to C_r(V)\) that assigns zero to every singular simplex that is not completely contained in \( V \). If we sheafify this then we are forced to consider singular chains that are (possibly) infinite sums of simplices. If the space \( X \) is paracompact then we may restrict to chains that are locally finite. The result is called Borel-Moore homology. For field coefficients \( k \), Borel and Moore defined a sheaf with sections \((U, C_{BM}^r) = \text{Hom}(C_r^c(U; k), k)\) where \( C_r^c \) denotes the cochains with compact support. For more general rings \( R \) it is necessary to replace \( \text{Hom}(\cdot, k) \) with a complex \( \text{Hom}(\cdot, I^\bullet) \) where \( R \to I^\bullet \) is an injective resolution of the ring. The resulting double complex is converted into a single complex by the usual diagonal sum trick. If \( X \) is compact then the Borel-Moore homology coincides with the usual (e.g. singular) homology.

Exercise. In the simplicial sheaf setting find a complex of injective sheaves that gives the Borel-Moore homology for a finite simplicial complex \( K \).

Let us say that a topological space \( X \) has finite type if it is homeomorphic to \( K - L \) where \( K \) is a finite simplicial complex and \( L \) is a closed subcomplex. In this case \( H^r_{BM}(X) \cong H_r(K, L) \) coincides with the relative homology which can then be expressed as the homology of a chain complex formed by the simplices that are contained only in \( K \), and by defining the differential so as to ignore all components of the boundary that may lie in \( L \). This gives a simple combinatorial construction of Borel-Moore homology for spaces of finite type.

Exercise. Find a complex of injective sheaves that gives the Borel-Moore homology for a space \( X = K - L \) of finite type, where \( K \) is a finite simplicial complex and \( L \) is a closed subcomplex.

5.2. Homotopy theory. Two morphisms \( f, g : A^\bullet \to B^\bullet \) of complexes are said to be homotopic if there is a collection of mappings \( h : A^r \to B^{r-1} \) so that \( hd_A + d_B h = f - g \). This is an equivalence relation. Equivalence classes are referred to as homotopy classes of maps; the set of which is denoted \([A^\bullet, B^\bullet]\). Define the complex of abelian groups (or \( R \) modules)

\[
\text{Hom}^n(A^\bullet, B^\bullet) = \prod_s \text{Hom}(A^s, B^{s+n})
\]

with differential \( df = d_B f + (-1)^{n+1} f d_A \) where \( f : A^s \to B^{s+n} \).

5.3. Lemma. Let \( f : A^\bullet \to B^\bullet \) be a morphism of complexes and let \( C(f) \) be the cone of \( f \). For any complex \( S^\bullet \) we have a morphism of complexes \( f_* : \text{Hom}^\bullet(S^\bullet, A^\bullet) \to \text{Hom}^\bullet(S^\bullet, B^\bullet) \). Then there is
a canonical isomorphism of complexes of abelian groups,

\[ C(f_*) \cong \text{Hom}^*(S^*, C(f)). \]

For any complex \( T^* \) we have a morphism of complexes \( f^* : \text{Hom}(B^*, T^*) \to \text{Hom}(A^*, T^*) \). Suppose the cohomology of \( S^*, T^* \) is bounded, that is, \( H^r(S^*) = H^r(T^*) = 0 \) if \( |r| \) is sufficiently large. Then there is a canonical quasi-isomorphism of complexes of abelian groups,

\[ C(f^*)[-1] \cong \text{Hom}^*(C(f), T^*). \]

The first statement is obvious because \( \text{Hom}(S^s, A^{t+1} \oplus B^t) = \text{Hom}(S^s, A^{t+1}) \oplus \text{Hom}(S^s, B^t) \).

The second statement is similar. The following exercise is crucial.

5.4. Proposition. \( H^n(\text{Hom}^*(A^*, B^*)) = [A^*, B^*[n]]. \)

In particular given \( f : A^* \to B^* \) let \( C^* = C(f) \) be the cone. Then for any complex \( S^* \) there is a long exact sequence

\[ \cdots [S^*, A^*] \to [S^*, B^*] \to [S^*, C^*] \to [S^*, A^*[1]] \to [S^*, B^*[1]] \to \cdots \]

We can do the same with sheaf-Hom. Recall that \( \text{Hom}_*(A, B)(U) = \text{Hom}_{\text{sh}(U)}(A|U, B|U) \). We obtain a complex of sheaves,

\[ \text{Hom}^n(A^*, B^*) = \prod_s \text{Hom}(A^s, B^{s+n}) \]

with the property that

\[ H^0(X, \text{Hom}^*(A^*, B^*)) = \Gamma(X, \text{Hom}^*(A^*, B^*)) = H^0(\text{Hom}^*(A^*, B^*)) = [A^*, B^*] \]

5.5. The bounded homotopy category \( K^b(X) \) of sheaves on \( X \) is the category whose objects are complexes of sheaves whose cohomology sheaves are bounded (meaning that \( H^r(A^*) = 0 \) for sufficiently large \( r \)), and whose morphisms are homotopy classes of morphisms, that is,

\[ \text{Hom}_{K^b(X)}(A^*, B^*) = [A^*, B^*] = H^0(X, \text{Hom}^*(A^*, B^*)). \]

5.6. Wonderful properties of injective sheaves. Roughly speaking, when we restrict to injective objects, then quasi-isomorphisms become homotopy equivalences. For sheaf theory, this is important because a homotopy of complexes of sheaves also gives a homotopy on global sections. In this way, quasi-isomorphisms of complexes of injective sheaves give isomorphisms on hypercohomology.

5.7. Lemma. Let \( C^* \) be a (bounded below) complex of sheaves and suppose that the cohomology sheaves \( H^r(C^*) = 0 \) for all \( r \). Let \( J^* \) be a complex of injective sheaves. Then any morphism \( f : C^* \to J^* \) is homotopic to zero, meaning that there exists \( h : C^* \to J^*[-1] \) such that \( d_J h + h d_C = f \).
Proof. It helps to think about the diagram of complexes:

$$
\begin{array}{cccccccc}
0 & \longrightarrow & C^0 & \overset{d^0}{\longrightarrow} & C^1 & \overset{d^1}{\longrightarrow} & C^2 & \overset{d^2}{\longrightarrow} & C^3 & \longrightarrow & \cdots \\
& & f & \downarrow & h^1 & \downarrow & f & \downarrow & h^2 & \downarrow & \cdots \\
0 & \longrightarrow & J^0 & \overset{d^0}{\longrightarrow} & J^1 & \overset{d^1}{\longrightarrow} & J^2 & \overset{d^2}{\longrightarrow} & J^3 & \longrightarrow & \cdots 
\end{array}
$$

The first step is easy, since \( C^0 \to C^1 \) is an injection and since \( J^0 \) is injective there exists \( h^1 : C^1 \to J^0 \) that makes the triangle commute, that is, \( h^1 d^0 = f \). Now let us define \( h^2 : C^2 \to J^1 \). Consider the map \( (f - d^0 h^1) : C^1 \to J^1 \). It vanishes on \( \text{Im}(d^0) = \ker(d^1) \) because

\[
(f - d^0 h^1)d^0 = f d^0 - d^0 h^1 d^0 = f d^0 - d^0 f = 0.
\]

Therefore it passes to a vertical mapping in this diagram:

$$
\begin{array}{ccc}
C^1 & \longrightarrow & C^1 / \ker(d^1) \\
& & \downarrow f - d^0 h^1 \\
& & J^1 \\
\end{array}
$$

where the second horizontal mapping is an injection. Since \( J^1 \) is injective we obtain an extension \( h^2 : C^2 \to J^1 \) such that \( h^2 \circ d^1 = f - d^0 h^1 \) so that \( h^2 d^1 + d^0 h^1 = f \). Continuing in this way, the other \( h^r \) can be constructed inductively. \( \square \)

Exactly the same argument may be used to prove the following:

5.8. **Lemma.** Let \( f : A^\bullet \to B^\bullet \) be a quasi-isomorphism of (bounded below) complexes. Then for any complex \( J^\bullet \) of injectives, the induced map \([B^\bullet, J^\bullet] \to [A^\bullet, J^\bullet]\) on homotopy classes is an isomorphism. \( \square \)

This result can also be proven by applying the previous lemma to the cone \( C(f) \) and using the long exact sequence on cohomology.

5.9. **Corollary.** The following statements hold.

1. Suppose \( J^\bullet \) is a complex of injective sheaves and \( H^n(J^\bullet) = 0 \) for all \( n \). Then \( J^\bullet \) is homotopy equivalent to the zero complex.
2. Let \( \phi : X^\bullet \to Y^\bullet \) be a quasi-isomorphism of sheaves of injective complexes. Then \( \phi \) admits a homotopy inverse \( g : Y^\bullet \to X^\bullet \) (meaning that \( g \phi \sim I_X \) and \( \phi g \sim I_Y \)).
3. Let \( A^\bullet \to I^\bullet \) and \( B^\bullet \to J^\bullet \) be injective resolutions of complexes \( A^\bullet, B^\bullet \). Then any morphism \( f : A^\bullet \to B^\bullet \) admits a lift \( \tilde{f} : I^\bullet \to J^\bullet \) and any two such lifts are homtopic.
4. Let \( f : A^\bullet \to B^\bullet \) be a quasi-isomorphism of complexes of sheaves. Then \( f \) induces an isomorphism on hypercohomology \( H^r(X, A^\bullet) \cong H^r(X, B^\bullet) \) for all \( r \).
Proof. For (1) consider the identity mapping $J^* \to J^*$ For (2), mapping $(X^* \xrightarrow{\phi} Y^*)$ to $X^*$ and using the lemma gives an isomorphism $[Y^*, X^*] \to [X^*, X^*]$, the map given by $f \mapsto f \circ \phi$. So the identity $X^* \to X^*$ corresponds to some $f$ such that $f \circ \phi \sim Id$, implying that $\phi$ has a left homotopy-inverse. Now consider mapping $Y^*$ into the triangle $X^* \to Y^* \to C(\phi) \to \cdots$, giving an exact sequence $\cdots \to [Y^*, X^*] \to [Y^*, Y^*] \to [Y^*, C(\phi)] \to \cdots$. Since $C(\phi)$ is injective and its cohomology vanishes, the identity is homotopic to zero, hence $[Y^*, C(\phi)] = 0$ so that $[Y^*, X^*] \cong [Y^*, Y^*]$ with the map given by $g \mapsto \phi \circ g$. Therefore there exists $g : Y^* \to X^*$ so that $\phi \circ g \sim Id$ meaning that $\phi$ has a right inverse in the homotopy category. If a mapping has both a left inverse and a right inverse then it has an inverse (in other words, $f$ and $g$ are homotopic, so either of them will behave as a homotopy inverse to $\phi$). For (3), the lemma gives an isomorphism $[A^*, J^*] \to [I^*, J^*]$. For (4), the hypercohomology is defined in terms of the global sections of an injective resolution. So we may assume that $A^*$ and $B^*$ are injective. By the lemma, the cone $C(f)$ is homotopic to zero. Let $h$ be such a homotopy. Now take global sections. The global sections of the cone coincides with the cone on the global sections, that is, we have a triangle of groups:

$$
\begin{array}{ccc}
\Gamma(X, A^*) & \xrightarrow{\alpha} & \Gamma(X, B^*) \\
\downarrow & & \downarrow \\
\Gamma(X, C(f)) & \xrightarrow{\beta} & \Gamma(X, C(f))
\end{array}
$$

The homotopy $h$ also gives a homotopy on the global sections so that $H^n(\Gamma(X, C(f))) = 0$. So the long exact sequence on cohomology implies that $H^n(X, A^*) \to H^n(X, B^*)$ is an isomorphism. □