15. LECTURE 15: COHOMOLOGY OF TORIC VARIETIES

15.1. In 1915 Emmy Noether proved that if a Hamiltonian system is preserved by an 1-parameter infinitesimal symmetry (that is to say, by the action of a Lie group) then a certain corresponding "conjugate" function, or "first integral" is preserved under the time evolution of the system. Time invariance gives rise to conservation of energy. Translation invariance gives rise to conservation of momentum. Rotation invariance gives rise to conservation of angular momentum.

Today, this is known as the moment map: Suppose $(M, \omega)$ is a symplectic manifold, and suppose a compact lie group $G$ acts on $M$ and preserves the symplectic form. The infinitesimal action of $G$ in the direction of $V$ defines a vector field $X$ on $M$. Contract this with the symplectic form to obtain a 1-form $\theta = \iota_X(\omega)$. It follows that $d\theta = 0$. If the action of $G$ is Hamiltonian then in fact, $\theta = df$ for some smooth function $f : M \to \mathbb{R}$ (defined up to a constant). This is the conserved quantity. In summary, if the action is Hamiltonian then there exists a moment map, that is, a smooth mapping $\mu : M \to \mathfrak{g}^*$ so that for each $X \in \mathfrak{g}$ the differential of the function $p \mapsto \langle \mu(p), X \rangle$ equals $\iota_X(\omega)$.

Now consider the Fubini Study metric $h(z, w) = \sum d\bar{z}_i \wedge dz_i$ on projective space. The real and imaginary part, $h = R + i\omega$ are respectively, positive definite and sympletic. Fix $a_0, a_1, \cdots, a_n \in \mathbb{Z}$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{noether.png}
\caption{Emmy Noether}
\end{figure}
If \( \lambda \in \mathbb{C}^x \) acts on \( \mathbb{C}^{n+1} \) by
\[
\lambda \cdot (z_0, z_1, \ldots, z_n) = (\lambda^{a_0} z_0, \lambda^{a_1} z_1, \ldots, \lambda^{a_n} z_n)
\]
then, restricting the action to \((S^1)^n\), the resulting moment map \( \mu : \mathbb{CP}^n \to \mathbb{R} \) is
\[
\mu([z_0 : z_1 : \cdots : z_n]) = \frac{a_0|z_0|^2 + a_1|z_1|^2 + \cdots + a_n|z_n|^2}{(|z_0|^2 + \cdots + |z_n|^2)}.
\]
If \( (\lambda_0, \lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^x)^n \) acts on \( \mathbb{C}^n \) by
\[
(\lambda_0, \ldots, \lambda_n) \cdot (z_0, \ldots, z_n) = (\lambda_0 z_0, \ldots, \lambda_n z_n)
\]
then, restricting the action to \((S^1)^n\) the resulting moment map \( \mu : \mathbb{CP}^n \to \mathbb{R}^n \)
\[
\mu([z_0 : \cdots : z_n]) = \frac{(|z_0|^2, |z_1|^2, \ldots, |z_n|^2)}{(|z_0|^2 + \cdots + |z_n|^2)}
\]
and it is the standard simplex contained in the hyperplane \( x_0 + \cdots + x_n = 1 \). These actions are Hamiltonian and the moment map collapses orbits of \((S^1)^n\).

15.2. Now let \( X \subset \mathbb{CP}^N \) be an \( n \)-dimensional subvariety on which a torus \( T = (\mathbb{C}^x)^n \) acts with finitely many orbits. In this case the action extends to a linear action on projective space of the sort described above and the moment map image (for the action of \((S^1)^n\), \( \mu(X) \subset \mu(\mathbb{CP}^n) \)) is convex. In fact, the convexity theorem of Atiyah, Kostant, Guillemin, Sternberg states that

15.3. Theorem. The moment map image \( \mu(X) \) is the convex hull of the images \( \mu(x_i) \) of the \( T \)-fixed points in \( X \). The image of each \( k \)-dimensional \( T \)-orbit is a single \( k \)-dimensional face of this polyhedron.

It turns out, moreover, that the toric variety is a rational homology manifold if and only \( \mu(X) \) is a simple polytope, meaning that each vertex is adjacent to exactly \( n \) edges.

Algebraic geometers prefer a presentation of a toric variety from a fan, a collection of homogeneous cones in Euclidean space. From a fan one constructs a convex polyhedron by intersecting the fan with a ball centered at the origin, and then flattening the faces. The resulting convex polyhedron is the dual of the moment map polyhedron. If the moment map polyhedron is simple then the fan-polyhedron is simplicial, meaning that the faces are simplices.

15.4. Definition. If \( Y \) is a complex algebraic variety define the intersection cohomology Poincaré polynomial
\[
h(Y, t) = h_0 + h_1 t + h_2 t^2 + \cdots + h_n t^n
\]
where \( h_r = \text{rank } \mathcal{H}^r(Y; \mathbb{Q}) \). If \( y \in Y \) define the local Poincaré polynomial \( h_y(Y, t) = \sum_{r \geq 0} \text{rank } (\mathcal{H}^r(\mathcal{IC}^*)_y) t^r \).

If \( Y \) is defined over \( \mathbb{F}_q \) we use the same notation for the Poincaré polynomial of the étale intersection cohomology.
15.5. Counting points. There is a very general approach to understanding the cohomology and intersection cohomology of an \(n\)-dimensional algebraic variety defined over a finite field \(\mathbb{F}_q\), provided its odd degree cohomology groups vanish. The variety \(Y/\mathbb{F}_q\) is said to be pure if the eigenvalues of Frobenius on \(H^r(Y; \mathbb{Q}_\ell)\) have absolute value \(\sqrt{q}\) with respect to any embedding into the complex numbers. The Weil conjectures (proven by Deligne) say that

\[
\sum_{r=0}^{2n} (-1)^r \text{Tr}(F_{\mathbb{F}_q}: H^r(Y) \to H^r(Y)) = |Y(\mathbb{F}_q)|
\]

the right hand side being the number of points that are fixed by the Frobenius morphism. The intersection cohomology of any projective algebraic variety is pure. If the variety \(Y\) is also nonsingular (so that \(IH^*(Y) = Y^*(Y)\) and Tate (which means that the eigenvalues on \(H^r\) are in fact equal to \(\sqrt{q}^r\)) then this gives

\[
h(Y, \sqrt{q}) = \sum_{r=0}^{n} \text{rank}H^{2r}(Y)q^r = |Y(\mathbb{F}_q)|.
\]

For example, if such a variety \(Y\) is defined over the integers, is nonsingular and \(Y(\mathbb{C})\) has an algebraic cell decomposition with \(m_r\) cells of (complex) dimension \(r\) then \(h_{2r+1} = 0\) and \(h_{2r} = m_r\) accounts for \(m_rq^r\) points over \(\mathbb{F}_q\). In the case of a nonsingular toric variety whose moment map image is a convex polyhedron with \(f_r\) faces of dimension \(r\) this gives

\[
h(Y, \sqrt{q}) = \sum_{s=0}^{n} \text{rank}H^{2s}(Y; \mathbb{C})q^s = \sum_{r=0}^{n} f_r(q-1)^r
\]

since each \(r\)-dimensional orbit is itself (isomorphic to) a torus of dimension \(r\). The hard Lefschetz theorem says \(h_{2s-2} \leq h_{2s}\) for \(2s \leq n\) which in turn gives inequalities between the face numbers, as observed by Stanley in 1980.

15.6. If we wish to use intersection cohomology rather than ordinary cohomology in the Weil conjectures then the formula becomes

\[
\sum_{r=0}^{2n} (-1)^r \text{Tr}(F_{\mathbb{F}_q}: IH^r(Y) \to IH^r(Y)) = |Y(\mathbb{F}_q)|_{\text{mult}}
\]

where each point \(y \in Y(\mathbb{F}_q)\) is counted with a multiplicity equal to the (alternating sum of) trace of Frobenius acting on the stalk of the intersection cohomology at \(y \in Y(\mathbb{F}_q)\). If this is pure and if the stalk cohomology vanishes in odd degrees, then this multiplicity equals the Poincaré polynomial \(h_y(Y, \sqrt{q})\) of the stalk of the intersection cohomology. In conclusion, if the intersection cohomology of \(Y\) is Tate and vanishes in odd degrees then
Let us now try to determine these multiplicities $h_y$. If $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ define the truncation $\tau_{\leq r} f$ to be the polynomial $a_0 + \cdots + a_r x^r$ consisting of those terms of degree $\leq r$.

**15.7. Lemma.** Let $Z \subset \mathbb{CP}^{N-1}$ be a projective algebraic variety of dimension $d$, with intersection cohomology Poincaré polynomial

$$g(t) = g_0 + g_1 t + \cdots + g_{2d} t^{2d} = \sum_{r=0}^{2d} \dim(IH^r(Z)) t^r.$$  

Then the stalk of the intersection cohomology of the complex cone $Y = \text{cone}_C(Z) \subset \mathbb{CP}^N$ at the cone point $y \in Y$ has Poincaré polynomial

$$h_y(Y, t) = \tau_{\leq d} (g(t)(1 - t^2)).$$

**Proof.** The complex projective space $\mathbb{CP}^N$ is the complex cone over $\mathbb{CP}^{N-1}$. In fact, if we remove the cone point then what remains is a line bundle $\mathcal{L} \to \mathbb{CP}^{N-1}$ whose first Chern class $c^1(\mathcal{L}) \in H^2(\mathbb{CP}^{N-1})$ is the class of a hyperplane section. This is to say that there exists a section of this bundle that vanishes precisely on a hyperplane; it may be taken to be

$$s([z_0 : \ldots : z_{N-1}]) = [z_0 : \ldots : z_{N-1}, \Sigma_j a_j z_j] \in \mathbb{CP}^N$$

for any fixed choice (not all zero) of $a_0, a_1, \ldots, a_{N-1} \in \mathbb{C}$. The vanishing of the last coordinate is a hyperplane in $\mathbb{CP}^{N-1}$. So this class may be used as a hard Lefschetz class.

If $Z \subset \mathbb{CP}^{N-1}$ is a projective algebraic variety then $\text{cone}_C(Z) \subset \mathbb{CP}^N$ is a singular variety and the link $L$ of the cone point can be identified with the sphere bundle of this line bundle $\mathcal{L}_Z \to Z$. The Gysin sequence becomes

$$IH^{i-2}(Z) \xrightarrow{\cup c^1} IH^i(Z) \longrightarrow IH^i(L) \longrightarrow IH^{i-1}(Z) \xrightarrow{\cup c^1} IH^{i+1}(Z) \longrightarrow$$

It follows from the hard Lefschetz theorem for $Z$ that $IH^i(L)$ is the primitive part of the intersection cohomology of $Z$ for $i \leq d$, that is,

$$IH^i(L) \cong IP^i(Z) = \text{coker}(c^1(\mathcal{L}) : IH^{i-2}(Z) \to IH^i(Z))$$

for $i \leq \dim(Z)$, and hence its Poincaré polynomial is given by

$$g_0 + g_1 t + (g_2 - g_0) t^2 + (g_3 - g_1) t^3 + (g_4 - g_2) t^4 + \cdots + (g_d - g_{d-2}) t^d = \tau_{\leq d} g(t)(1 - t^2). \quad \square$$

\[ (15.6.1) \quad \sum_{s=0}^{n} \text{Tr}(Fr_q|(IH^{2s}(Y))) = \sum_{s=0}^{n} \text{rank } IH^{2s}(Y) q^s = h(Y, \sqrt{q}) = \sum_{y \in Y(F_q)} h_y(Y, \sqrt{q}). \]
15.8. **Some geometry.** Let $\mu : Y \to P \subset \mathbb{R}^m$ be the moment map corresponding to the action of a torus $T \cong (\mathbb{C}^\times)^m$ on a toric variety $Y$. If $F$ is a face of $P$, the link of $F$ can be realized as another conex polyhedron, $L_F = P \cap V$ where $V \subset \mathbb{R}^N$ is a linear subspace such that $\dim(V) + \dim(F) = N - 1$, which passes near $F$ and through $P$. (For example, $V$ may be taken to lie completely in the plane $F^\perp$.) In fact, $L_F$ is the moment map image of a sub-toric variety $Y_F$ on which a sub-torus $T_F$ acts.

15.9. In the case of a toric variety $Y$, a given face $F$ corresponds to a stratum $S_F$ of the toric variety. The link of this stratum is therefore isomorphic to a circle bundle over a toric variety whose moment map image is the link $L_F$ of the face $F$. Let $h(Y_F, t)$ be the intersection cohomology Poincaré polynomial of this “link” toric variety. Then equations (15.7.1) and (15.6.1) give:

\[
15.10. \textbf{Theorem.} \textit{The IH Poincaré polynomial of } Y \textit{ is}
\]
\[
h(Y, t) = \sum_F (t^2 - 1)^{\dim(F)} \cdot \sum_{r \leq n - \dim(F)} ((1 - t^2) h(Y_F, t))
\]

15.11. In particular, the intersection cohomology only depends on the combinatorics of the moment map image $P = \mu(Y)$, and moreover, the functions $h(Y_F, t)$ may be determined (inductively) from the moment map images $L_F = \mu_F(Y_F)$. The hard Lefschetz theorem (which says that $h_{2r} \geq h_{2r-2}$ for all $r \leq \dim(Y)$) then implies a collection of inequalities among the numbers of chains of faces.

15.12. **Remarks.** This formula simplifies if $P$ is a simple polyhedron, to:

\[
h(Y, t) = \sum_F (t^2 - 1)^{\dim(F)} = f(t^2 - 1)
\]

where $f(s) = f_0 + f_1 s + \cdots + f_d s^d$ and $f_j$ is the number of faces of dimension $j$. The polytopes considered here are always rational, meaning that the vertices are rational points in $\mathbb{R}^d$, or equivalently, the faces are the kernels of linear maps $\mathbb{R}^d \to \mathbb{R}$ with rational coefficients. Any simple (or simplicial) polytope can be perturbed by moving the faces (resp. the vertices) so as to make them rational. Therefore the inequalities arising from hard Lefschetz apply to all simple polytopes. However a general polytope cannot necessarily be perturbed into a rational polytope with the same face combinatorics. The Egyptian pyramid, for example, has a square face. Lifting one of the vertices on this face, an arbitrarily small amount, will force the face to “break”. In order to prove that the inequalities arising from hard Lefschetz for intersection cohomology can be applied to any polytope it was necessary to construct something like intersection cohomology in the non-rational case. This was accomplished by (Barthel, Brasselet, Fieseler, Kaup) and K. Karu [11] (who proved that it satisfies the hard Lefschetz formula).
There is another way to prove this result using the decomposition theorem (which does not involve passing to varieties over a finite field). The singularities of the toric variety $Y$ can be resolved by a sequence of steps, each of which is toric with moment maps that correspond to ‘cutting off the faces" that are singular. For example, the Egyptian pyramid has a single singular point. The singularity is resolved by a mapping $\pi : \hat{Y} \to Y$ as illustrated in this diagram:

Let us examine the decomposition theorem for this mapping. The mapping is an isomorphism everywhere except over the singular point $y \in Y$ and $\pi^{-1}(y) \cong \mathbb{P}^1 \times \mathbb{P}^1$. The stalk cohomology of the pushforward $R\pi_*(\mathbb{Q}_Y)$ is $(\mathbb{Q}, 0, \mathbb{Q} \oplus \mathbb{Q}, 0, \mathbb{Q})$. Put this into the support diagram for a 3 dimensional variety:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>cod0</th>
<th>cod2</th>
<th>cod4</th>
<th>cod6</th>
<th>$H^\ast(\pi^{-1}(y))$</th>
</tr>
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<tbody>
<tr>
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<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td></td>
</tr>
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<tr>
<td>2</td>
<td>-1</td>
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<td></td>
<td>$\mathbb{Q} \oplus \mathbb{Q}$</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
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<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>$\mathbb{Q}$</td>
</tr>
</tbody>
</table>

(Here, the degree $i$ is the “usual” cohomology degree notation and the degree $j$ is the “perverse degree” notation.) From the decomposition theorem we know that one term will be $\mathcal{IC}^\ast(Y)$ and that there are additional terms supported at the singular point $y$. From the support condition it is clear that $\mathbb{Q}[3]$ (on the bottom row) is part of the $\mathcal{IC}$ sheaf. It is not so clear how much of the $(\mathbb{Q} \oplus \mathbb{Q})[1]$ belongs to $\mathcal{IC}^\ast(Y)$ and how much belongs to the other terms. However the $\mathbb{Q}[-1]$ (at the top of the column; in degree $j = 1$) is definitely not part of $\mathcal{IC}$. By Poincaré and especially by Hard Lefschetz, it must be paired with one copy of $\mathbb{Q}$ in degree $-1$. So this leaves $\mathbb{Q}[3] \oplus \mathbb{Q}[1]$ (in degrees $-3$ and $1$ respectively) for the IC sheaf. A closer inspection of this argument shows that these two terms constitute the primitive cohomology of the fiber, as we saw earlier.

Thus, the decomposition theorem singles out the primitive cohomology of the fiber as belonging to the IC sheaf. Now, assuming by induction that the formula holds for $IH^\ast(\hat{Y}) = H^\ast(\hat{Y})$ (which is less singular that $Y$) and knowing how these terms decompose, it is easy to conclude that the formula must also hold for $IH^\ast(Y)$. 