12. Lecture 12: Examples of perverse sheaves

12.1. IC of subvarieties. As above we consider the middle perversity $\bar{m}$ and a Whitney stratified space of dimension $n$ with even dimensional strata. Let $Y$ denote the closure of a single stratum, $Y^\circ$. Let $L_Y$ be a local system on the stratum $Y^\circ$. Then the intersection complex $IC^\bar{m}_Y(L_Y)[{-\text{cod}(Y)/2}]$ is $\bar{m}$-perverse. Here are the support diagrams for an 8 dimensional stratified space with strata of dimension 0, 2, 4, 6, 8 where, as above, “x” denotes possibly nonzero stalk cohomology and “c” denotes possibly nonzero stalk cohomology with compact support.

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$\text{dim}(Y) = 8$

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$\text{dim}(Y) = 6$

**Figure 6.** Shifted IC of subvarieties

Adding these up gives the support diagram (Figure 8) for a perverse sheaf. (It is hoped that the Reader will appreciate the manner in which the Author coaxed latex into lining these up correctly.)

12.2. Logarithmic perversity. Because the support conditions for (middle) perverse sheaves are relaxed slightly from those for $IC^\bullet$, there are several other perversities for which intersection cohomology forms a (middle) perverse sheaf. These include the logarithmic perversity $\tilde{\ell}$, given by $\tilde{\ell}(k) = k/2 = \bar{m}(k) + 1$ and its Verdier dual, the sublogarithmic perversity, $\tilde{s}$ given by $\tilde{s}(k) = \bar{m}(k) - 1$.

12.3. Let $Y^\circ$ be a stratum of $W$ (which is stratified by even dimensional strata). Let $Y$ be its closure with inclusion $j_Y : Y \to W$. It is stratified by even dimensional strata. Let $A^\bullet$ be a perverse sheaf on $Y$. Then $Rj_*(A^\bullet)[-\text{cod}(Y)/2]$ is a perverse sheaf on $W$.

12.4. Hyperplane complements. Let $\{H_1, H_2, \cdots, H_r\}$ be a collection of complex affine hyperplanes in $W = \mathbb{C}^n$. Stratify $W$ according to the multi-intersections of the hyperplanes. The largest stratum is

$$W^\circ = W - \bigcup_{j=1}^{r} H_j$$
and it has a highly nontrivial fundamental group. Let $\mathcal{L}$ be a local coefficient system on this hyperplane complement. Then $IC^*_i(\mathcal{L})$, $IC^{\bullet}_m(\mathcal{L})$ and $IC^{\bullet}_t(\mathcal{L})$ are perverse sheaves on $W = \mathbb{C}^n$. These are surprisingly complicated objects, and even the case of middle perversity, when the hyperplanes are the coordinate hyperplanes, has been extensively studied. Notice, in this case, that the space $W = \mathbb{C}^n$ is nonsingular, the hyperplane complement $W^o$ is nonsingular, and the sheaf $IC(\mathcal{L})$ is constructible (with respect to this chosen stratification) but to analyze this sheaf we are forced to consider the singularities of the multi-intersections of the hyperplanes.

In the simplest case, $(\mathbb{C}, \{0\})$ the category of perverse sheaves is equivalent to the category of representations of the following quiver

$$
\begin{array}{c}
\bullet \\
\alpha \\
\beta \\
\bullet
\end{array}
$$

where $I - \alpha \beta$ and $I - \beta \alpha$ are invertible.

For $\mathbb{C}^2$, $xy = 0$ (the coordinate axes) the perverse category is equivalent to the category of representations of the quiver

$$
\begin{array}{c}
\bullet \\
\alpha \\
\beta \\
\bullet
\end{array}
$$

with the same conditions on each of the horizontal and vertical pairs, such that all possible ways around the outside of the square commute.
12.5. **Small and semismall maps.** Let $M$ be a compact complex algebraic manifold and let $\pi : M \to W$ be an algebraic mapping. Then $\pi$ is said to be *semismall* if

$$\text{cod}_W(\{x \in X | \dim \pi^{-1}(x) \geq k\}) \geq 2k.$$ 

In other words, if the map has been stratified then for each stratum $S \subset W$ the dimension of the fiber over $S$ is $\leq \frac{1}{2}$ the codimension of $S$. The map is *small* if, for each singular stratum $S$, $\dim \pi^{-1}(x) < \frac{1}{2}\text{cod}(S)$ (for all $x \in S$).

If $\pi$ is small then $R\pi_*(\mathbb{Q})$ is a self dual sheaf on $W$ whose support satisfies the support conditions of (middle) intersection cohomology. It follows from the axiomatic characterization that there is a canonical isomorphism (in $D^b_c(W)$), $R\pi_*(\mathbb{Q}) \cong IC_{\pi}(W)$. In other words, the intersection cohomology of $W$ is canonically isomorphic to the ordinary cohomology of $M$.

If $\pi$ is semi-small then $R\pi_*(\mathbb{Q})$ is (middle) perverse.

Let $W = \{P \subset \mathbb{C}^4 | \dim(P) = 2, \dim(P \cap \mathbb{C}^2) \geq 1\}$ be the singular Schubert variety in the Grassmannian of 2-planes in 4-space. It has a singularity when $P = \mathbb{C}^2$. A resolution of singularities is $\tilde{W} = \{(P, L) | P \in W, \text{ and } L \subset P \cap \mathbb{C}^2 \subset \mathbb{C}^4\}$. Then $\pi : \tilde{W} \to W$ is a small map so $R\pi_*(\mathbb{Q}) \cong IC_{\pi}(W)$ hence $IH^*(W) \cong H^*(\tilde{W})$.

12.6. **Sheaves on $\mathbb{P}^1$.** Let us stratify $\mathbb{P}^1$ with a single zero dimensional stratum, $N$ (the north pole, say). The support diagram is the following:

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So the skyscraper sheaf supported at the point, $\mathbb{Q}_N[-1]$ is perverse. We also have the following:

$\mathbb{Q}_N[-1]:$ $\begin{array}{|c|ccc|}
  2 & 1 & \text{cx} \\
  0 & \text{cx} & x & x
\end{array}$ $\begin{array}{|c|ccc|}
  2 & 1 & c & c \\
  0 & x & x & x
\end{array}$

$\mathbb{Q}_\mathbb{P}^1:$$\begin{array}{|c|ccc|}
  2 & 1 & c & c \\
  0 & x & x & x
\end{array}$ $\begin{array}{|c|ccc|}
  2 & 1 & c & c \\
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\end{array}$

The first sheaf is self dual. The second sheaf is self dual. The third and fourth sheaves are dual to each other. It turns out that there is one more indecomposable perverse sheaf on this space, which is not an IC sheaf, and its support diagram is the full diagram. It is self dual. Here is how to construct it. Take a closed disk and put the constant sheaf on the interior, zero on the boundary, except for one point (or even one segment). Then map this disk to the 2-sphere, collapsing the boundary to the N pole, and push this sheaf forward.

If we started with zero on the boundary and pushed forward we would get the sheaf $Rj_!(\mathbb{Q})$. If we started with the full constant sheaf on the disk and pushed forward we would get the sheaf $Rj_*(\mathbb{Q})$. This new sheaf has both stalk cohomology and compact support stalk cohomology in degree 1, at the singular point. Verdier duality switches these two types of boundary conditions, so when we have a mixed boundary condition as in this case, we obtain a self dual sheaf.
In this case the category of perverse sheaves is equivalent to the category of representations of the quiver

\[
\begin{array}{c}
\bullet \\
\alpha \\
\beta \\
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\]

where \(\alpha \beta = \beta \alpha = I\). There are five indecomposable objects, one of which is has \(\mathbb{Q} \oplus \mathbb{Q}\) on one of the vertices of the graph.

12.7. **Deligne’s numbering system.** In their book [?] Beilinson, Bernstein and Deligne modified the indexing system for cohomology in a way that vastly reduces the amount of notation and arithmetic involving indices. Although the new system is extremely simple, it is deceptively so, because it takes us one step further away from any intuition concerning perverse sheaves. The new system works best in the case of a complex algebraic (or analytic) variety \(W\), stratified with complex algebraic (or analytic) strata, and counted according to their complex dimensions. The idea is simply to shift all degrees by \(\dim_{\mathbb{C}}(W) = \dim(W)/2\). So the support conditions look like this:

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In symbols,

\[
\dim \text{spt}H^{-r}(A^\bullet) \leq r \quad \text{and} \quad \dim \text{spt}H^{-r}(D(A^\bullet)) \leq r
\]

or equivalently,

\[
\dim \{x \in W \mid H^i_x(A^\bullet) \neq 0\} \leq -i \quad \text{for all } i \in \mathbb{Z}
\]

\[
\dim \{x \in W \mid H^i_x(A^\bullet) \neq 0\} \leq i \quad \text{for all } i \in \mathbb{Z}
\]

12.8. **Perversity zero.** Let \(W\) be a stratified pseudomanifold of dimension \(n\) (with a fixed stratification). The category of Perverse sheaves on \(W\) with perversity zero, constructible with respect to this stratification, is equivalent to the category of sheaves on \(W\) (nb: this means sheaves, rather than complexes of sheaves) that are constructible with respect to this stratification, that is, sheaves whose restriction to each stratum is locally trivial. In this case, the “abelian subcategory” defined
by the perversity condition simply coincides with the abelian category structure of the category of sheaves. Here are support diagrams for intersection cohomology with perversity zero, and for perverse sheaves with perversity zero.

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perverse sheaf support

**Figure 8.** Support/cosupport for $p = 0$ IC sheaf and perverse sheaf

**Technical interlude**

The following lemma that provides lifts of morphisms in the derived category, see [GMII] §?

**12.9. Lemma.** Let $A^\bullet, B^\bullet$ be objects in the derived category. Suppose $H^r(A^\bullet) = 0$ for all $r > p$ and suppose that $H^r(B^\bullet) = 0$ for all $r < p$. Then the natural map

$\text{Hom}_{D_b(X)}(A^\bullet, B^\bullet) \to \text{Hom}_{\mathcal{S}h(X)}(H^p(A^\bullet), H^p(B^\bullet))$

is an isomorphism.

**Proof.** When we wrote IH II, Verdier (who was one of the referees) showed us how to replace our 4 page proof with the following simple proof. Up to quasi-isomorphism it is possible to replace the complexes $A^\bullet, B/b$ with complexes

$$\cdots \longrightarrow A^{p-1} \xrightarrow{d_A} A^p \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \longrightarrow I^p \xrightarrow{d_B} I^{p+1} \longrightarrow I^{p+2} \longrightarrow \cdots$$

where $I^r$ are injective. This means that a morphism in the derived category is represented by an honest morphism between these complexes, that is, a mapping

$$\phi : H^p(A^\bullet) = \text{coker}(d_A) \to \ker(d_b) = H^p(B^\bullet).$$
12.10. **Proof of Theorem 11.1.** We have a Whitney stratification of $W$ and inclusions

\[ U_2 \to U_3 \to \cdots \to U_n \to U_{n+1} = W \]

Let us suppose that $A^\bullet$ is constructible with respect to this stratification and that it satisfies the support (but not necessarily the co-support) conditions, that is

\[ H^r(A^\bullet)_x = 0 \text{ for } r \geq p(c) + 1 \]

whenever $x \in X^{n-c}$ lies in a stratum of codimension $c$. Fix $k \geq 2$ and consider the situation

\[ U_k \to U_{k+1} \leftarrow X^{n-k} \]

where $X^{n-k}$ is the union of the codimension $k$ strata. Let $A^\bullet_k = A^\bullet|U_k$. Let $\bar{q}$ be the complementary perversity, $q(c) = c - 2 - p(c)$. The following proposition says that the vanishing of the stalk cohomology with compact supports $H^r(i_x^! A^\bullet|b)$ is equivalent to the condition that the attaching map is an isomorphism:

12.11. **Proposition.** The following statements are equivalent.

1. $A^\bullet_{k+1} \cong \tau_{\leq p(k)} Rj_{k *} A^\bullet_k$
2. $H^r(A^\bullet_{k+1})_x \to H^r(Rj_{k *}, A^\bullet)_x$ is an isomorphism for all $x \in X^{n-k}$
3. $H^r(i_x^! A^\bullet_{k+1}) = 0$ for all $r \leq p(k) + 1$
4. $H^r(i_x^! A^\bullet_{k+1}) = 0$ for all $r < n - q(k)$ for all $x \in X^{n-k}$

**Proof.** Items (1) and (2) are equivalent because there is a canonical morphism

\[ A^\bullet_{k+1} \to Rj_{k *}, A^\bullet = Rj_{k *}, A^\bullet_k \]

truncation $\tau_{\leq p(k)}$ leaves an isomorphism in degrees $\leq p(k)$. Items (3) and (4) are equivalent because $i_x : \{x\} \to X^{n-k}$ is the inclusion into a manifold so $i_x^! = i_x^! [n-k]$, and because $r < p(k) + 2 + (n-k) = n - (k - 2 - p(k)) = n - q(k)$. Items (2) and (3) are equivalent because there is a distinguished triangle,

\[
\begin{array}{ccc}
Ri_{k *}, i_x^! (A^\bullet_{k+1}) & \xrightarrow{\alpha} & A^\bullet_{k+1} \\
\downarrow & & \downarrow \\
Rj_{k *}, i_x^! (A^\bullet_{k+1}) & & \\
\end{array}
\]
and therefore an exact sequence on stalk cohomology as follows:

\[ H^{p+2}(i_k^* A_{k+1}^*) \rightarrow H^{p+2}(A_{k+1}^*) \rightarrow H^{p+2}(Rj_k^* A_k^*) \]

\[ H^{p+1}(i_k^* A_{k+1}^*) \rightarrow H^{p+1}(A_{k+1}^*) \rightarrow H^{p+1}(Rj_k^* A_k^*) \]

\[ H^p(i_k^* A_{k+1}^*) \rightarrow H^p(A_{k+1}^*) \]

\[ H^{p-1}(i_k^* A_{k+1}^*) \rightarrow H^{p-1}(A_{k+1}^*) \]

Now use the fact that the yellow highlighted terms are zero and the green highlighted morphisms are isomorphisms to conclude the proof of the Proposition.

Since the sheaf $IC_p^*[−n]$ satisfies these conditions, this proposition proves (by induction) that it is isomorphic (in the derived category) to the sheaf $P_p^*$ that is defined by Deligne’s construction. We gave an intuitive argument for this statement a few lectures ago, but the above constitutes a proof.

12.12. Continuation of the proof of Theorem 11.1. Now let us show that if $\mathcal{L}_1, \mathcal{L}_2$ are local systems on $U_2$ and if $A^*_k = P^*_p(\mathcal{L}_1)$ and if $B^*_k = P^*_p(\mathcal{L}_2)$ then we have an isomorphism

\[ \text{Hom}_{\text{Sh}}(\mathcal{L}_1, \mathcal{L}_2) \cong \text{Hom}_{D^b(X)}(A^*, B^*). \]

As before, let $A_{k+1}^* = A^*|U_{k+1} = \tau_{\leq p(k)}^* Rj_k^* A_k^*$. Assume by induction that we have established an isomorphism

\[ \text{Hom}(\mathcal{L}_1, \mathcal{L}_2) \cong \text{Hom}_{D^b(U_k)}(A_k^*, B_k^*). \]

Using the above triangle for $B^*$ we get an exact triangle of $\mathbf{R}\text{Hom}$ sheaves,

\[ \mathbf{R}\text{Hom}^*(A_{k+1}^*, Rj_k^* i_k^* B_{k+1}^*) \rightarrow \mathbf{R}\text{Hom}^*(A_{k+1}^*, B_{k+1}^*) \]

By Lemma 12.3 and the support conditions, we see that $\alpha$ is an isomorphism in degree zero,

\[ \text{Hom}_{D^b(X)}(A_{k+1}^*, B_{k+1}^*) = H^0(U_{k+1}; \mathbf{R}\text{Hom}^*(A_{k+1}^*, B_{k+1}^*)) \cong H^0(U_{k+1}; \mathbf{R}\text{Hom}^*(A_{k+1}^*, Rj_k^* B_k^*)). \]

Moreover,

\[ Rj_k^* \mathbf{R}\text{Hom}^*(A_k^*, B_k^*) \cong Rj_k^* \mathbf{R}\text{Hom}^*(j_k^* A_{k+1}^*, B_k^*) \cong \mathbf{R}\text{Hom}^*(A_{k+1}^*, Rj_k^* B_k^*) \]
by the standard identities (above), whose cohomology is

\[ H^0(U_{k+1}; Rj_\ast \mathbf{R} \mathbf{Hom}^\bullet(A_k^\bullet, B_k^\bullet)) \cong H^0(U_k; \mathbf{R} \mathbf{Hom}^\bullet(A_k^\bullet, B_k^\bullet)) \cong \mathbf{Hom}_{D^b(U_k)}(A_k^\bullet, B_k^\bullet). \]

So, putting these together we have a canonical isomorphism

\[ \mathbf{Hom}_{D^b(U_k)}(A_k^\bullet, B_k^\bullet) \cong \mathbf{Hom}_{D^b(U_{k+1})}(A_{k+1}^\bullet, B_{k+1}^\bullet) \]

which was canonically isomorphic to \( \mathbf{Hom}_{Sh}(\mathcal{L}_1, \mathcal{L}_2) \) by induction. This completes the proof of the theorem, but the main point is that the depth of the argument is the moment in which Lemma 12.9 was used in order to lift a morphism \( A_{k+1}^\bullet \to Rj_\ast B_k^\bullet \) to a morphism \( A_{k+1}^\bullet \to B_{k+1}^\bullet \).
\[ \Box \]