11. Perverse sheaves

In the next lecture we will prove the following result. It is actually the same as the proof that $P_n^\bullet[n] \cong IC_p^\bullet$.

11.1. Theorem. Let $W$ be an $n$-dimensional Whitney stratified set with biggest stratum $U$ and let $\bar{p}$ be a perversity. Then Deligne’s construction

$$\mathcal{L} \mapsto P^\bullet(\mathcal{L}) = \tau_{\leq \bar{p}(n)} Rj_{ns} \cdots \tau_{\leq \bar{p}(3)} Rj_{3s} \tau_{\leq \bar{p}(2)} Rj_{2s} \mathcal{L}$$

defines an equivalence of categories between the category of local systems of $K$-vector spaces ($K = \mathbb{Q}, \mathbb{R}, \mathbb{C}$) on the nonsingular part $U$, and the full subcategory of $D^b_c(W)$ consisting of “IC sheaves”, that is, complexes of sheaves $A^\bullet$, constructible with respect to the given stratification, such that the following conditions hold

1. $A^j|_{U_2} \cong \mathcal{L}$ is isomorphic to a local coefficient system
2. $\check{H}^r(A^\bullet) = 0$ for $r < 0$
3. $\check{H}^r(i_x^! A^\bullet) = 0$ for $r > \bar{p}(k)$
4. $\check{H}^r(i_x^! A^\bullet) = 0$ for $r < n - q(k)$

for all points $x \in W$, where $i_x : \{x\} \to W$ is the inclusion of the point and $k$ denotes the codimension of the stratum containing $x$ and where $q(k) = k - 2 - \bar{p}(k)$ is the complementary perversity.

This says, in particular, that if $\mathcal{L}$ is a local system on $U_2$ and if $A^\bullet$ is a constructible complex of sheaves that satisfies the above conditions, then there is a canonical isomorphism $A^\bullet \cong IC_p^\bullet(\mathcal{L})$. Moreover, it says that if $\mathcal{L}_1, \mathcal{L}_2$ are local systems on $U_2$ then

$$R\text{Hom}(IC_p^\bullet(\mathcal{L}_1), IC_p^\bullet(\mathcal{L}_2)) \cong \text{Hom}(\mathcal{L}_1, \mathcal{L}_2).$$

If $\mathcal{L}$ is an indecomposable local system (which is not isomorphic to a direct sum of two nontrivial local systems) then $IC_p^\bullet(\mathcal{L})$ is an indecomposable complex of sheaves (and is not isomorphic to a direct sum of two nontrivial complexes of sheaves).

For a perversity $\bar{p}$ let $p^{-1}(j) = \min\{c | p(c) \geq j\}$ and $p^{-1}(j) = \infty$ if $j > p(n)$. We can reformulate these conditions (2,3,4) in a way that does not refer to a particular stratification as follows:

(S1) $\dim\{x \in W | H^r(i_x^! A^\bullet) \neq 0\} \leq n - p^{-1}(j)$ for all $j > 0$
(S2) $\dim\{x \in W | H^r(i_x^! A^\bullet) \neq 0\} \leq n - q^{-1}(n - j)$ for all $j < n$.

As above, the condition (S2) is the Verdier dual of condition (S1) and may be expressed as

(S2') $\dim\{x \in W | H^r(i_x^! D(A^\bullet)) \neq 0\} \leq n - q^{-1}(j)$ for all $j > 0$.

11.2. Let $W$ be a Whitney stratified space with a given stratification. We have two notions of the constructible derived category,

1. As complexes of sheaves that are cohomologically constructible with respect to the given stratification,
As complexes of sheaves that are cohomologically constructible with respect to some stratification

In order to reduce the total number of words in these notes, we shall simply refer to “the constructible derived category”, meaning either one of these two possibilities.

Perverse sheaves are defined for any perversity but the indexing conventions are messy in general. In the following we will give the definition for the middle perversity, which is the only case of real importance. In the examples we will discuss a few others as well.

11.3. Definition. Let $W$ be a $n$-dimensional Whitney stratified (or stratifiable) space that can be stratified with strata of even dimension. Let $K = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. Fix a perversity $\bar{p}$. A (middle perversity) perverse sheaf on $W$ is a complex of sheaves $A^\bullet$ in the bounded constructible derived category $\mathcal{D}^b_c(W)$ of $K$-vector spaces, such that (see Figure 5):

- (P1) $\dim \{ x \in W | H^r(j^*_x A^\bullet) \neq 0 \} \leq n - 2r$
- (P2) $\dim \{ x \in W | H^r(j^*_x A^\bullet) \neq 0 \} \leq 2r - n$

\begin{table}
\begin{tabular}{|c|c|c|c|c|}
\hline
$i$ & cod0 & cod2 & cod4 & cod6 & cod8 \\
\hline
8 & c & c & c & c & c \\
7 & c & c & c & c \\
6 & c & c & c & c \\
5 & c & c & c & c \\
4 & c & c & c & c \\
3 & x & x & x & x \\
2 & x & x & x & x \\
1 & x & x & x & x \\
0 & x & x & x & x \\
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\end{tabular}
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\begin{tabular}{|c|c|c|c|c|}
\hline
$i$ & cod0 & cod2 & cod4 & cod6 & cod8 \\
\hline
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5 & c & c & c & c \\
4 & c & c & c & c \\
3 & x & x & x & x \\
2 & x & x & x & x \\
1 & x & x & x & x \\
0 & x & x & x & x \\
\hline
\end{tabular}
\end{table}

**Figure 5.** Stalk and co-stalk cohomology of $IC^\bullet$ and perverse sheaves

In these figures, “x” denotes regions of possibly nontrivial stalk cohomology and “c” denotes regions of possibly nontrivial stalk cohomology with compact support, $H^r(j^*_x A^\bullet)$.

11.4. Definition. Fix a Whitney stratified (or stratifiable) space. Fix a perversity $\bar{p}$. The category of perverse sheaves (with perversity $\bar{p}$) is the full subcategory of $\mathcal{D}^b_c(W)$ whose objects are perverse sheaves with perversity $\bar{p}$.

11.5. Theorem (Beilinson, Bernstein, Deligne). The category of perverse sheaves with perversity $\bar{p}$ forms an abelian subcategory of the derived category $\mathcal{D}^b_c(W)$. If $A^\bullet$ is $\bar{p}$-perverse then its Verdier dual $\mathcal{D}(A^\bullet)$ is $\bar{q}$-perverse, where $\bar{q}$ is the perversity complementary to $\bar{p}$.
There is something very mysterious about this theorem. If \( \phi : A^\bullet \to B^\bullet \) is a morphism (in \( D^b_c(W) \)) between two perverse sheaves, it can be lifted to an honest morphism of complexes \( A^\bullet \to B^\bullet \). As such, it has a kernel and a cokernel. These are unlikely to be perverse, and moreover, they may change if we choose different (but quasi-isomorphic) representative complexes for \( A^\bullet, B^\bullet \). However, the kernel and cokernel of \( \phi \) in the category of perverse sheaves are again perverse sheaves, and are well defined as elements of the derived category. Moreover, various constructions from the theory of abelian categories can be implemented. For example, suppose \( A_0^d \to A_1^d \to A_2^d \to \cdots \) is a complex of perverse sheaves, that is a complex such that \( d \circ d = 0 \) in the derived category. Then \( \ker(d)/\im(d) \) makes sense as a perverse sheaf, so we obtain the perverse cohomology \( p\, H^r(A^\bullet) \) of such a complex. (Clearly these bullets will start to get in the way so it is customary to drop them at this point.)

11.6. **Historical comment.** Around 1980 Kazhdan and Lusztig realized that certain questions involving representations of Hecke algebras, Verma modules, and Kazhdan Lusztig polynomials were related to the failure of Poincaré duality for Schubert varieties. On the advice of Raoul Bott they spoke at length with MacPherson, who replied with a long letter which, at the end, suggested the use of intersection cohomology as a solution to their problem. Consequently the Kazhdan Lusztig polynomials were shown to coincide with the intersection cohomology local Poincaré polynomial of one Schubert variety at a point in another Schubert variety. This resulted in a further series of conjectures, by Kazhdan and Lusztig concerning representations of Verma modules and their relation to Kazhdan-Lusztig polynomials. These conjectures were eventually proven independently by Beilinson-Bernstein and by Brylinski-Kashiwara. On an algebraic manifold (such as the flag manifold) there is a ring \( D \) (or rather, a sheaf of rings) of differential operators. To each \( D \)-module there corresponds a sheaf of solutions, which is a constriktible sheaf. B-B and K-L showed that each Verma module can be associated to a certain \( D \)-module whose sheaf of solutions turns out to be the IC sheaf. This provided the link between the Kazhdan-Lusztig polynomials and Verma modules. However, the category of \( D \)-modules is an abelian category, whereas the (derived) category of constriktible sheaves is not abelian, so it was conjectured that there might correspond an abelian subcategory of the derived category that “receives” the solutions of \( D \)-modules. This turned out to be the category of perverse sheaves, with middle perversity. On the other hand, intersection homology is a topological invariant, so then the question arose as to whether this category of perverse sheaves could be constructed purely topologically, and for other perversities as well. The book of BBD completely answers this question, giving a very general setting in which the category of perverse sheaves, an abelian subcategory of the derived category, could be constructed.