ABELIAN SURFACES WITH ANTI-HOLOMORPHIC MULTIPLICATION

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1. Introduction

Let $\mathfrak{h}_2 = \mathbf{Sp}(4,\mathbb{R})/\mathbf{U}(2)$ be the Siegel upperhalf space of rank 2. The quotient $\mathbf{Sp}(4,\mathbb{Z})\backslash\mathfrak{h}_2$ has three remarkable properties: (a) it is the moduli space of principally polarized abelian surfaces, (b) it has the structure of a quasi-projective complex algebraic variety which is defined over the rational numbers $\mathbb{Q}$, and (c) it has a natural compactification (the Baily-Borel Satake compactification) which is defined over the rational numbers.

Fix a square-free integer $d < 0$. One might ask whether similar statements hold for the arithmetic quotient

$$W = \mathbf{SL}(2,\mathcal{O}_d)\backslash\mathbf{H_3} \tag{1.0.1}$$

where $\mathbf{H_3} = \mathbf{SL}(2,\mathbb{C})/\mathbf{SU}(2)$ is the hyperbolic 3-space and where $\mathcal{O}_d$ is the ring of integers in the quadratic imaginary number field $\mathbb{Q}(\sqrt{d})$. One might first attempt to interpret (1.0.1) as a moduli space for abelian surfaces with complex multiplication, but this is wrong. Moreover, the space W is (real) 3-dimensional and it does not have an algebraic structure. Nevertheless, in this paper we show that there are analogues to all three statements (a), (b), and (c) if we introduce the appropriate "real" structure and level structure on the abelian varieties.

Consider a 2-dimensional abelian variety A with a principal polarization H and with a homomorphism $\psi : \mathcal{O}_d \to \mathrm{End}_{\mathbb{R}}(A)$ such that $\kappa = \psi(\sqrt{d})$ acts as an anti-holomorphic endomorphism of A. Such a triple (A, H, κ) will be referred to (in §4) as a principally polarized abelian surface with anti-holomorphic multiplication by $\mathcal{O}_d$ There is an associated notion of a level N structure on such a triple. (See §7.)

If N is a positive integer, let

$$\Lambda_N = \mathbf{SL}(2,\mathcal{O}_d)[N] \text{ and } \Gamma(N) = \mathbf{Sp}(4,\mathbb{Z})[N]$$

be the principal congruence subgroups of $\mathbf{SL}(2,\mathcal{O}_d)$ and $\mathbf{Sp}(4,\mathbb{Z})$ respectively, with level N. Fix $d < 0$ and assume that $\mathbb{Q}(\sqrt{d})$ has class number one. Suppose $N \geq 3$ and, if $d \equiv 1 (\mathrm{mod}\ 4)$, then assume also that N is even. For simplicity (and for the purposes of this introduction only), assume that d is invertible (mod N). In Theorems 6.3 and 7.5 we prove the following analogues of statements (a) and (b) above.

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1.1. **Theorem.** *Given d, N as above, the moduli space $V(d,N)$ of principally polarized abelian surfaces with anti-holomorphic multiplication by $\mathcal{O}_d$ and level N structure consists of finitely many copies of the arithmetic quotient $\Lambda_N \backslash \mathbf{H_3}$. These copies are indexed by a certain (nonabelian cohomology) set $H^1(\mathbb{C}/\mathbb{R}, \Gamma(N))$. Moreover this moduli space $V(d,N)$ coincides with the set of real points $X_{\mathbb{R}}$ of a quasi-projective complex algebraic variety which is defined over $\mathbb{Q}$.*

The key observation is that there is an involution $\widehat{\ }$ on $\mathbf{Sp}(4,\mathbb{R})$ whose fixed point set is $\mathbf{SL}(2,\mathbb{C})$ and which has the property that the fixed subgroup of $\Gamma(N)$ is exactly Λ_N. This observation was inspired by Nygaard's earlier article [N]. The key technical tool, Proposition 5.5, an analog of the lemma of Comessatti and Silhol ([Si2]), describes "normal forms" for the period matrix of an abelian surface with anti-holomorphic multiplication.

The algebraic variety $X_{\mathbb{C}}$ is just the Siegel moduli space $\Gamma(N)\backslash\mathfrak{h}_2$ of principally polarized abelian surfaces with level N structure. The involution $\widehat{\ }$ extends to an anti-holomorphic involution of the Baily-Borel Satake compactification $\overline{X}$ of $X_{\mathbb{C}}$ and hence defines a real structure on $\overline{X}$. In §9 we prove an analogue of statement (c) above by showing that $\overline{X}$ admits a rational structure which is compatible with this real structure.

One might ask whether the topological closure $\overline{V}$ of $V(d,N)$ in $\overline{X}$ coincides with the set of real points $\overline{X}_{\mathbb{R}}$ of the Baily-Borel compactification. At the moment, we do not know the answer to this question. However, in Theorem 8.1 we show that the difference $\overline{X}_{\mathbb{R}} - \overline{V}$ consists at most of finitely many isolated points.

The results in this article are, to a large extent, parallel to those of [GT], in which similar phenomena are explored for $\mathbf{GL}(n,\mathbb{R})$. However the techniques used here are completely different from those in [GT] and we have been unable to find a common framework for both papers. Nevertheless we believe they are two examples of a more general phenomenon in which an arithmetic quotient of a (totally real) Jordan algebra admits the structure of a (connected component of a) real algebraic variety which classifies real abelian varieties with polarization, endomorphism, and level structures.

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2. An involution on the symplectic group

2.1. We identify the symplectic group $\mathbf{Sp}(4,\mathbb{R})$ with all 4×4 real matrices $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ such that

$$ {}^tAD - {}^tCB = I;\ \ {}^tAC \text{ and } {}^tBD \text{ are symmetric.} \tag{2.1.1} $$

The inverse of the symplectic matrix g is $\begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}$. The symplectic group acts transitively on the Siegel upper halfspace

$$ \mathfrak{h}_2 = \left\{ Z = X + iY \in M_{2\times 2}(\mathbb{C}) \;\middle|\; {}^tZ = Z \text{ and } Y > 0 \right\} $$

by fractional linear transformations:

$$g \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

The stabilizer of the basepoint

$$e_0 = iI$$

is the unitary group U(2) which is embedded in $\mathbf{Sp}(4, \mathbb{R})$ by $A + iB \mapsto \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$.

2.2. **The number field.** Fix a square-free integer $d < 0$ and let $\mathcal{O}_d$ be the ring of integers in the quadratic imaginary number field $\mathbb{Q}(\sqrt{d})$, that is,

$$\mathcal{O}_d = \begin{cases} \mathbb{Z}[\sqrt{d}] & \text{if } d \not\equiv 1 (\text{mod } 4) \\ \mathbb{Z}[\frac{1+\sqrt{d}}{2}] & \text{if } d \equiv 1 (\text{mod } 4) \end{cases}$$

Throughout the rest of this paper we will need to consider the two cases $d \not\equiv 1(\text{mod } 4)$ and $d \equiv 1(\text{mod } 4)$ separately.

2.3. **The embeddings.** If $d \not\equiv 1(\text{mod } 4)$ define $\sigma = I_2$ to be the 2×2 identity matrix and $M = I_4$ to be the 4×4 identity matrix. If $d \equiv 1(\text{mod } 4)$ define $\sigma = \left(\begin{smallmatrix} 2 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ and

$$M = \begin{pmatrix} \sigma & 0 \\ 0 & 2\,{}^t\sigma^{-1} \end{pmatrix} \in \mathbf{GL}(4, \mathbb{R}).$$

Let

$$\mathbf{H}_3 = \{(z, r) \in \mathbb{C} \times \mathbb{R} |\ \text{Im}(z) > 0\}$$

denote the hyperbolic 3-space. Define the embedding $\phi : \mathbf{H}_3 \hookrightarrow \mathfrak{h}_2$ by

$$\phi(z, r) = \sigma \begin{pmatrix} z & r \\ r & d\bar{z} \end{pmatrix} {}^t\sigma.$$

Define the embedding $\phi : \mathbf{SL}(2, \mathbb{C}) \to \mathbf{Sp}(4, \mathbb{R})$ by

$$\phi \begin{pmatrix} a_1 + b_1\sqrt{d} & a_2 + b_2\sqrt{d} \\ a_3 + b_3\sqrt{d} & a_4 + b_4\sqrt{d} \end{pmatrix} = M \begin{pmatrix} a_1 & b_1 & b_2 & a_2 \\ b_1 d & a_1 & a_2 & b_2 d \\ b_3 d & a_3 & a_4 & b_4 d \\ a_3 & b_3 & b_4 & a_4 \end{pmatrix} M^{-1}. \tag{2.3.1}$$

The matrix M normalizes $\mathbf{Sp}(4, \mathbb{R})$ and was chosen so that Proposition 2.10 (below) holds.

2.4. **The involutions.** If $d \not\equiv 1 (\text{mod } 4)$ define $\beta = \left(\begin{smallmatrix} 0 & 1 \\ d & 0 \end{smallmatrix}\right)$. If $d = 4m+1$ define $\beta = \left(\begin{smallmatrix} 1 & 2 \\ 2m & -1 \end{smallmatrix}\right)$. Define

$$N_\beta = \begin{pmatrix} \beta & 0 \\ 0 & {}^t\beta \end{pmatrix}. \tag{2.4.1}$$

Then $\beta^2 = dI_2$ and $N_\beta^2 = dI_4$ and $\sigma^{-1}\beta\sigma = \left(\begin{smallmatrix} 0 & 1 \\ d & 0 \end{smallmatrix}\right)$. The element N_β normalizes $\mathbf{Sp}(4, \mathbb{R})$ and gives rise to an involution on $\mathbf{Sp}(4, \mathbb{R})$,

$$\widehat{g} = N_\beta g N_\beta^{-1} = N_\beta^{-1} g N_\beta. \tag{2.4.2}$$

It also gives rise to an involution on $\mathfrak{h}_2$,

$$\widehat{Z} = \beta \overline{Z}\, {}^t\beta^{-1} = \frac{1}{d} \beta \overline{Z}\, {}^t\beta.$$

2.5. **Proposition.** *The embeddings (2.3) and involutions (2.4) are related as follows.*

(1) *For any $g \in \mathbf{Sp}(4, \mathbb{R})$ and $x \in \mathfrak{h}_2$ we have: $\widehat{gx} = \widehat{g}\widehat{x}$.*
(2) *The fixed point set of the involution $\widehat{}$ on $\mathbf{Sp}(4, \mathbb{R})$ is $\phi(\mathbf{SL}(2, \mathbb{C}))$.*
(3) *The fixed point set of the involution $\widehat{}$ on $\mathfrak{h}_2$ is $\phi(\mathbf{H}_3)$.*
(4) *There is a unique (transitive) action of $\mathbf{SL}(2, \mathbb{C})$ on $\mathbf{H}_3$ so that $\phi(gx) = \phi(g) \cdot \phi(x)$ for all $g \in \mathbf{SL}(2, \mathbb{C})$ and all $x \in \mathbf{H}_3$.*

2.6. **Proof.** The proof is a direct calculation for which it helps to observe that

$$M^{-1} N_\beta M = \left(\begin{array}{cc|cc} 0 & 1 & & \\ d & 0 & & \\ \hline & & 0 & d \\ & & 1 & 0 \end{array}\right). \quad \square$$

2.7. **The basepoint.** If $h = \left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right) \in \mathbf{GL}(4, \mathbb{R})$ normalizes $\mathbf{Sp}(4, \mathbb{R})$, if $e \in \mathfrak{h}_2$ and if $XeY^{-1} \in \mathfrak{h}_2$ then we will write $h \cdot e = XeY^{-1}$ and observe that

$$\text{Stab}(XeY^{-1}) = h\text{Stab}(e)h^{-1}$$

where $\text{Stab}(e)$ denotes the stabilizer of e in $\mathbf{Sp}(4, \mathbb{R})$. Let $\tau = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \sqrt{-d} \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{smallmatrix}\right) \in \mathbf{Sp}(4, \mathbb{R})$. Then MT normalizes $\mathbf{Sp}(4, \mathbb{R})$. Define the basepoint

$$e_1 = MT \cdot e_0$$

where $e_0 = iI$ as in §2.1. Define $u_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$; with ${}^tu_0 = u_0^{-1} = -u_0$. The proof of the following lemma (which will be needed in §3.2) is a direct calculation.

2.8. **Lemma.** *The basepoint e_1 satisfies $\widehat{e_1} = e_1$. If $s = (MT)c(MT)^{-1} \in Stab(e_1)$ with $c \in U(2)$ then*

$$\widehat{s} = (MT)\dot{c}(MT)^{-1}$$

where $c \mapsto \dot{c}$ is the involution on $U(2)$ which is given by

$$\dot{c} = u_0 \bar{c} u_0^{-1} = -u_0\, {}^tc^{-1} u_0. \quad \square \tag{2.8.1}$$

2.9. **Arithmetic subgroups.** Fix a (rational) integer $N \in \mathbb{Z}$. Let $(N) \subset \mathcal{O}_d$ be the ideal generated by N. Let $z = a + b\sqrt{d} \in \mathcal{O}_d$. If $d \not\equiv 1(\text{mod } 4)$ then a, b are integers, and

$$z \equiv 1(\text{mod } (N)) \text{ iff } N|(a-1) \text{ and } N|b. \tag{2.9.1}$$

If $d \equiv 1(\text{mod } 4)$ then a, b are both integers or both half-integers, and

$$z \equiv 1(\text{mod } (N)) \text{ iff } N|(a-b-1) \text{ and } N|2b \tag{2.9.2}$$

(which may be seen by writing $z = u + v\left(\frac{1+\sqrt{d}}{2}\right)$ with $N|(u-1)$ and $N|v$).

Define the principal congruence subgroup

$$\Lambda_N = \mathbf{SL}(2, \mathcal{O}_d)(N) = \left\{ \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \mathbf{SL}(2, \mathbb{C}) \middle| z_i \in \mathcal{O}_d, \begin{matrix} z_1, z_4 \equiv 1(\text{mod } (N)) \\ z_2, z_3 \equiv 0(\text{mod } (N)) \end{matrix} \right\}. \tag{2.9.3}$$

For $N = 1$ set $\Lambda = \mathbf{SL}(2, \mathcal{O}_d)(N) = \mathbf{SL}(2, \mathcal{O}_d)$.

Let $\Gamma(N) \subset \mathbf{Sp}(4, \mathbb{Z})$ denote the principal congruence subgroup of level N. (For $N = 1$ set $\Gamma(N) = \mathbf{Sp}(4, \mathbb{Z})$.) Let $\widehat{\Gamma}(N) = \{\widehat{\gamma} |\ \gamma \in \Gamma(N)\}$. Define

$$\Gamma_N = \Gamma(N) \cap \widehat{\Gamma}(N). \tag{2.9.4}$$

If d is invertibile (mod N) then $\Gamma_N = \Gamma(N)$. For, if $g \in \Gamma(N)$ then

$$d\widehat{g} = N_\beta g N_\beta \equiv N_\beta^2 \equiv dI(\text{mod } N)$$

hence $\widehat{g} \in \Gamma(N)$. In general, Γ_N has finite index in $\Gamma(N)$ because $\Gamma(dN) \subset \Gamma_N$ has finite index in $\Gamma(N)$.

2.10**. Proposition.** *The embedding* $\phi : \mathbf{SL}(2, \mathbb{C}) \to \mathbf{Sp}(4, \mathbb{R})$ *of §2.3 satisfies:*

$$\phi^{-1}(\Gamma_N) = \Lambda_N = \mathbf{SL}(2, \mathcal{O}_d)(N)$$

for any $N \geq 1$.

2.11. **Proof.** Let

$$h = \begin{pmatrix} a_1 + b_1\sqrt{d} & a_2 + b_2\sqrt{d} \\ a_3 + b_3\sqrt{d} & a_4 + b_4\sqrt{d} \end{pmatrix} \in \mathbf{SL}(2, \mathbb{C}).$$

Set

$$A = \begin{pmatrix} a_1 & b_1 \\ db_1 & a_1 \end{pmatrix}, \ B = \begin{pmatrix} b_2 & a_2 \\ a_2 & db_2 \end{pmatrix}, \ C = \begin{pmatrix} db_3 & a_3 \\ a_3 & b_3 \end{pmatrix}, \ D = \begin{pmatrix} a_4 & db_4 \\ b_4 & a_4 \end{pmatrix} \tag{2.11.1}$$

First suppose that $d \not\equiv 1(\text{mod } 4)$. Then $\phi(h) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. It follows that $\phi(h) \in \Gamma(N)$ iff $h \in \mathbf{SL}(2, \mathcal{O}_d)(N)$. This shows that $\phi(\mathbf{SL}(2, \mathbb{C})) \cap \Gamma(N) = \phi(\mathbf{SL}(2, \mathcal{O}_d)(N))$. But the image of ϕ is fixed under the involution $g \mapsto \widehat{g}$ so this intersection is also contained in $\widehat{\Gamma}(N)$ as claimed.

Now suppose that $d \equiv 1(\text{mod } 4)$. Then

$$\phi(h) = \begin{pmatrix} \sigma A \sigma^{-1} & \frac{1}{2}\sigma B\, {}^t\sigma \\ 2\, {}^t\sigma^{-1} C \sigma^{-1} & {}^t\sigma^{-1} D\, {}^t\sigma \end{pmatrix}. \tag{2.11.2}$$

Moreover, $h \in \mathbf{SL}(2, \mathcal{O}_d)$ iff for each i $(1 \leq i \leq 4)$, either a_i, b_i are both integral or they are both half-integral. A simple calculation using (2.9.2) shows that $\phi(h) \in \Gamma(N) \iff h \in \mathbf{SL}(2, \mathcal{O}_d)(N)$. As in the preceding paragraph, the image of ϕ is fixed under the involution $g \mapsto \widehat{g}$, which implies that $\phi(\mathbf{SL}(2, \mathbb{C})) \cap \Gamma(N)$ is also contained in $\widehat{\Gamma}(N)$. □

2.12. **Γ-real points.** Let $\gamma \in Sp_4(\mathbb{R})$ and define the locus of $(\gamma, \widehat{\ })$-real points,

$$E_\gamma = \left\{ Z \in \mathfrak{h}_2 |\ \widehat{Z} = \gamma \cdot Z \right\}.$$

Then $E_I = \phi(\mathbf{H}_3)$ is the fixed point set of $Z \mapsto \widehat{Z}$. If $\Gamma \subset Sp_4(\mathbb{R})$ is an arithmetic subgroup, define the $(\Gamma, \widehat{\ })$-real points of $\mathfrak{h}_2$ to be the union $\cup_\gamma E_\gamma$ over all $\gamma \in \Gamma$.

3. Galois cohomology

3.1. The involution $g \mapsto \widehat{g}$ may be considered to be an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \widehat{\ }\}$ on $\mathbf{Sp}(4, \mathbb{R})$. For any $\gamma \in \mathbf{Sp}(4, \mathbb{R})$ let $f_\gamma : \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathbf{Sp}(4, \mathbb{R})$ be the mapping $f_\gamma(1) = I$ and $f_\gamma(\widehat{\ }) = \gamma$. Then f_γ is a 1-cocycle iff $\gamma\widehat{\gamma} = I$ and it is a coboundary iff there exists $h \in \mathbf{Sp}(4, \mathbb{R})$ such that $\gamma = \widehat{h}h^{-1}$.

3.2. **Lemma.** *With respect to the above action, the (nonabelian) Galois cohomology set* $H^1(\mathbb{C}/\mathbb{R}, Stab(e_1))$ *is trivial.*

3.3. **Proof.** By Lemma 2.8 it suffices to show that $H^1(\mathbb{C}/\mathbb{R}, \mathbf{U}(2)) = 0$ where $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathbf{U}(2)$ by $c \mapsto \dot{c}$. Assume $c\dot{c} = I$. We must show there exists $w \in \mathbf{U}(2)$ so that $c = \dot{w}w^{-1}$.

Embed $U(1) \hookrightarrow \mathbf{U}(2)$ by $e^{i\theta} \mapsto \left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{smallmatrix}\right)$. From (2.8.1), $c\dot{c} = I$ if and only if $cu_0 = u_0\,{}^t c$ which holds iff $c \in U(1)$. (Write $c = \left(\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right)$ and compute both sides to get $c_1 = c_4$ and $c_2 = c_3 = 0$.) If $c = e^{i\theta}$ then it suffices to take $w = e^{-i\theta/2}$. □

3.4. **Lemma.** *Let* $\gamma \in \mathbf{Sp}(4, \mathbb{R})$ *and suppose that* f_γ *is a 1-cocycle. Then* f_γ *is a coboundary iff* $E_\gamma \neq \phi$.

3.5. **Proof.** If f_γ is a coboundary, say $\gamma = \widehat{w}w^{-1}$ then $E_\gamma \supset w\phi(\mathbf{H}_3)$ is not empty. On the other hand, suppose $\widehat{Z} = \gamma Z$. Choose $k \in \mathbf{Sp}(4, \mathbb{R})$ so that $Z = ke_1$ (hence $\widehat{Z} = \widehat{k}e_1$). Therefore $u := \widehat{k}^{-1}\gamma k \in \mathrm{Stab}(e_1)$ and $u\widehat{u} = I$. By §3.2 there exists $w \in \mathbf{Sp}(4, \mathbb{R})$ so that $u = \widehat{w}w^{-1}$, hence $\gamma = \widehat{kw}(kw)^{-1}$. □

3.6. **Proposition.** *With respect to the above action, the (nonabelian) Galois cohomology set* $H^1(\mathbb{C}/\mathbb{R}, \mathbf{Sp}_4(\mathbb{R}))$ *is trivial.*

3.7. **Proof.** Suppose $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbf{Sp}(4,\mathbb{R})$ and that $\gamma\widehat{\gamma} = I$. Then

$$\beta A \beta^{-1} = {}^tD; \ \beta D \text{ and } {}^t\beta C \text{ are skew symmetric.} \tag{3.7.1}$$

Write ${}^t\beta C = \left(\begin{smallmatrix} 0 & \lambda \\ -\lambda & 0 \end{smallmatrix}\right)$. We consider two cases: $\lambda \neq 0$ and $\lambda = 0$.

First suppose that $\lambda \neq 0$. Then C is invertible and $\left({}^t\beta C\right)^{-1} = -\frac{1}{\lambda^2}\,{}^t\beta C$. Set

$$\gamma' = \widehat{h}\gamma h^{-1} \text{ where } h = \begin{pmatrix} I & -C^{-1}D \\ 0 & I \end{pmatrix}.$$

Then $f_{\gamma'}$ is also a 1-cocycle which is cohomologous to f_γ. In fact,

$$\gamma' = \begin{pmatrix} 0 & -\,{}^tC^{-1} \\ C & 0 \end{pmatrix}$$

which is most easily seen by writing $\gamma' = \left(\begin{smallmatrix} A' & B' \\ C & 0 \end{smallmatrix}\right)$, using (3.7.1) to get $A' = 0$; hence $B' = -\,{}^tC^{-1}$. By Lemma 3.4 it suffices to find $Z \in \mathfrak{h}_2$ so that $\widehat{Z} = \gamma' Z$. Such a point is given by $Z = i\left(\begin{smallmatrix} y_1 & 0 \\ 0 & y_2 \end{smallmatrix}\right)$ where $y_1, y_2 > 0$ are chosen so that $y_1 y_2 = -d/\lambda^2$.

Now suppose instead that $\lambda = 0$. Set

$$\gamma' = \widehat{h}\gamma h^{-1} = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \text{ where } h = \begin{pmatrix} I & \frac{1}{2}A^{-1}B \\ 0 & I \end{pmatrix}.$$

It suffices to find $Z \in \mathfrak{h}_2$ so that $\gamma' \cdot Z = \widehat{Z}$, that is,

$$AZ\,{}^tA = \beta \overline{Z}\,{}^t\beta^{-1}. \tag{3.7.2}$$

In fact we will find such an element $Z = iY$ where $Y \in \mathbf{GL}(2,\mathbb{R})$ is positive definite and symmetric.

Since $\beta^2 = dI$, (3.7.1) gives $(\beta A)^2 = dI$ hence the matrix

$$A' = \frac{1}{\sqrt{-d}}\beta A$$

has characteristic polynomial $x^2 + 1$ so it is $\mathbf{GL}(2,\mathbb{R})$-conjugate to the matrix $j = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, say, $wA'w^{-1} = j$. Let

$$W = \begin{pmatrix} w & 0 \\ 0 & {}^tw^{-1} \end{pmatrix} \text{ and } J = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$$

be the corresponding elements of the symplectic group. Take $Z = W^{-1} \cdot iI = iw^{-1}\,{}^tw^{-1}$. Then

$$A'Z\,{}^tA' = w^{-1}jwZ\,{}^tw\,{}^tj\,{}^tw^{-1} = w^{-1}iI\,{}^tw^{-1} = Z = -\overline{Z}.$$

Hence

$$AZ\,{}^tA = -d\beta^{-1}(-\overline{Z})\,{}^t\beta^{-1} = \beta\overline{Z}\,{}^t\beta^{-1}. \quad \square$$

3.8. **Corollary.** *Let $\Gamma \subset \mathbf{Sp}(4,\mathbb{R})$ be a torsion-free arithmetic group such that $\widehat{\Gamma} = \Gamma$. Fix $\gamma \in \Gamma$. Then the following statements are equivalent:*

(1) $E_\gamma \neq \phi$

(2) f_γ *is a cocycle (i.e.* $\gamma\widehat{\gamma} = I$*).*
(3) *There exists* $g \in \mathbf{Sp}(4,\mathbb{R})$ *such that* $\gamma = \widehat{g}g^{-1}$.

In this case, $E_\gamma = g \cdot \phi(\mathbf{H}_3)$*. If* $\gamma_1, \gamma_2 \in \Gamma$ *are distinct then* $E_{\gamma_1} \cap E_{\gamma_2} = \phi$.

3.9. **Proof.** Suppose $Z \in E_\gamma$. Then $\widehat{\gamma}\gamma Z = \widehat{\gamma Z} = Z$ so $\widehat{\gamma}\gamma$ is torsion. By hypothesis this implies $\widehat{\gamma}\gamma = I$ hence (1) implies (2). If f_γ is a 1-cocycle then by Proposition 3.6 it is a coboundary, so (2) implies (3). If $\gamma = \widehat{g}g^{-1}$ then $E_\gamma = g \cdot \phi(\mathbf{H}_3)$ so (3) implies (1). Finally, if $Z \in E_{\gamma_1} \cap E_{\gamma_2}$ then $\gamma_1 Z = \widehat{Z} = \gamma_2 Z$ so $\gamma_2^{-1}\gamma_1$ fixes Z, hence $\gamma_1 = \gamma_2$. □

3.10. If $\Gamma \subset \mathbf{Sp}(4,\mathbb{Z})$ is a subgroup, define

$$\widetilde{\Gamma} = \left\{ g \in \mathbf{Sp}(4,\mathbb{Z}) |\ \widehat{g}g^{-1} \in \Gamma \right\}.$$

If $\widehat{\Gamma} = \Gamma$ then $\widehat{\widetilde{\Gamma}} = \widetilde{\Gamma}$. It is easy to see that $\widetilde{\Gamma}_N$ is a group and that Γ_N is a normal subgroup.

3.11**. Proposition.** *Assume that* $\mathcal{O}_d$ *is a principal ideal domain (cf. §5.1). Let* $N \geq 3$*. If* $d \equiv 1 (\mathrm{mod}\ 4)$*, assume also that* N *is even. Then the short exact sequence*

$$1 \longrightarrow \Gamma_N \longrightarrow \widetilde{\Gamma}_N \longrightarrow \Gamma_N \backslash \widetilde{\Gamma}_N \longrightarrow 1 \tag{3.11.1}$$

induces a bijection

$$H^1(\mathbb{C}/\mathbb{R}, \Gamma_N) \cong \Gamma_N \backslash \widetilde{\Gamma}_N / \phi(\mathbf{SL}(2,\mathcal{O}_d)). \tag{3.11.2}$$

3.12. **Proof.** We claim that $H^1(\mathbb{C}/\mathbb{R},\Gamma_N) \to H^1(\mathbb{C}/\mathbb{R},\widetilde{\Gamma}_N)$ is trivial. For, if $\gamma \in \Gamma_N$ and if f_γ is a 1-cocycle then by Corollary 3.8 the set $E_\gamma \neq \phi$. By Proposition 5.8 (below), there exists $g \in \mathbf{Sp}(4,\mathbb{Z})$ so that $\gamma = \widehat{g}g^{-1}$, which proves the claim. The long exact cohomology sequence associated to (3.11.1) is

$$\begin{array}{ccccccccc} H^0(\Gamma_N) & \to & H^0(\widetilde{\Gamma}_N) & \to & H^0(\Gamma_N\backslash\widetilde{\Gamma}_N) & \to & H^1(\Gamma_N) & \to & 1 \\ || & & || & & || & & & & \\ \phi(\Lambda_N) & & \phi(\Lambda) & & \Gamma_N\backslash\widetilde{\Gamma}_N & & & & \end{array}$$

which completes the proof of (3.11.2). □

4. Anti-holomorphic multiplication

4.1. Recall [La] that a symplectic form Q on $\mathbb{C}^2$ is *compatible* with the complex structure if $Q(iu,iv) = Q(u,v)$ for all $u, v \in \mathbb{C}^2$. A compatible form Q is *positive* if the symmetric form $R(u,v) = Q(iu,v)$ is positive definite. If Q is compatible and positive then it is the imaginary part of a unique positive definite Hermitian form $H = R + iQ$. Let $L \subset \mathbb{C}^2$ be a lattice and let $H = R + iQ$ be a positive definite Hermitian form on $\mathbb{C}^2$. A basis of L is *symplectic* if the matrix for Q with respect to this basis is $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. The lattice L is symplectic if it admits a symplectic basis. A principally polarized abelian surface is a pair $(A = \mathbb{C}^2/L, H = R + iQ)$ where H is a positive definite Hermitian form on $\mathbb{C}^2$ and where $L \subset \mathbb{C}^2$ is a symplectic lattice relative to $Q = \mathrm{Im}(H)$.

Each $Z \in \mathfrak{h}_2$ determines a principally polarized abelian surface (A_Z, H_Z) as follows. Let Q_0 be the standard symplectic form on $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$ with matrix $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ (with respect to the standard basis of $\mathbb{R}^2 \oplus \mathbb{R}^2$). Let $F_Z : \mathbb{R}^2 \oplus \mathbb{R}^2 \to \mathbb{C}^2$ be the real linear mapping with matrix (Z, I), that is,

$$F_Z \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = Zx + y.$$

Then $(F_Z)_*(Q_0)$ is a compatible, positive symplectic form and $L_Z = F_Z(\mathbb{Z}^2 \oplus \mathbb{Z}^2)$ is a symplectic lattice with symplectic basis F_Z(standard basis). The Hermitian form corresponding to Q_Z is

$$H_Z(u, v) = Q_Z(iu, v) + iQ_Z(u, v) = {}^t u(\mathrm{Im}(Z))^{-1}\bar{v} \text{ for } u, v \in \mathbb{C}^2.$$

The pair $(A_Z = \mathbb{C}^2/L_Z, H_Z)$ is the desired principally polarized abelian surface. If z_1, z_2 are the standard coordinates on $\mathbb{C}^2$ then, with respect to the above symplectic basis of L, the differential forms dz_1, dz_2 have period matrix (Z, I).

The principally polarized abelian surfaces $(A_Z = \mathbb{C}^2/L_Z, H_Z)$ and $(A_\Omega = \mathbb{C}^2/L_\Omega, H_\Omega)$ are isomorphic iff there exists a complex linear mapping $\psi : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\psi(L_\Omega) = L_Z$ and $\psi_*(H_\Omega) = H_Z$. Set $h = {}^t(F_Z^{-1}\psi F_\Omega) = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Then: $h \in \mathbf{Sp}(4, \mathbb{Z})$, $\Omega = h \cdot Z$, and $\psi(M) = {}^t(CZ + D)M$ for all $M \in \mathbb{C}^2$, which is to say that the following diagram commutes:

$$\begin{array}{ccccc} \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) & \mathbb{R}^2 \oplus \mathbb{R}^2 & \xrightarrow[F_{h\cdot Z}]{} & \mathbb{C}^2 & M \\ \downarrow & \downarrow & & \downarrow{\scriptstyle \psi} & \downarrow \\ {}^t h \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) & \mathbb{R}^2 \oplus \mathbb{R}^2 & \xrightarrow[F_Z]{} & \mathbb{C}^2 & {}^t(CZ + D)M \end{array} \tag{4.1.1}$$

(since $h \cdot Z$ is symmetric). If $Z \in \mathfrak{h}_2$ and $\widehat{Z} = \beta\overline{Z}\,{}^t\beta^{-1}$ then this diagram commutes:

$$\begin{array}{ccccc} \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) & \mathbb{R}^2 \oplus \mathbb{R}^2 & \xrightarrow[F_Z]{} & \mathbb{C}^2 & M \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ {}^t N_\beta \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) & \mathbb{R}^2 \oplus \mathbb{R}^2 & \xrightarrow[F_{\widehat{Z}}]{} & \mathbb{C}^2 & \beta\overline{M} \end{array} \tag{4.1.2}$$

4.2. As in §2.2, fix a square-free integer $d < 0$ and let $\mathcal{O}_d$ denote the ring of integers in the number field $\mathbb{Q}[\sqrt{d}]$. Let us say that a principally polarized abelian surface $(A = \mathbb{C}^2/L, H)$ admits *anti-holomorphic multiplication* by the ring $\mathcal{O}_d$ if there is a homomorphism $\Psi : \mathcal{O}_d \to \mathrm{End}_{\mathbb{R}}(A)$ such that the endomorphism $\Psi(\sqrt{d})$ is anti-holomorphic and is compatible with the polarization H. To be precise, suppose $H = R + iQ$. Then Ψ is an anti-holomorphic multiplication by $\mathcal{O}_d$ iff

$$\kappa = \Psi(\sqrt{d}) : \mathbb{C}^2 \to \mathbb{C}^2$$

is complex anti-linear and satisfies

(1) $\kappa^2 = dI$
(2) $Q(\kappa(u), \kappa(v)) = dQ(u, v)$ for all $u, v \in \mathbb{C}^2$

(3) $\kappa(L) \subset L$ and,

(4) if $d \equiv 1 (\text{mod } 4)$ then $\frac{1}{2}(\kappa + I)(L) \subset L$

in which case, Ψ is determined by κ.

4.3. **Lemma.** *Let $\Gamma \subset \mathbf{Sp}(4, \mathbb{Z})$ be a torsion-free arithmetic subgroup. If $d \equiv 1 (\text{mod } 4)$ then assume also that $\Gamma \subset \Gamma(2)$. Suppose $Z \in \mathfrak{h}_2^{\Gamma}$. Then there is a unique $\gamma \in \Gamma$ such that $\widehat{Z} = \gamma \cdot Z$. Moreover, $\gamma \in \Gamma \cap \widehat{\Gamma}$, and the mapping*

$$\kappa_Z = F_Z \circ {}^t(N_\beta \gamma) \circ F_Z^{-1} : \mathbb{C}^2 \to \mathbb{C}^2 \tag{4.3.1}$$

defines an anti-holomorphic multiplication by $\mathcal{O}_d$ on the principally polarized abelian surface (A_Z, H_Z).

4.4. **Proof.** The element γ is unique since Γ is torsion-free. Since $\widehat{\gamma}\gamma Z = Z$ we get: $\widehat{\gamma}\gamma = I$. Set $\eta = {}^t(N_\beta \gamma)$. Then

$$\eta^2 = {}^t(N_\beta \gamma N_\beta \gamma) = {}^t(N_\beta \gamma d N_\beta^{-1} \gamma) = d\,{}^t(\widehat{\gamma}\gamma) = dI$$

so the same is true of κ_Z. Also

$$\eta = {}^t\gamma \begin{pmatrix} {}^t\beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & dI \end{pmatrix}.$$

The first two factors are in $\mathbf{Sp}(4, \mathbb{R})$ so

$$Q_0(\eta u, \eta v) = d Q_0(u, v)$$

for all $u, v \in \mathbb{R}^4$. Hence $Q_Z(\kappa_Z u, \kappa_Z v) = d Q_Z(u, v)$ for all $u, v \in \mathbb{C}^2$ which verifies conditions (1) and (2) above. Condition (3) holds since η preserves the lattice $\mathbb{Z}^2 \oplus \mathbb{Z}^2$. Now suppose that $d \equiv 1 (\text{mod } 4)$ and that $\gamma \equiv I (\text{mod } 2)$. Then ${}^t(N_\beta \gamma) + I \equiv 0 (\text{mod } 2)$ (since $\beta \equiv I (\text{mod } 2)$), which shows that $\frac{1}{2}(I + \eta)$ preserves the lattice $\mathbb{Z}^2 \oplus \mathbb{Z}^2$, and verifies (4).

Finally we check that $\kappa_Z : \mathbb{C}^2 \to \mathbb{C}^2$ is anti-linear. Let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. By (4.1.1) and (4.1.2) the following diagram commutes,

$$\begin{array}{ccccc} \mathbb{R}^2 \oplus \mathbb{R}^2 & \xrightarrow[F_Z]{} & \mathbb{C}^2 & & M \\ {\scriptstyle {}^tN_\beta}\downarrow & & \downarrow & & \downarrow \\ \mathbb{R}^2 \oplus \mathbb{R}^2 & \xrightarrow[F_{\widehat{Z}}]{} & \mathbb{C}^2 & & \beta\overline{M} \\ {\scriptstyle {}^t\gamma}\downarrow & & \downarrow & & \downarrow \\ \mathbb{R}^2 \oplus \mathbb{R}^2 & \xrightarrow[F_{\gamma^{-1}\cdot\widehat{Z}}]{} & \mathbb{C}^2 & & {}^t(CZ + D)\beta\overline{M} \end{array} \tag{4.4.1}$$

But $Z = \gamma^{-1} \cdot \widehat{Z}$ so the bottom arrow is also F_Z. Then κ_Z is the composition along the right hand vertical column and it is given by $M \mapsto {}^t(CZ + D)\beta\overline{M}$ which is anti-linear. $\square$

5. The Comessatti lemma

5.1. Throughout §5 we fix a square-free integer $d < 0$. If $d \equiv 1 (\text{mod } 4)$ set $m = (d-1)/4$. Recall [St] that $\mathcal{O}_d \subset \mathbb{Q}(\sqrt{d})$ is a principal ideal domain iff $d \in \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}$. Define the matrix N_β by (2.4.1).

5.2**. Lemma.** *Let M be a free $\mathcal{O}_d$ module of rank 2. Let $\Psi(x) : M \to M$ denote the action by $x \in \mathcal{O}_d$. Then M has a $\mathbb{Z}$-basis with respect to which the matrix for $\Psi(\sqrt{d})$ is ${}^tN_\beta$.*

5.3. **Proof.** Set $M = \mathcal{O}_d v \oplus \mathcal{O}_d w$. If $d \not\equiv 1 (\text{mod } 4)$ then $\mathcal{O}_d = \mathbb{Z}[\sqrt{d}]$ and the desired basis is given by $a_1 = v$,$a_2 = \Psi(\sqrt{d})v$, $b_1 = \Psi(\sqrt{d})w$, and $b_2 = w$.

If $d \equiv 1 (\text{mod } 4)$ let $\alpha = (-1+\sqrt{d})/2$ and $\alpha' = (1+\sqrt{d})/2$. Then $\mathcal{O}_d = \mathbb{Z}[\alpha] = \mathbb{Z}[\alpha']$, $m = \alpha^2 + \alpha = (\alpha')^2 - \alpha'$ and $\sqrt{d} = 2\alpha + 1 = 2\alpha' - 1$. The desired basis is given by $a_1 = v$, $a_2 = \Psi(\alpha)v$, $b_1 = \Psi(\alpha')w$, and $b_2 = w$. □

5.4. Let $A = \mathbb{C}^2/L$ be a complex torus. If ω_1, ω_2 form a basis for the space of holomorphic 1-forms and if v_1, v_2, v_3, v_4 are a basis for L, then the corresponding period matrix Ω has entries $\Omega_{ij} = \int_{v_j} \omega_i$. If $v_i' = \sum_j A_{ij} v_j$ and if $\omega_i' = \sum_j B_{ij} \omega_j$ then the resulting period matrix is

$$\Omega' = B\Omega\, {}^tA. \tag{5.4.1}$$

The following proposition is an analog of the lemma ([Si2], [C]) of Comessatti and Silhol.

5.5**. Proposition.** *Suppose $\mathcal{O}_d$ is a principal ideal domain. Let $(A = \mathbb{C}^2/L, H = R + iQ)$ be a principally polarized abelian surface which admits an anti-holomorphic multiplication $\kappa : \mathbb{C}^2 \to \mathbb{C}^2$ by $\mathcal{O}_d$. Then there exists a basis for the holomorphic 1-forms on A and a symplectic basis for the lattice L such that the resulting period matrix is (Z, I) for some $Z \in \phi(\mathbf{H_3}) \subset \mathfrak{h}_2$, and such that the matrix for κ with respect to this basis (of L) is ${}^tN_\beta$.*

5.6. **Proof.** The mapping κ preserves L and defines on L the structure of an $\mathcal{O}_d$ module. Since L is torsion-free and since $\mathcal{O}_d$ is principal, it follows that L is a free $\mathcal{O}_d$ module of rank 2, to which we may apply Lemma 5.2.

First consider the case $d \not\equiv 1 (\text{mod } 4)$. Let $\{a_1, a_2, b_1, b_2\}$ be the basis of L constructed in Lemma 5.2. Then $\kappa(a_1) = a_2$, $\kappa(a_2) = da_1$, $\kappa(b_1) = db_2$, and $\kappa(b_2) = b_1$. We will now show that this basis is symplectic with respect to the symplectic form Q, or else it can be modified to give a symplectic basis. Let $x = Q(a_1, b_1) \in \mathbb{Z}$ and $y = Q(a_1, b_2) \in \mathbb{Z}$. Since (§4.2) $Q(\kappa u, \kappa v) = dQ(u, v)$, we find that the matrix for Q is $\left(\begin{smallmatrix} 0 & M \\ -M & 0 \end{smallmatrix}\right)$ where $M = \left(\begin{smallmatrix} x & y \\ dy & x \end{smallmatrix}\right)$. Since $\{a_1, a_2, b_1, b_2\}$ form a basis of L, we have: $\pm 1 = \det(M) = x^2 - dy^2 > 0$. Hence this determinant is 1 and there are four possibilities

- $x = 1$ and $y = 0$
- $x = -1$ and $y = 0$
- $x = 0$ and $y = 1$ and $d = -1$
- $x = 0$ and $y = -1$ and $d = -1$.

In the second case, replace a_1 by $a_1' = -a_1$ and replace a_2 by $a_2' = -a_2$. In the third case replace a_1 by $a_1' = a_2$ and replace a_2 by $a_2' = -a_1$. In the fourth case replace a_1 by $a_1' = -a_2$ and replace a_2 by $a_2' = a_1$. Then in all four cases, the resulting basis $\{a_1', a_2', b_1' = b_1, b_2' = b_2\}$ is symplectic and the matrix for κ is ${}^tN_\beta$.

Choose a holomorphic 1-form $\omega_1 \in \Gamma(\Omega_A^1)$ such that $\int_{b_1'} \omega_1 = 1$ and $\int_{b_2'} \omega_1 = 0$. Define $\omega_2 = \overline{\kappa^* \omega_1}$ to be the complex conjugate of the pullback of ω_1 under the mapping κ. Set $z = \int_{a_1'} \omega_1$ and $r = \int_{a_2'} \omega_1$. Then

$$\int_{b_i'} \omega_2 = \int_{\kappa(b_i')} \overline{\omega_1} = \begin{cases} 0 & \text{if } i = 1 \\ 1 & \text{if } i = 2 \end{cases}$$

$$\int_{a_i'} \omega_2 = \int_{\kappa(a_i')} \overline{\omega_1} = \begin{cases} \bar{r} & \text{if } i = 1 \\ d\bar{z} & \text{if } i = 2 \end{cases}$$

Therefore the period matrix for $\{\omega_1, \omega_2, a_1', a_2', b_1', b_2'\}$ is (Ω, I) where $\Omega = \left(\begin{smallmatrix} z & r \\ \bar{r} & d\bar{z} \end{smallmatrix}\right)$. The Riemann relations imply that this matrix is symmetric (so $\bar{r} = r$) and that $\mathrm{Im}(\Omega) > 0$. This completes the proof if $d \not\equiv 1 \pmod 4$.

Now consider the case $d \equiv 1 (\mathrm{mod}\ 4)$. Let $\{a_1, a_2, b_1, b_2\}$ be the basis of L which was constructed in Lemma 5.2. The matrix for κ with respect to this basis is ${}^tN_\beta$. Set

$$\begin{array}{rcrcr} a_1' & = & \frac{1}{2}a_1 & & \\ a_2' & = & \frac{1}{2}a_1 & + & a_2 \\ b_1' & = & 2b_1 & - & b_2 \\ b_2' & = & & & b_2 \end{array} \tag{5.6.1}$$

Then

$$\kappa(a_1') = a_2',\ \kappa(a_2') = da_1',\ \kappa(b_1') = db_2',\ \text{and } \kappa(b_2') = b_1'. \tag{5.6.2}$$

We will modify this basis so as to obtain a symplectic basis of L. Set $x' = Q(a_1', b_1') \in \frac{1}{2}\mathbb{Z}$ and $y' = Q(a_1', b_2') \in \frac{1}{2}\mathbb{Z}$. The matrix for Q with respect to the basis $\{a_1', a_2', b_1', b_2'\}$ of $\mathbb{C}^2$ is $\left(\begin{smallmatrix} 0 & M' \\ -M' & 0 \end{smallmatrix}\right)$ where $M' = \left(\begin{smallmatrix} x' & y' \\ dy' & x' \end{smallmatrix}\right)$. The determinant of (5.6.1) is 1 so $\pm 1 = \det(M') = (x')^2 - d(y')^2 > 0$. Since $d \leq -3$ there are the following possibilities,

- $x' = 1$ and $y' = 0$
- $x' = -1$ and $y' = 0$
- $x' = \pm\frac{1}{2}$, $y' = \pm\frac{1}{2}$, and $d = -3$.

In the second case, replace a_1' by $-a_1'$ and a_2' by $-a_2'$. In the third case replace a_1' by ua_1' and a_2' by ua_2' where

$$u = \pm\left(\frac{1 \pm \sqrt{-3}}{2}\right)$$

is the appropriately chosen unit in $\mathcal{O}_{-3}$. In all three cases the resulting basis, which we will continue to denote by $\{a_1', a_2', b_1', b_2'\}$ is a symplectic basis for $\mathbb{C}^2$ such that (5.6.2) holds, although it is not a basis of the lattice L.

Choose a holomorphic 1-form ω_1' so that $\int_{b_1'} \omega_1' = 1$ and $\int_{b_2'} \omega_1' = 0$. Set $\omega_2' = \overline{\kappa^* \omega_1'}$. Set $z = \int_{a_1'} \omega_1'$ and $r = \int_{a_2'} \omega_1'$. Then the period matrix for $\{\omega_1', \omega_2', a_1', a_2', b_1', b_2'\}$ is (Ω, I) where $\Omega = \left(\begin{smallmatrix} z & \bar{r} \\ r & d\bar{z} \end{smallmatrix}\right)$ as in the case $d \not\equiv 1 (\mathrm{mod}\ 4)$. Now define a further change of coordinates,

$$\begin{array}{rclcrcl}
\omega_1'' & = & 2\omega_1 & & & & \\
\omega_2'' & = & -\omega_1 & + & \omega_2 & & \\
a_1'' & = & 2a_1' & & & = & ua_1 \\
a_2'' & = & -a_1' & + & a_2' & = & ua_2 \\
b_1'' & = & \frac{1}{2}b_1' & + & \frac{1}{2}b_2' & = & b_1 \\
b_2'' & = & & & b_2' & = & b_2
\end{array}$$

(Here, u is a unit in $\mathcal{O}_d$ and it equals 1 except possibly in the case $d = -3$.) Then $\{a_1'', a_2'', b_1'', b_2''\}$ is an integral symplectic basis for L. By (5.4.1) the period matrix for $\{\omega_1'', \omega_2'', a_1'', a_2'', b_1'', b_2''\}$ is given by

$$\sigma\,(\Omega, I)\begin{pmatrix} {}^t\sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} = \left(\sigma\Omega\,{}^t\sigma, I\right)$$

where $\sigma = \left(\begin{smallmatrix} 2 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. It follows that Ω is symmetric, $\mathrm{Im}(\Omega) > 0$, and the matrix for κ with respect to the basis $\{a_1'', a_2'', b_1'', b_2''\}$ is ${}^tN_\beta$. □

5.7. The main application of the Comessatti lemma is to strengthen the conclusion of Proposition 3.8. Fix a square-free integer $d < 0$ and let $\mathcal{O}_d$ be the ring of integers in $\mathbb{Q}(\sqrt{d})$. Let $\Gamma \subset \mathbf{Sp}(4, \mathbb{Z})$ be a torsion free subgroup such that $\widehat{\Gamma} = \Gamma$. If $d \equiv 1 (\mathrm{mod}\ 4)$ then suppose also that $\Gamma \subset \Gamma(2)$.

5.8**. Proposition.** *Let $\gamma \in \Gamma$ and suppose $\mathcal{O}_d$ is a principal ideal domain. Then $E_\gamma \neq \phi$ if and only if there exists $g \in \mathbf{Sp}(4, \mathbb{Z})$ such that $\gamma = \widehat{g}g^{-1}$. In this case the element g is uniquely determined up to multiplication from the right by an element of $\phi(\mathbf{SL}(2, \mathcal{O}_d))$, and*

$$E_\gamma = g \cdot \phi(\mathbf{H}_3).$$

5.9. **Proof.** Suppose $\gamma = \widehat{g}g^{-1}$ for some $g \in \mathbf{Sp}(4, \mathbb{Z})$. Then $Z \in E_\gamma$ iff $\gamma \cdot Z = \widehat{Z}$ iff $\widehat{g}g^{-1} \cdot Z = \widehat{Z}$ iff $g^{-1}Z \in \phi(\mathbf{H}_3)$ since it is fixed under the $\widehat{\ }$ involution. This proves the "if" part of the first statement.

Now suppose that $E_\gamma \neq \phi$. Choose $Z \in E_\gamma$ which is not fixed by any element of $\mathbf{Sp}(4, \mathbb{Z})$ other than $\pm I$. Such points exist and are even dense in E_γ since, by Proposition 3.8 the set E_γ is a translate of $\phi(\mathbf{H}_3)$. Let $L_Z = F_Z(\mathbf{Z}^2 \oplus \mathbf{Z}^2)$ be the corresponding lattice in $\mathbb{C}^2$ and let $(A_Z = \mathbb{C}^2/L_Z, H_Z)$ be the corresponding polarized abelian surface. By Lemma 4.3 this variety admits anti-holomorphic multiplication κ_Z by the ring $\mathcal{O}_d$. By Proposition 5.5

there is a basis for the holomorphic 1-forms on A_Z and a symplectic basis for L_Z so that the resulting period matrix is (Ω, I) where

$$\Omega = \sigma \begin{pmatrix} z & r \\ r & d\bar{z} \end{pmatrix} {}^t\sigma \quad \text{with } r \in \mathbb{R} \text{ and } \operatorname{Im} \begin{pmatrix} z & r \\ r & d\bar{z} \end{pmatrix} > 0.$$

Then $(\mathbb{C}^2/L_Z, H_Z)$ and $(\mathbb{C}^2/L_\Omega, H_\Omega)$ are isomorphic principally polarized abelian surfaces, so by §4.1 there exists a linear mapping $\psi : \mathbb{C}^2 \to \mathbb{C}^2$ so that $\psi(L_\Omega) = L_Z$ and $\psi_*(H_\Omega) = H_Z$. Define $h = {}^t(F_Z^{-1}\psi F_\Omega) \in \mathbf{Sp}(4, \mathbb{Z})$. Then $\Omega = h \cdot Z$. By Proposition 2.5, $\Omega = \widehat{\Omega} = \widehat{h}\widehat{Z} = \widehat{h}\gamma Z$. So $h^{-1}\widehat{h}\gamma = \pm I$. If the plus sign occurs, then take $g = h^{-1}$ to get $\gamma = \widehat{g}g^{-1}$. If the minus sign occurs, take $g = h^{-1} \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix} \right)$ to get $\gamma = \widehat{g}g^{-1}$.

Finally suppose $\gamma = \widehat{h}h^{-1}$ for some $h \in \mathbf{Sp}(4, \mathbb{Z})$. Then the element $a = h^{-1}g \in \mathbf{Sp}(4, \mathbb{Z})$ satisfies $\widehat{a} = a$. By Propositions 2.5 and 2.10,

$$a \in \mathbf{Sp}(4, \mathbb{Z}) \cap \phi(\mathbf{SL}(2, \mathbb{C})) = \phi(\mathbf{SL}(2, \mathcal{O}_d)).$$

$\square$.

6. Arithmetic quotients

As in §2.2 fix a square-free integer $d < 0$, let $\mathcal{O}_d$ be the ring of integers in $\mathbb{Q}(\sqrt{d})$. To simplify notation we will write $\Lambda = \mathbf{SL}(2, \mathcal{O}_d)$. As in §2.9, fix an integer $N \geq 1$, let $\Gamma_N = \Gamma(N) \cap \widehat{\Gamma}(N)$ and denote by Λ_N the principal level subgroup

$$\Lambda_N = \mathbf{SL}(2, \mathcal{O}_d)(N).$$

As in Proposition 2.5, the group $\mathbf{SL}(2, \mathbb{C})$ acts on hyperbolic space $\mathbf{H}_3$ in a way which is compatible with the embeddings $\phi : \mathbf{SL}(2, \mathbb{C}) \to \mathbf{Sp}(4, \mathbb{R})$ and $\phi : \mathbf{H}_3 \to \mathfrak{h}_2$. Let $W = \Lambda_N \backslash \mathbf{H}_3$ and let $X = \Gamma_N \backslash \mathfrak{h}_2$. Let $\pi : \mathbf{H}_3 \to W$ and $\pi : \mathfrak{h}_2 \to X$ denote the projections.

For any $g \in \widetilde{\Gamma}_N$ define

$${}^g\phi : W = \Lambda_N \backslash \mathbf{H}_3 \to \Gamma_N \backslash \mathfrak{h}_2 = X$$

by $\Lambda_N w \mapsto \Gamma_N g\phi(w)$.

6.1. **Lemma.** *Let $g \in \widetilde{\Gamma}_N$ and set $\gamma = \widehat{g}g^{-1} \in \Gamma_N$. Then the mapping ${}^g\phi$ is well defined and injective. Its image*

$${}^g\phi(W) = \pi(E_\gamma)$$

coincides with the projection of E_γ. If $h, g \in \widetilde{\Gamma}_N$ determine the same double coset

$$\Gamma_N h\phi(\Lambda) = \Gamma_N g\phi(\Lambda) \in \Gamma_N \backslash \widetilde{\Gamma}_N / \phi(\Lambda)$$

then ${}^g\phi(W) = {}^h\phi(W)$.

6.2. **Proof.** The mapping ${}^g\phi$ is well defined, for if $\lambda \in \Lambda_N$ then

$${}^g\phi(\lambda w) = \Gamma_N g\phi(\lambda)g^{-1}g\phi(w) = \Gamma_N g\phi(w) = {}^g\phi(w)$$

because $\phi(\lambda) \in \Gamma_N$ which is normal in $\widetilde{\Gamma}_N$. It is easy to see that $E_\gamma = g \cdot \phi(\mathbf{H}_3) \subset \mathfrak{h}_2$, so we have a commutative diagram

$$\begin{array}{ccccc} \mathbf{H}_3 & \xrightarrow{g\cdot\phi} & E_\gamma & \subset & \mathfrak{h}_2 \\ {\scriptstyle\pi}\downarrow & & \downarrow & & \downarrow{\scriptstyle\pi} \\ W & \xrightarrow{{}^g\phi} & \pi(E_\gamma) & \subset & X \end{array}$$

from which it follows that ${}^g\phi(W) = \pi(E_\gamma)$. We need to show that ${}^g\phi$ is injective.

We claim the normalizer N_γ of E_γ in Γ_N is $g\phi(\Lambda_N)g^{-1}$. For, if $h \in \Gamma_N$, $x \in E_\gamma$, and $hx \in E_\gamma$ then $\widehat{h}\widehat{x} = \gamma hx$ so $h^{-1}\gamma^{-1}\widehat{h}\gamma \in \Gamma_N$ fixes x. This holds for all $x \in E_\gamma$ so $\widehat{h}\gamma = \gamma h$, or $\widehat{g}^{-1}\widehat{h}\widehat{g} = g^{-1}hg$. This implies that $g^{-1}hg \in \phi(\mathbf{SL}(2,\mathbb{C}))$ hence $h \in g\phi(\mathbf{SL}(2,\mathbb{C}))g^{-1} \cap \Gamma_N$ which proves the claim.

Now let $w_1, w_2 \in \mathbf{H}_3$ and suppose that ${}^g\phi(\Lambda_N w_1) = {}^g\phi(\Lambda_N w_2) \in \pi(E_\gamma)$. Since $\pi(E_\gamma) = N_\gamma \backslash E_\gamma$ there exists $h \in N_\gamma$ so that $hg\phi(w_1) = g\phi(w_2)$. By the preceding claim, $h = g\phi(\lambda)g^{-1}$ for some $\lambda \in \Lambda_N$, which implies $g\phi(\lambda w_1) = g\phi(w_2)$. But ϕ is injective so $w_2 = \lambda w_1$ and $\Lambda_N w_2 = \Lambda_N w_1 \in W$.

Finally, if $h = \gamma g\phi(\lambda)$ represents the same double coset as g (where $\gamma \in \Gamma_N$ and $\lambda \in \Lambda$) then

$${}^h\phi(\Lambda_N w) = \Gamma_N \gamma g\phi(\lambda)\phi(w) = {}^g\phi(\lambda w)$$

which shows that ${}^h\phi(W) = {}^g\phi(W)$. □

The involution $\widehat{\ }: \mathfrak{h}_2 \to \mathfrak{h}_2$ passes to an anti-holomorphic involution on X and hence defines a real structure on X.

6.3. **Theorem.** *Suppose $\mathcal{O}_d$ is a principal ideal domain. Fix $N \geq 3$. If $d \equiv 1 (\text{mod } 4)$ assume also that N is even. Then the set of real points $X_\mathbb{R}$ is the disjoint union*

$$X_\mathbb{R} = \coprod_g {}^g\phi(W) \tag{6.3.1}$$

of finitely many copies of $W = \Lambda_N \backslash \mathbf{H}_3$, indexed by elements

$$g \in \Gamma_N \backslash \widetilde{\Gamma}_N / \phi(\Lambda) = H^1(\mathbb{C}/\mathbb{R}, \Gamma_N).$$

The copy indexed by $g \in \widetilde{\Gamma}$ corresponds to the cohomology class f_γ where $\gamma = \widehat{g}g^{-1} \in \Gamma_N$.

6.4. **Proof.** A point $x \in X$ is real iff it is the image of a Γ_N-real point $Z \in \mathfrak{h}_2$. So

$$X_\mathbb{R} = \bigcup_{\gamma \in \Gamma_N} \pi(E_\gamma).$$

By Lemma 6.1, for each $g \in \widetilde{\Gamma}_N$ the mapping ${}^g\phi : W \to X_{\mathbb{R}}$ is injective. By Proposition 5.8 the union (6.3.1) contains $X_{\mathbb{R}}$. It remains to show that the individual components are disjoint.

Let $g_1, g_2 \in \widetilde{\Gamma}_N$ and set $\gamma_i = \widehat{g}_i g_i^{-1} \in \Gamma_N$. Suppose the corresponding components have a nontrivial intersection, say

$$x' \in \pi(E_{\gamma_1}) \cap \pi(E_{\gamma_2}).$$

We claim that there exists $\gamma \in \Gamma_N$ so that

$$\widehat{\gamma}\gamma_1 = \gamma_2\gamma \tag{6.4.1}$$

and hence $\pi(E_{\gamma_1}) = \pi(E_{\gamma_2})$. There is a lift $x \in E_{\gamma_1}$ of x' so that $\gamma x \in E_{\gamma_2}$ for some $\gamma \in \Gamma_N$. This implies $\widehat{\gamma}\gamma_1 x = \widehat{\gamma}\widehat{x} = \gamma_2\gamma x$. Therefore $\gamma^{-1}\gamma_2^{-1}\widehat{\gamma}\gamma_1 \in \Gamma_N$ fixes x. But Γ_N is torsion-free, so $\widehat{\gamma}\gamma_1 = \gamma_2\gamma$ as claimed.

Equation (6.4.1) gives $\widehat{g_2}^{-1}\widehat{\gamma}\widehat{g}_1 = g_2^{-1}\gamma g_1 \in \mathbf{Sp}(4, \mathbb{Z})$ which implies by Propositions 2.5 and 2.10 that $g_2^{-1}\gamma g_1 \in \phi(SL_2(\mathcal{O}_d))$. In other words, $g_2 = \gamma g_1 \phi(h)$ for some $h \in \mathbf{SL}(2, \mathcal{O}_d)$, so the elements g_1 and g_2 represent the same double coset in $\Gamma_N \backslash \widetilde{\Gamma}/\phi(\Lambda)$. □

7. The moduli space of abelian surfaces with anti-holomorphic multiplication

7.1. **Level structures.** Let $(A = \mathbb{C}^2/L, H = R + iQ)$ be a principally polarized abelian surface. A level N structure on A is a choice of basis $\{U_1, U_2, V_1, V_2\}$ for the N-torsion points of A which is symplectic, in the sense that there exists a symplectic basis $\{u_1, u_2, v_1, v_2\}$ for L such that

$$U_i \equiv \frac{u_i}{N} \text{ and } V_i \equiv \frac{v_i}{N} \pmod{L}$$

(for $i = 1, 2$). For a given leven N structure, such a choice $\{u_1, u_2, v_1, v_2\}$ determines a mapping

$$F : \mathbb{R}^2 \oplus \mathbb{R}^2 \to \mathbb{C}^2 \tag{7.1.1}$$

such that $F(\mathbb{Z}^2 \oplus \mathbb{Z}^2) = L$, by $F(e_i) = u_i$ and $F(f_i) = v_i$ where $\{e_1, e_2, f_1, f_2\}$ is the standard basis of $\mathbb{R}^2 \oplus \mathbb{R}^2$. The choice $\{u_1, u_2, v_2, v_2\}$ (or equivalently, the mapping F) will be referred to as a *lift* of the level N structure. It is well defined modulo the principal congruence subgroup $\Gamma(N)$, that is, if $F' : \mathbb{R}^2 \oplus \mathbb{R}^2 \to \mathbb{C}^2$ is another lift of the level structure, then $F' \circ F^{-1} \in \Gamma(N)$.

Fix a square-free integer $d < 0$ and let (A, H, κ) be a principally polarized abelian surface with anti-holomorphic multiplication by $\mathcal{O}_d$ as in §4.2. A level N structure $\{U_1, U_2, V_1, V_2\}$ on A is *compatible* with κ if for some (and hence for any) lift F of the level structure, the following diagram commutes (mod L) :

$$\begin{array}{ccc} \frac{1}{N}\left(\mathbb{Z}^2 \oplus \mathbb{Z}^2\right) & \xrightarrow[F]{} & \frac{1}{N}L \\ {}^tN_\beta \big\downarrow & & \big\downarrow \kappa \\ \frac{1}{N}\left(\mathbb{Z}^2 \oplus \mathbb{Z}^2\right) & \xrightarrow[F]{} & \frac{1}{N}L \end{array} \tag{7.1.2}$$

We will refer to the collection $\mathcal{A} = (A = \mathbb{C}^2/L, H = R + iQ, \kappa, \{U_i, V_j\})$ as a principally polarized abelian surface with anti-holomorphic multiplication and level N structure. If $\mathcal{A}' = (A' = \mathbb{C}^2/L', H' = R + iQ, \kappa', \{U_i', V_j'\})$ is another such, then an isomorphism $\mathcal{A} \cong \mathcal{A}'$ is a complex linear mapping $\psi : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\psi(L) = L'$, $\psi_*(H) = H'$, $\psi_*(\kappa) = \kappa'$, and such that for some (and hence for any) lift $\{u_1, u_2, v_1, v_2\}$ and $\{u_1', u_2', v_1', v_2'\}$ of the level structures,

$$\psi\left(\frac{u_i}{N}\right) \equiv \frac{u_i'}{N} \quad \text{and} \quad \psi\left(\frac{v_j}{N}\right) \equiv \frac{v_j'}{N} \pmod{L}.$$

7.2. If $Z \in \mathfrak{h}_2$, then for any $N \geq 1$ we define the *standard level N structure* on the abelian surface (A_Z, H_Z) to be the basis $\{F_Z(e_i/N), F_Z(f_j/N)\}$ (mod L) where $\{e_1, e_2, f_1, f_2\}$ is the standard basis of $\mathbb{R}^2 \oplus \mathbb{R}^2$.

7.3**. Lemma.** *Suppose $Z \in \mathfrak{h}_2$, $\gamma \in \mathbf{Sp}(4, \mathbb{Z})$, and $\widehat{Z} = \gamma \cdot Z$. Let $N \geq 3$. Then the standard level N structure on the abelian surface (A_Z, H_Z) is compatible with the anti-holomorphic multiplication κ_Z iff $\gamma \in \Gamma_N$.*

7.4. **Proof.** It follows immediately from diagram (4.4.1) that $\gamma \in \Gamma(N)$ iff the standard level N structure on (A_Z, H_Z) is compatible with κ_Z. Since $\Gamma(N)$ is torsion-free, $\widehat{\gamma}\gamma = I$ which implies $\gamma \in \widehat{\Gamma}(N)$; hence $\gamma \in \Gamma_N$. ☐

By Lemma 7.3, each point $Z \in \mathfrak{h}_2^{\Gamma(N)}$ determines a principally polarized abelian surface $\mathcal{A}_Z = (A_Z, H_Z, \kappa_Z, \{F_Z(e_i/N), F_Z(f_j/N)\})$ with anti-holomorphic multiplication and (compatible) level N structure.

7.5**. Theorem.** *Suppose $\mathcal{O}_d$ is principal. Fix $N \geq 3$. If $d \equiv 1 \pmod 4$, assume also that N is even. Then the association $Z \mapsto \mathcal{A}_Z$ determines a one to one correspondence between the real points (6.3.1) $X_{\mathbb{R}}$ of $X = \Gamma_N \backslash \mathfrak{h}_2$ and the moduli space $V(d, N)$ consisting of isomorphism classes of principally polarized abelian surfaces with anti-holomorphic multiplication by $\mathcal{O}_d$ and (compatible) level N structure.*

7.6. **Proof.** A point $x \in X$ is real iff it is the image of a Γ_N-real point $Z \in \mathfrak{h}_2$. If two Γ_N-real points $Z, \Omega \in \mathfrak{h}_2$ determine isomorphic varieties, say $\psi : \mathcal{A}_\Omega \cong \mathcal{A}_Z$ then by (4.1.1) there exists $h \in \mathbf{Sp}(4, \mathbb{Z})$ such that $\Omega = h \cdot Z$. Since the isomorphism ψ preserves the level N structures, it follows also from (4.1.1) that $h \in \Gamma(N)$. We claim that $h \in \Gamma_N$. Let $\widehat{Z} = \gamma_Z \cdot Z$ and $\widehat{\Omega} = \gamma_\Omega \cdot \Omega$, with $\gamma_Z, \gamma_\Omega \in \Gamma_N$. Putting diagram (4.4.1) for Z together with the analogous diagram for Ω and diagram (4.1.1), and using the fact that $\psi_*(\kappa_\Omega) = \kappa_Z$ gives a diagram

$$\begin{array}{ccc} \mathbb{R}^2 \oplus \mathbb{R}^2 & \xrightarrow[F_\Omega]{} & \mathbb{C}^2 \\ {}^t(\gamma_\Omega^{-1}\widehat{h}\gamma_Z)\big\downarrow & & \big\downarrow \kappa_Z \psi \kappa_\Omega^{-1} = \psi \\ \mathbb{R}^2 \oplus \mathbb{R}^2 & \xrightarrow[F_Z]{} & \mathbb{C}^2 \end{array}$$

from which it follows that ${}^t(\gamma_\Omega \widehat{h} \gamma_Z) \in \Gamma(N)$, hence $\widehat{h} \in \Gamma(N)$, hence $h \in \Gamma_N$.

So it remains to show that every principally polarized abelian surface with anti-holomorphic multiplication and level N structure, $\mathcal{A} = (A, H, \kappa, \{U_i, V_j\})$ is isomorphic to some $\mathcal{A}_Z$. By the Comessatti lemma (Proposition 5.5) there exists $Z' \in \mathfrak{h}_2$, such that $\widehat{Z'} = Z'$, (cf. Proposition 2.5) and there exists an isomorphism

$$\psi' : (A_{Z'}, H_{Z'}, \kappa_{Z'}) \cong (A, H, \kappa)$$

between the principally polarized abelian surfaces with anti-holomorphic multiplication. However the isomorphism ψ' must be modified because it does not necessarily take the standard level N structure on $(A_{Z'}, H_{Z'}, \kappa_{Z'})$ to the given level N structure on (A, H, κ).

Choose a lift $\{u_1, u_2, v_1, v_2\}$ of the level N structure and let $F : \mathbb{R}^2 \oplus \mathbb{R}^2 \to \mathbb{C}^2$ be the corresponding mapping (7.1.1). Define

$$ {}^t g^{-1} = F^{-1} \circ \psi' \circ F_{Z'} \in \mathbf{Sp}(4, \mathbb{Z}) \tag{7.6.1}$$

$$ Z = g \cdot Z' \tag{7.6.2}$$

$$ \gamma = \widehat{g} g^{-1} = N_\beta^{-1} g N_\beta g^{-1}. \tag{7.6.3}$$

If $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ define $\xi : \mathbb{C}^2 \to \mathbb{C}^2$ by $\xi(M) = {}^t(CZ + D)M$. Define $\psi = \psi' \circ \xi$. We will show that $\gamma \in \Gamma_N$, that $\widehat{Z} = \gamma \cdot Z$, and that ψ induces an isomorphism $\psi : \mathcal{A}_Z \to \mathcal{A}$ of principally polarized abelian surfaces with anti-holomorphic multiplication and compatible level N structures.

In the following diagram, F is the mapping (7.1.1) associated to the lift of the level N structure. The bottom square commutes by the definition of g, while the top square commutes by (4.1.1).

$$\begin{array}{ccc} \mathbb{R}^2 \oplus \mathbb{R}^2 & \xrightarrow{F_Z} & \mathbb{C}^2 \\ {}^t g \big\downarrow & & \big\downarrow \xi \\ \mathbb{R}^2 \oplus \mathbb{R}^2 & \xrightarrow[F_{Z'}]{} & \mathbb{C}^2. \\ {}^t g^{-1} \big\downarrow & & \big\downarrow \psi' \\ \mathbb{R}^2 \oplus \mathbb{R}^2 & \xrightarrow{F} & \mathbb{C}^2 \end{array} \tag{7.6.4}$$

First let us verify that $\xi : (A_Z, H_Z, \kappa_Z) \to (A_{Z'}, H_{Z'}, \kappa_{Z'})$ is an isomorphism of principally polarized varieties with anti-holomorphic multiplication by $\mathcal{O}_d$. It follows from (7.6.4) that $\xi_*(L_Z) = L_{Z'}$ and $\xi_*(H_Z) = H_{Z'}$. We claim that $\xi_*(\kappa_Z) = \kappa_{Z'}$, that is, $\kappa_{Z'} = \xi\kappa_Z\xi^{-1}$. But this follows from direct calculation using $\xi = F_{Z'}\,{}^t g F_Z$, $\kappa_Z = F_Z\,{}^t(N_\beta\gamma)F_Z^{-1}$, $\kappa_{Z'} = F_{Z'}\,{}^t N_\beta F_{Z'}$ and (7.6.3) (and it is equivalent to the statement that the pushforward by ${}^t g$ of the involution ${}^t(N_\beta\gamma)$ on $\mathbb{R}^2 \oplus \mathbb{R}^2$ is the involution ${}^t N_\beta$). It follows that

$$\psi_*(\kappa_Z) = \kappa. \tag{7.6.5}$$

We claim that the standard level N structure on (A_Z, H_Z) is compatible with κ_Z. By construction, the mapping ψ takes the standard level N structure on (A_Z, H_Z) to the given leven N structure on (A, H). By assumption, the diagram (7.1.2) commutes (mod L). By (7.6.4), $F = \psi \circ F_Z$. Using (7.6.5) it follows that the diagram

$$\begin{array}{ccc} \frac{1}{N}(\mathbb{Z}^2 \oplus \mathbb{Z}^2) & \xrightarrow[F_Z]{} & \frac{1}{N}L_Z \\ {\scriptstyle {}^t N_\beta}\downarrow & & \downarrow{\scriptstyle \kappa_Z} \\ \frac{1}{N}(\mathbb{Z}^2 \oplus \mathbb{Z}^2) & \xrightarrow[F_Z]{} & \frac{1}{N}L_Z \end{array}$$

commutes (mod L_Z), which proves the claim. It also follows from Lemma 7.3 that $\gamma \in \Gamma_N$.

In summary, we have shown that $(A_Z, H_Z, \kappa_Z, \{F_Z(e_i/N), F_Z(f_j/N)\})$ is a real principally polarized abelian surface with anti-holomorphic multiplication and (compatible) level N structure, and that the isomorphism ψ preserves both the anti-holomorphic multiplication and the level structures. $\square$

8. Baily-Borel Compactification

Throughout this section we fix a square-free integer $d < 0$. Let $g \mapsto \widehat{g} = N_\beta g N_\beta^{-1}$ be the resulting involution on $\mathbf{Sp}(4, \mathbb{R})$ with fixed point set $\phi(\mathbf{SL}(2, \mathbb{C}))$. Let $\Gamma \subset \mathbf{Sp}(4, \mathbb{Z})$ be a torsion-free subgroup of finite index such that $\widehat{\Gamma} = \Gamma$. Let $\overline{\mathfrak{h}}_2$ be the partial Satake compactification which is obtained by attaching rational boundary components of (complex) dimension 0 and 1 (with the Satake topology). The quotient $\overline{X} = \Gamma\backslash\overline{\mathfrak{h}}_2$ is the Baily-Borel compactification of X. It is a complex projective algebraic variety. Denote by

$$\partial\overline{X} = \overline{X} - X = \partial_0\overline{X} \cup \partial_1\overline{X}$$

the decomposition of the singular set into its (complex-) 0 and 1 dimensional strata. The involution $g \mapsto \hat{g}$ passes to an involution on $\overline{X}$ and defines a real structure on $\overline{X}$, which we refer to as the $(\Gamma, \widehat{\ })$-real structure. Its fixed point set is the set $\overline{X}(\mathbb{R})$ of real points of $\overline{X}$.

Throughout the rest of §8 we fix a level $N \geq 1$, set $\Gamma = \Gamma_N$ of §2.9, and $X = \Gamma\backslash\mathfrak{h}_2$.

8.1. **Theorem.** *Suppose $\mathcal{O}_d$ is principal. If $d \equiv 1 \pmod 4$, assume that N is divisible by 4. If $d = -2$, assume that N is even. If $d = -1$, assume that N is even and $N \geq 4$. Then the 1-dimensional boundary strata of $\overline{X}$ contain no real points, that is, $\overline{X}(\mathbb{R}) \cap \partial_1\overline{X} = \phi$.*

In the next few sections we give separate proofs for $d \equiv 1 \pmod 4$, $d = -1$, and $d = -2$. Let $\mathbf{GSp}(4, \mathbb{R})$ be the set of real matrices g such that $gJ\,{}^tg = \lambda g$ for some $\lambda \in \mathbb{R}^\times$ (where $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ is the standard symplectic form). It acts on $\mathfrak{h}_2$ by fractional linear transformations and the center acts trivially. Identify the upper halfplane $\mathfrak{h}_1$ with the *standard* 1-dimensional boundary component F_1 of $\mathfrak{h}_2$ by mapping $z = x + iy \in \mathfrak{h}_1$ to $\left(\begin{smallmatrix} z & 0 \\ 0 & i\infty \end{smallmatrix}\right)$ (as a limit of 2×2 complex matrices). Let $P_1 \subset \mathbf{GSp}(4, \mathbb{R})$ denote the maximal parabolic subgroup which normalizes this boundary component. It acts on $\mathfrak{h}_1$ via the projection $\nu_h : P_1 \to \mathbf{GL}(2, \mathbb{R})$ which is given by

$$\begin{pmatrix} a & 0 & b & * \\ * & t & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & s \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{8.1.1}$$

For any $p \in P_1$ denote by $p_h = \nu_h(p)$. If $pJ\,{}^tp = \lambda J$ then $\det(p_h) = ad - bc = st = \lambda$.

Define $L_\beta : \mathfrak{h}_2 \to \mathfrak{h}_2$ by $L_\beta(y) = -\beta y\,{}^t\beta^{-1}$, cf. §2.4. It is the fractional linear tansformation corresponding to the matrix

$$L_\beta = \begin{pmatrix} -\beta & 0 \\ 0 & {}^t\beta \end{pmatrix} \in \mathbf{GSp}(4, \mathbb{Z}). \tag{8.1.2}$$

(Note that $L_\beta J\,{}^tL_\beta = -dJ$.) Denote by $\tilde{Z} = L_\beta \hat{Z} = L_\beta \cdot N_\beta \cdot \bar{Z} = -\bar{Z}$ for $Z \in \mathfrak{h}_2$. This is an anti-holomorphic involution which extends to the partial compactification $\overline{\mathfrak{h}}_2$ and preserves the standard boundary component F_1. Let $g \mapsto \tilde{g}$ denotes the involution $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} A & -B \\ -C & D \end{smallmatrix}\right)$ on $\mathbf{Sp}(4, \mathbb{R})$. Then $\widetilde{gZ} = \tilde{g}\tilde{Z}$ for all $g \in \mathbf{Sp}(4, \mathbb{R})$ and all $Z \in \mathfrak{h}_2$.

Suppose $x' \in \partial_1 \overline{X}(\mathbb{R})$. Then there is a lift $x \in \overline{\mathfrak{h}}_2$ which lies in some 1-dimensional rational boundary component, and there exists $\gamma \in \Gamma$ such that $\hat{x} = \gamma x$. Since $\mathbf{Sp}(4, \mathbb{Z})$ acts transitively on the set of all rational boundary components of a given dimension there exists $g \in \mathbf{Sp}(4, \mathbb{Z})$ and there exists $z \in F_1$ so that $x = gz$. Then $\hat{g}\hat{z} = \hat{x}$ and $\tilde{g}\tilde{z} = \tilde{x}$. It follows that $pz = \tilde{z} = -\bar{z} \in F_1$ where

$$p = \tilde{g}^{-1} L_\beta \gamma g \in \mathbf{GSp}(4, \mathbb{Z}). \tag{8.1.3}$$

Hence $p \in P_1$, p is integral, and by (8.1.1), its Hermitian part

$$p_h = \begin{pmatrix} p_{11} & p_{13} \\ p_{31} & p_{33} \end{pmatrix} \in \mathbf{GL}(2, \mathbb{R})$$

is also integral and has determinant $-d$.

8.2. **The case $d \equiv 1 \pmod 4$.** Since $p \cdot z = -\bar{z}$ we find

$$p_{11}z + p_{13} = -\bar{z}(p_{31}z + p_{33}) = -p_{31}z\bar{z} - p_{33}\bar{z}.$$

Comparing imaginary parts, $p_{11} = p_{33}$ so

$$-d = \det(p_h) = p_{11}^2 - p_{13}p_{31}. \tag{8.2.1}$$

If $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbf{Sp}(4,\mathbb{Z})$ then $\tilde{g} = g - \left(\begin{smallmatrix} 0 & 2B \\ 2C & 0 \end{smallmatrix}\right)$. Therefore $g^{-1}\tilde{g} \equiv I \pmod 2$. Since $\beta \equiv I \pmod 2$, the same holds for L_β. Moreover $\gamma \in \Gamma_N \subset \Gamma(N/2) \subset \Gamma(2)$ so

$$p = (\tilde{g}^{-1}g)g^{-1}(L_\beta\gamma)g \equiv I \pmod 2$$

and the same holds for p_h. In other words, p_{11} is odd while p_{13}, p_{31} are even. Therefore $p_{11}^2 - p_{13}p_{31} \equiv 1 \pmod 4$ which contradicts (8.2.1) since $-d \equiv -1 \pmod 4$. This completes the proof of Theorem 8.1 in the case $d \equiv 1 \pmod 4$.

8.3**. Lemma.** *Suppose $d = -1$ or $d = -2$. Let $g, \gamma \in \mathbf{Sp}_4(\mathbb{Z})$ and suppose that $\gamma \equiv I \pmod 2$. Suppose $p = \tilde{g}^{-1}L_\beta\gamma g \in P_1$, cf.* (8.1.3). *Denote the Hermitian part, cf.* (8.1.1), *by*

$$p_h = \begin{pmatrix} p_{11} & p_{13} \\ p_{31} & p_{33} \end{pmatrix}.$$

Then p_{13} and p_{31} are even. If $d = -1$ then $p_h \in \mathrm{SL}_2(\mathbb{Z})(2)$ lies in the principal congruence subgroup of level 2.

8.4. **Proof.** Set $q = \tilde{g}^{-1}L_\beta g$. Then $p \equiv q \pmod 2$ so $p_h \equiv q_h \pmod 2$. If $\tilde{g}^{-1} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ then

$$q = \begin{pmatrix} -A\beta\,{}^tD + B\,{}^t\beta\,{}^tC & -A\beta\,{}^tB + B\,{}^t\beta\,{}^tA \\ -C\beta\,{}^tD + D\,{}^t\beta\,{}^tC & -C\beta\,{}^tB + D\,{}^t\beta\,{}^tA \end{pmatrix}$$

Suppose $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$ and similarly for B. Direct computation gives $q_{13} = -a_1b_2 - da_2b_1 + db_1a_2 + b_2a_1 = 0$ and similarly $q_{31} = 0$. Hence p_{13} and p_{31} are even. If $d = -1$ then $p_{11}p_{33} - p_{13}p_{31} = 1$ which implies that p_{11} and p_{33} are odd. Hence $p_h \equiv I \pmod 2$. □

8.5. **Proof of Theorem 8.1 in the case $d = -2$.** The same argument in §8.2 leads to equation (8.2.1): $2 = p_{11}^2 - p_{13}p_{31}$. By Lemma 8.3, p_{13} and p_{31} are even. This leads to a contradiction whether p_{11} is even or odd.

8.6. **Proof of Theorem 8.1 in the case $d = -1$.** Return to (8.1.3) and assume that $\gamma \in \Gamma_N \subset \Gamma(N)$ for some even integer $N \geq 4$. By Lemma 8.3, $p_h \in \mathrm{SL}_2(\mathbb{Z})(2)$. Since $p_h z = -\bar{z}$ we can apply [GT] §5 to find $Y \in \mathbb{R}_+$ and $h \in \mathbf{SL}(2,\mathbb{Z})$ such that $z = h \cdot iY$. Using (8.1.1), the element h may also be regarded as lying in $P_1 \cap \mathbf{Sp}(4,\mathbb{Z})$. Define $g_1 = gh$. Then the element

$$v = (\tilde{g}_1)^{-1}L_\beta\gamma g_1 \in P_1 \cap \mathbf{Sp}(4,\mathbb{Z})$$

fixes the point $iY \in \mathfrak{h}_1$. Again by Lemma 8.3 its Hermitian part $v_h \in \Gamma(2)$ (which is torsion-free). Since it fixes iY it must equal $\pm I$. In summary we may write

$$v = \begin{pmatrix} \alpha & * \\ 0 & {}^t\alpha^{-1} \end{pmatrix}$$

where

$$\alpha = \begin{pmatrix} \epsilon_1 & 0 \\ * & \epsilon_2 \end{pmatrix}$$

with $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$. If $g_1 = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$ (and similarly for B, C, and D), then $\tilde{g}_1 v \equiv L_\beta g_1 \pmod{N}$ gives

$$A\alpha \equiv -\beta A \pmod{N} \text{ and } C\alpha \equiv \beta C \pmod{N}.$$

From this we obtain $a_2 \equiv \epsilon_2 a_4 \equiv -a_2 \pmod{N}$ and $-c_2 \equiv \epsilon_2 c_4 \equiv c_2 \pmod{N}$. Provided $N \geq 3$ this implies that the second column of the matrix g_1 is 0 (mod N), which contradicts the fact that $\det(g_1) = 1$. This completes the proof of Theorem 8.1 in the case $d = -1$.

9. Rational Structure

As in §2.2, fix a square-free integer $d < 0$, let $\mathcal{O}_d$ be the full ring of integers in $\mathbb{Q}(\sqrt{d})$ and let $g \mapsto \hat{g} = N_\beta g N_\beta^{-1}$ be the resulting involution on $\mathbf{Sp}(4, \mathbb{R})$ and $Z \mapsto \hat{Z} = \beta \overline{Z}\,{}^t\beta^{-1}$ the resulting anti-holomorphic involution on $\mathfrak{h}_2$. In this section we make no further assumptions on d. Fix a level $N \geq 1$ and, as in §2.9 let $\Gamma = \Gamma_N = \Gamma(N) \cap \widehat{\Gamma}(N)$.

9.1**. Theorem.** *The Baily-Borel compactification $\overline{X}$ admits the structure of a complex projective variety which is defined over $\mathbb{Q}$ such that the resulting real structure, when restricted to X agrees with the real structure of Theorem 6.3.*

The proof will occupy the rest of this section.

9.2**. Proposition.** *The complex vectorspace of (holomorphic) Γ-modular forms on $\mathfrak{h}_2$ is spanned by modular forms with rational Fourier coefficients.*

9.3. **Proof.** This follows directly from [Sh] who proves the following in Theorem 3 (ii). Let $\mathbf{G} = \mathbf{GSp}_4$, let $\mathbb{A}$ be the adeles of $\mathbb{Q}$ and let $S \subset \mathbf{G}(\mathbb{A})^+$ be an open subgroup containing $\mathbb{Q}^\times \mathbf{G}(\mathbb{R})^+$ such that $S/\mathbb{Q}^\times \mathbf{G}(\mathbb{R})^+$ is compact. (Here, $+$ denotes the identity component.) Let $\Gamma = S \cap \mathbf{G}(\mathbb{Q})$. Suppose that

$$\Delta = \left\{ \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \middle| \; t \in \prod_p \mathbb{Z}_p^\times \right\} \subset S.$$

Then the complex vectorspace of Γ-modular forms with weight k on $\mathfrak{h}_2$ is spanned by those forms whose Fourier coefficients are in the field $k_S = \mathbb{Q}$. To apply this to our setting, define

$$\begin{aligned} S(N) &= \left\{ x \in \mathbf{G}(\mathbb{A})^+ \mid \; x_p \equiv \left(\begin{smallmatrix} I & 0 \\ 0 & a_p I \end{smallmatrix}\right) (\mathrm{mod}\ N \cdot \mathbb{Z}_p), \exists a_p \in \mathbb{Z}_p^\times \right\} \cdot \mathbb{Q}^\times \\ S^\beta(N) &= N_\beta S(N) N_\beta^{-1}. \end{aligned}$$

(Here, x_p denotes the p-component of x.) It is easy to see that each of these contains Δ, hence Shimura's hypothesis is satisfied. If $S = S(N) \cap S^\beta(N)$ then $\Gamma_S = S \cap \mathbf{G}(\mathbb{Q}) = (\Gamma(N) \cap \widehat{\Gamma}(N)) \cdot \mathbb{Q}^\times$. □

9.4. Let $I_- = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$. Its action by fractional linear transformations maps the Siegel lower halfspace $\mathfrak{h}_2^-$ to the upper halfspace $\mathfrak{h}_2$, that is, $I_- \cdot Z = -Z$. Hence, for any holomorphic mapping $f : \mathfrak{h}_2 \to \mathbb{C}$ we may define $f^\beta : \mathfrak{h}_2 \to \mathbb{C}$ by

$$f^\beta(Z) = f(I_- \cdot N_\beta \cdot Z) = f(-\beta Z\, {}^t\beta^{-1}).$$

9.5. **Proposition.** *If $f : \mathfrak{h}_2 \to \mathbb{C}$ is a holomorphic Γ-modular form of weight k then f^β is also a holomorphic Γ-modular form of weight k, and*

$$f^\beta(\widehat{Z}) = \overline{f^\beta(Z)} \tag{9.5.1}$$

for all $Z \in \mathfrak{h}_2$.

9.6. **Proof.** Suppose that $f(\gamma \cdot Z) = j(\gamma, Z)^k f(Z)$ for all $\gamma \in \Gamma$ and all $Z \in \mathfrak{h}_2$ where

$$j\left(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), Z\right) = \det(CZ + D)$$

is the standard automorphy factor. Then $j(I_- N_\beta, Z) = \det(-\,{}^t\beta)$ is independent of Z. Let $\gamma \in \Gamma$ and set

$$\gamma' = I_- N_\beta \gamma N_\beta^{-1} I_-^{-1} \in \Gamma.$$

Then

$$\begin{aligned} f^\beta(\gamma \cdot Z) &= f(\gamma' \cdot I_- N_\beta \cdot Z) \\ &= j(\gamma', I_- N_\beta \cdot Z)^k f^\beta(Z) \\ &= \det(-\,{}^t\beta)^k j(\gamma, Z)^k \det(-\,{}^t\beta)^{-k} f^\beta(Z) \\ &= j(\gamma, Z)^k f^\beta(Z). \end{aligned}$$

which shows that f^β is Γ-modular of weight k. Next, with respect to the standard maximal parabolic subgroup P_0 (which normalizes the standard 0-dimensional boundary component), the modular form f has a Fourier expansion,

$$f(Z) = \sum_s a_s \exp\left(2\pi i \langle s, Z\rangle\right)$$

which is a sum over lattice points $s \in L^*$ where $L = \Gamma \cap Z(\mathcal{U}_0)$ is the intersection of Γ with the center of the unipotent radical $\mathcal{U}_0$ of P_0 and where $a_s \in \mathbb{Q}$. Then

$$\begin{aligned} f(\hat{Z}) &= \sum_s a_s \exp\left(2\pi i \langle s, \beta \overline{Z}\, {}^t\beta^{-1}\rangle\right) \\ &= \overline{\sum_s a_s \exp\left(2\pi i \langle s, -\beta Z\, {}^t\beta^{-1}\rangle\right)} \\ &= \overline{f(-\beta Z\, {}^t\beta^{-1})} = \overline{f^\beta(Z)}. \quad \square \end{aligned}$$

9.7. The Baily-Borel compactification $\overline{X}$ of X is the obtained by embedding X holomorphically into $\mathbb{CP}^m$ using $m+1$ (Γ-)modular forms (say $f_0, f_1, \ldots, f_m$) of some sufficiently high weight k, and then taking the closure of the image. Define an embedding $\Phi : X \to \mathbb{CP}^{2m+1}$ by

$$\Phi(Z) = \Big(f_0(Z) : f_1(Z) : \ldots : f_m(Z) : f_0^\beta(Z) : f_1^\beta(Z) \ldots : f_m^\beta(Z)\Big).$$

Denote these homogeneous coordinate functions by $x_j = f_j(Z)$ and $y_j = f_j^\beta(Z)$. Define an involution $\sigma : \mathbb{CP}^{2m+1} \to \mathbb{CP}^{2m+1}$ by $\sigma(x_j) = \bar{y}_j$ and $\sigma(y_j) = \bar{x}_j$. Then equation (9.5.1) says that this involution is compatible with the embedding Φ, that is, for all $Z \in X$ we have:

$$\sigma\Phi(Z) = \Phi(\widehat{Z}).$$

Define $\Psi : \mathbb{CP}^{2m+1} \to \mathbb{CP}^{2m+1}$ by setting $\xi_j = x_j + y_j$ and $\eta_j = i(x_j - y_j)$ for $0 \le j \le m$. Let $Y = \Psi\Phi(X)$ and let $\overline{Y}$ denote its closure.

9.8**. Proposition.** *The composition $\Psi\Phi : X \to \mathbb{CP}^{2m+1}$ is a holomorphic embedding which induces an isomorphism of complex algebraic varieties $\overline{X} \to \overline{Y}$. The variety $\overline{Y}$ is defined over the rational numbers, and the real points of Y are precisely the image of those points $Z \in X$ such that $\hat{Z} = Z$.*

9.9. **Proof.** The image $\Psi\Phi(X)$ is an algebraic subvariety of projective space, which is preserved by complex conjugation so it is defined over $\mathbb{R}$. The real points are obtained by setting $\bar{\xi}_j = \xi_j$ and $\bar{\eta}_j = \eta_j$ which gives $\bar{x}_j = y_j$ and $\bar{y}_j = x_j$ hence $\Phi(Z) = \sigma\Phi(Z)$, or $Z = \hat{Z}$. The Fourier coefficients of ξ_j and η_j are in $\mathbb{Q}[i]$ so the image $\Psi\Phi(X)$ is defined over $\mathbb{Q}[i]$. Since it is also invariant under $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, it follows that $\Psi\Phi(X)$ is defined over $\mathbb{Q}$. $\square$

9.10. **Remark.** The usual embedding $(f_0 : f_1 : \ldots : f_m) : X \hookrightarrow \mathbb{CP}^m$ determines the usual rational structure on $\overline{X}$, and the resulting complex conjugation is that induced by $Z \mapsto -\overline{Z}$ for $Z \in \mathfrak{h}_2$.

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