Sur les groupes d'homotopie des sphères en théorie des types homotopiques

On the homotopy groups of spheres in homotopy type theory

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Contents

Homotopy type theory

Homotopy type theory (HoTT) is

- A foundation of mathematics, based on the principle of univalence: isomorphic structures are equal
- An understanding of the identity types in Martin–Löf type theory
- An internal language for $(\infty, 1)$-toposes
- A framework in which we can do synthetic homotopy theory

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Homotopy theory

Intuition

Homotopy theory is the study of spaces up to homotopy.

A space-up-to-homotopy is

- a topological space up to weak homotopy equivalence, or
- a CW-complex up to homotopy equivalence, or
- a simplicial set up to weak equivalence, or
- an ∞-groupoid up to weak equivalence, or
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Synthetic homotopy theory

Homotopy theory in a purely homotopy-invariant way and constructively.
Basic concepts are primitive instead of being defined.

- spaces (types)
- points (elements of types)
- fibrations

- function spaces
- product spaces
- path spaces
- the space of all small spaces
- many "cell complexes"

Path spaces

Formation and introduction rules

Given a type A and two points $u, v : A$, there is a type

$$\mathrm{Path}_A(u, v).$$

Given a point $a : A$, there is a path

$$\mathrm{idp}_a : \mathrm{Path}_A(a, a).$$

Contractibility of singletons

Given $a : A$, for every $x : A$ and $p : \mathrm{Path}_A(a, x)$ there is a path

$$s_{a,x,p} : \mathrm{Path}((x, p), (a, \mathrm{idp}_a)).$$

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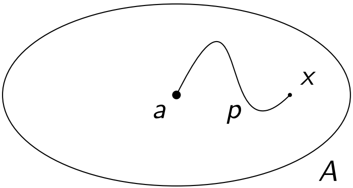
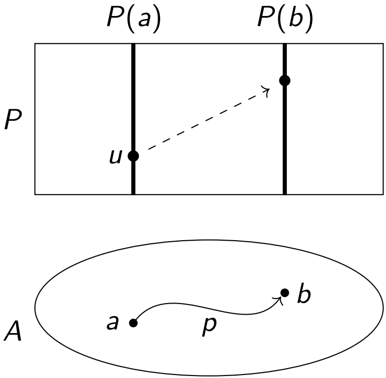


Transport

Intuition

The type Type is the type of all small types.
A function $P : A \to \text{Type}$ corresponds to a fibration.



Transport

Given $P : A \to \text{Type}$,
$p : \text{Path}_A(a, b)$ and $u : P(a)$,
there is an element

$$\text{transport}^P(p, u) : P(b).$$

Univalence

Definition

An equivalence between two types A and B is a pair of functions $f : A \to B$ and $g : B \to A$ such that the composites are homotopic to the identity functions.

Univalence axiom

Given two types A and B, the canonical map

$$\mathrm{Path}_{\mathrm{Type}}(A, B) \to (A \simeq B)$$

is an equivalence, where $A \simeq B$ is the type of equivalences from A to B.

Higher inductive types

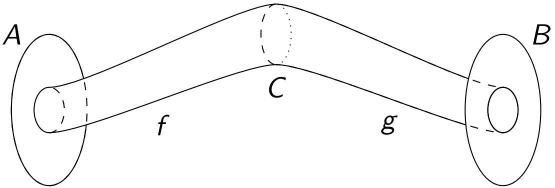
Definition

The pushout of a diagram

$$A \xleftarrow{f} C \xrightarrow{g} B$$

is the higher inductive type $A \sqcup^C B$ generated by

$$\mathsf{inl} : A \to A \sqcup^C B$$
$$\mathsf{inr} : B \to A \sqcup^C B$$
$$\mathsf{push} : (c : C) \to \mathsf{Path}_{A \sqcup^C B}(\mathsf{inl}(f(c)), \mathsf{inr}(g(c)))$$



Homotopy groups

Definition

Given a type A and $n : \mathbb{N}$, there is a universal n-truncated type $\|A\|_n$ called the *n-truncation* of A, together with a map

$$|-|_n : A \to \|A\|_n.$$

Definition

The n^{th} *homotopy group* of a pointed type A is

$$\pi_n(A) := \|\Omega^n A\|_0.$$

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Statement of the problem

Definition

The sphere $\mathbb{S}^{n+1}$ is the pushout of

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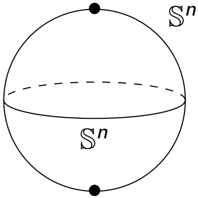

General goal

Compute the $\pi_k(\mathbb{S}^n)$ in homotopy type theory.
Here we will compute $\pi_4(\mathbb{S}^3)$.

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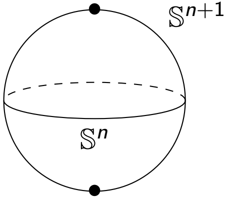

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Table of homotopy groups of spheres

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Contents

Fibrations

Intuition

A fibration is a family of spaces parametrized by another space.

In classical mathematics:

- We define E, B and $p : E \to B$
- We prove that p is a fibration

$$
\begin{array}{c}
F \\
\big\downarrow \\
E \\
\big\downarrow p \\
B
\end{array}
$$

In HoTT:

- We define B and $P : B \to \text{Type}$
- We figure out what the total space E looks like

The universal cover of the circle

Definition

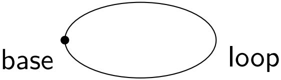
The circle $\mathbb{S}^1$ is generated by

$$\text{base} : \mathbb{S}^1,$$
$$\text{loop} : \text{Path}_{\mathbb{S}^1}(\text{base}, \text{base}).$$

Definition

The universal cover of the circle is defined by

$$U : \mathbb{S}^1 \to \text{Type},$$
$$U(\text{base}) := \mathbb{Z},$$
$$\text{ap}_U(\text{loop}) := \text{ua}(n \mapsto n + 1).$$



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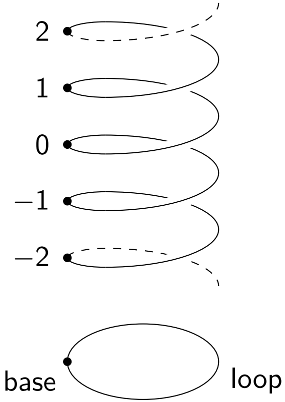
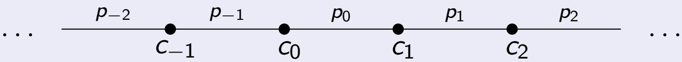


The universal cover of the circle

Proposition (flattening lemma)

The total space of U is equivalent to



Moreover one can prove that this type is contractible.

Corollary (Shulman 2011)

We have a fibration $\mathbb{Z} \hookrightarrow \mathbf{1} \twoheadrightarrow \mathbb{S}^1$ and

$$\pi_1(\mathbb{S}^1) \simeq \mathbb{Z},$$
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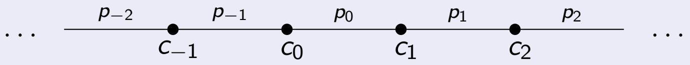

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The sphere $\mathbb{S}^2$ is generated by

$$\begin{aligned} \text{north} &: \mathbb{S}^2, \\ \text{south} &: \mathbb{S}^2, \\ \text{merid} &: \mathbb{S}^1 \to \text{Path}_{\mathbb{S}^2}(\text{north}, \text{south}). \end{aligned}$$

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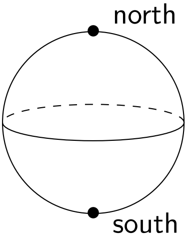

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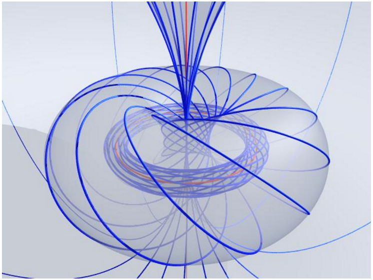

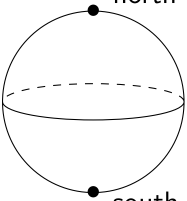

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The total space of Hopf is equivalent to the pushout of

$$\mathbb{S}^1 \xleftarrow{\text{fst}} \mathbb{S}^1 \times \mathbb{S}^1 \xrightarrow{\mu} \mathbb{S}^1$$

Moreover one can prove that this type is equivalent to $\mathbb{S}^3$.

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We have a fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \twoheadrightarrow \mathbb{S}^2$ and

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The James construction

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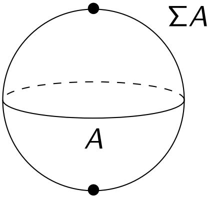
The suspension of a type A is the pushout ΣA of

$$\mathbf{1} \longleftarrow A \longrightarrow \mathbf{1}$$



Goal

Given a pointed type A, we want to approximate $\Omega\Sigma A$.
For instance if $A = \mathbb{S}^n$, understanding $\Omega\Sigma A$ helps understanding $\mathbb{S}^{n+1}$.

The James construction

If A is a k-connected pointed type, for $k \geq 0$, we have

$$\begin{array}{ccccccccc}
J_0A & \xrightarrow[(k-1)]{i_0} & J_1A & \xrightarrow[(2k)]{i_1} & J_2A & \xrightarrow[(3k+1)]{i_2} & \dots & & JA \\
\| & & \| & & \| & & & & \| \\
\mathbf{1} & & A & & (A \times A) \sqcup^{A \vee A} A & & \dots & & \Omega\Sigma A
\end{array}$$

where the maps i_n are more and more connected.

Freudenthal suspension theorem

Freudenthal suspension theorem (Lumsdaine, 2013)

If A is k-connected, then the map $A \to \Omega\Sigma A$ is $2k$-connected.

Corollary

$$\pi_2(\mathbb{S}^2) \simeq \pi_3(\mathbb{S}^3) \simeq \cdots \simeq \pi_n(\mathbb{S}^n) \simeq \ldots$$

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Second approximation

Second approximation of $\Omega\Sigma A$

If A is k-connected, then the map $(A \times A) \sqcup^{A \vee A} A \to \Omega\Sigma A$ is $(3k+1)$-connected.

Corollary

For $A = \mathbb{S}^2$, the map $(\mathbb{S}^2 \times \mathbb{S}^2) \sqcup^{\mathbb{S}^2 \vee \mathbb{S}^2} \mathbb{S}^2 \to \Omega\mathbb{S}^3$ is 4-connected. In particular,

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Proposition

There is a natural number n such that

$$\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/n\mathbb{Z}.$$

Proof: There is a map $W_{2,2} : \mathbb{S}^3 \to \mathbb{S}^2 \vee \mathbb{S}^2$, such that

$$(\mathbb{S}^2 \times \mathbb{S}^2) \simeq (\mathbf{1} \sqcup^{\mathbb{S}^3} (\mathbb{S}^2 \vee \mathbb{S}^2)).$$

We then get

$$(\mathbb{S}^2 \times \mathbb{S}^2) \sqcup^{\mathbb{S}^2 \vee \mathbb{S}^2} \mathbb{S}^2 \simeq \mathbf{1} \sqcup^{\mathbb{S}^3} \mathbb{S}^2,$$

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- The proof is constructive (no use of the axiom of choice or excluded middle)
- There is an algorithm computing the actual value of n (implemented in `cubicaltt` by Coquand et al.)
- We haven't managed to run this algorithm yet (!)
- I will give instead a mathematical proof that $n = 2$

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Contents

Eilenberg–MacLane spaces

Definition

We define the Eilenberg–MacLane spaces by

$$K_0 := \mathbb{Z},$$
$$K_n := \|\mathbb{S}^n\|_n \quad \text{for } n \geq 1.$$

Proposition

For every $n : \mathbb{N}$ we have

$$K_n \simeq \Omega K_{n+1}.$$

Proof: Uses the universal cover of the circle for $n = 0$, the Hopf fibration for $n = 1$, and the Freudenthal suspension theorem for $n \geq 2$. □

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The n^{th} cohomology group of X is

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The cohomology groups of $\mathbb{S}^n$ are

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Given two pointed types A and B, the smash product of A and B is the pushout of

$$A \times B \longleftarrow A \vee B \longrightarrow \mathbf{1}$$

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$$\wedge_{n,m} : \mathbb{S}^n \wedge \mathbb{S}^m \xrightarrow{\sim} \mathbb{S}^{n+m}.$$

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We define

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\mathbb{S}^i \wedge \mathbb{S}^j & \xrightarrow{\ \wedge_{i,j}\ } & \mathbb{S}^{i+j} \\
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K_i \wedge K_j & \underset{\smile}{\dashrightarrow} & K_{i+j}
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The cup product is distributive, associative, and graded-commutative.

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Products of spheres

Proposition

The cohomology of $\mathbb{S}^n \times \mathbb{S}^m$ is generated by

$$\begin{aligned} 1 &: H^0(\mathbb{S}^n \times \mathbb{S}^m), \\ \mathbf{x} &: H^n(\mathbb{S}^n \times \mathbb{S}^m), \\ \mathbf{y} &: H^m(\mathbb{S}^n \times \mathbb{S}^m), \\ \mathbf{z} &: H^{n+m}(\mathbb{S}^n \times \mathbb{S}^m) \end{aligned}$$

and we have

$$\begin{aligned} \mathbf{x} \smile \mathbf{y} &= \mathbf{z}, \\ \mathbf{x} \smile \mathbf{x} &= 0, \\ \mathbf{y} \smile \mathbf{y} &= 0. \end{aligned}$$

The Hopf invariant

Definition

Given $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$, we define

$$C_f := \mathbf{1} \sqcup^{\mathbb{S}^{2n-1}} \mathbb{S}^n,$$
$$\alpha_f : H^n(C_f) \quad \text{(generator)},$$
$$\beta_f : H^{2n}(C_f) \quad \text{(generator)}.$$

The Hopf invariant of f is $H(f) : \mathbb{Z}$ such that

$$\alpha_f \smile \alpha_f = H(f)\beta_f.$$

Proposition

The Hopf invariant $H : \pi_{2n-1}(\mathbb{S}^n) \to \mathbb{Z}$ is a group homomorphism.

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Hopf invariant of $\nabla \circ W_{n,n} : \mathbb{S}^{2n-1} \to \mathbb{S}^n$

Proposition

If n is even, we have $H(\nabla \circ W_{n,n}) = 2$.

Proof: There is a map $q : \mathbb{S}^n \times \mathbb{S}^n \to C_{\nabla \circ W_{n,n}}$ such that

$$q^*(\alpha) = \mathbf{x} + \mathbf{y},$$
$$q^*(\beta) = \mathbf{z}.$$

We have

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Consequences for the homotopy groups of spheres

Proposition

For every n, the group $\pi_{4n-1}(\mathbb{S}^{2n})$ is infinite.

Proof: All the multiples of $|\nabla \circ W_{2n,2n}|$ are different elements of $\pi_{4n-1}(\mathbb{S}^{2n})$. □

Proposition

The group $\pi_4(\mathbb{S}^3)$ is equivalent to

- $\mathbb{Z}/2\mathbb{Z}$ if there exists a map $f : \mathbb{S}^3 \to \mathbb{S}^2$ such that $H(f) = \pm 1$,
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Proposition

The map

$$g_i : K_i \to (\mathbb{S}^n \to_\star K_{i+n}),$$
$$g_i(x) := (y \mapsto (x \smile \mathbf{c}_n(y)))$$

is an equivalence.

Proof: By induction on i. It's immediate for $i = 0$, and for $i + 1$ the map g_{i+1} is an equivalence iff Ωg_{i+1} is, and $\Omega g_{i+1} \approx g_i$. □

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Given a 1-connected type B and a fibration

$$\mathbb{S}^{n-1} \hookrightarrow E \xrightarrow{p} B,$$

there is an element $e : H^n(B)$ and a long exact sequence

$$\ldots \longrightarrow H^{i-1}(E) \longrightarrow H^{i-n}(B) \xrightarrow{\smile e} H^i(B) \longrightarrow H^i(E) \longrightarrow \ldots$$

Proof: It follows from the previous proposition applied pointwise that $H^{i-n}(B)$ is isomorphic to $\tilde{H}^i(\tilde{E})$ via the cup product, where

$$\tilde{E} := \mathbf{1} \sqcup^E B.$$

The long exact sequence of the cofiber of p gives the result. □

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Definition

We define

$$\mathbb{C}P^2 := \mathbf{1} \sqcup^{\mathbb{S}^3} \mathbb{S}^2$$

for the map $\eta : \mathbb{S}^3 \to \mathbb{S}^2$ coming from the Hopf fibration.

Proposition

There is a fibration $P : \mathbb{C}P^2 \to \mathrm{Type}$ with fiber $\mathbb{S}^1$ and total space $\mathbb{S}^5$.

Proof: The construction is similar to the construction of the Hopf fibration and uses the fact that the multiplication on $\mathbb{S}^1$ is associative. □

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Cohomology of $\mathbb{C}P^2$

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The type $\mathbb{C}P^2$ satisfies $\alpha \smile \alpha = \pm\beta$. In particular, $H(\eta) = \pm 1$.

Proof: We have two short exact sequences:

$$0 \simeq H^1(\mathbb{S}^5) \longrightarrow H^0(\mathbb{C}P^2) \xrightarrow{\smile e} H^2(\mathbb{C}P^2) \longrightarrow H^2(\mathbb{S}^5) \simeq 0,$$

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Therefore e is a generator of $H^2(\mathbb{C}P^2)$ and $e \smile e$ is a generator of $H^4(\mathbb{C}P^2)$. □

Corollary

We have $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2\mathbb{Z}$.

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Table

	$\mathbb{S}^1$	$\mathbb{S}^2$	$\mathbb{S}^3$	$\mathbb{S}^4$	$\mathbb{S}^5$	$\mathbb{S}^6$	$\mathbb{S}^7$
π_1	$\mathbb{Z}$	0	0	0	0	0	0
π_2	0	$\mathbb{Z}$	0	0	0	0	0
π_3	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0
π_4	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
π_5	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
π_6	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
π_7	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
π_8	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
π_9	0	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$

Conclusion

Main result

We obtained a constructive and purely homotopy-theoretic proof in homotopy type theory of

$$\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2\mathbb{Z},$$

and $\pi_{n+1}(\mathbb{S}^n) \simeq \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$.

This proof required new homotopy-theoretic formulations of

- the Hopf fibration
- the James construction
- the cup product
- the Gysin sequence