Invariant Homotopy Theory in the Univalent Foundations

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Homotopy theory

Homotopy theory is the study of homotopy types (“spaces up to homotopy”), i.e.,

- topological spaces up to weak homotopy equivalences, or
- CW-complexes up to homotopy equivalences, or
- simplicial sets up to weak equivalences.

There is a formal system introduced by Vladimir Voevodsky, the univalent foundations (UF), whose basic objects behave just like homotopy types:

- There is a model of UF in simplicial sets
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Invariant homotopy theory

Invariant homotopy theory (or synthetic homotopy theory) is the study of the basic objects of UF, with intuition coming from homotopy theory

- It is *not* the study of topological spaces, simplicial sets, etc
- It is the "direct" study of homotopy types
- In particular, everything is homotopy-invariant.

What we do not have (non-exhaustive list)

- No notion of subspace
  e.g. no complement of a point
- No notion of a map $f : E \to B$ being a fibration
  e.g. $\exp : \mathbb{R} \to \mathbb{S}^1$ homotopic to a constant map
- Quotients often do not work
  e.g. projective spaces
- Matrix groups are tricky
  e.g. $\mathrm{SO}(n)$, grassmanians
- Equality is tricky
- (optional but usually assumed)
  Intuitionistic logic, no excluded middle, no axiom of choice

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What we do have (almost exhaustive list)

- Function spaces
- Path spaces
- Homotopy limits and homotopy colimits
  give many nice cell complexes, e.g. $\mathbb{S}^n$, $\mathbb{R}\mathrm{P}^n$ (Buchholtz, Rijke)
- Truncations
  give for instance π_k, $K(G, n)$
- Universes ("the (big) space of all (small) spaces")

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Fibrations and the univalence axiom

Intuition

A fibration is a family of spaces parametrized by another space.

A fibration over B is a map $P : B \to \mathcal{U}$, where $\mathcal{U}$ is a universe, and its fibers are the $P(b)$.

If B is defined as a cell complex/homotopy colimit, we define such a map by giving the images of all of the cells. In particular we need:

Univalence axiom (Voevodsky)

A path in the universe is the same thing as a homotopy equivalence between its endpoints.

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The universal cover of the circle

Definition

The circle $\mathbb{S}^1$ is generated by

$$\text{base} : \mathbb{S}^1,$$
$$\text{loop} : \text{Path}_{\mathbb{S}^1}(\text{base}, \text{base}).$$

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$$P : \mathbb{S}^1 \to \mathcal{U},$$
$$P(\text{base}) := \mathbb{Z},$$
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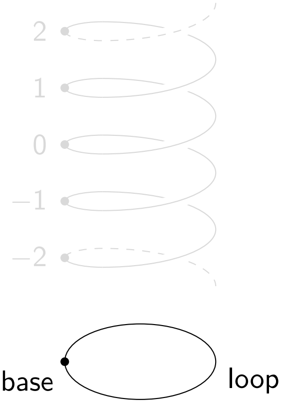

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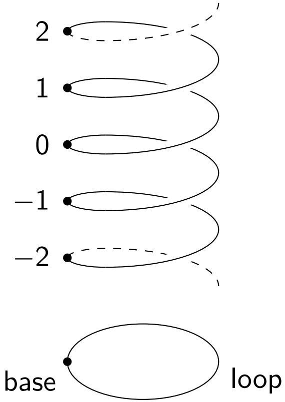

Some results

$\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2\mathbb{Z}$

→ universal cover of $\mathbb{S}^1$ (Shulman)

→ Hopf fibration

→ James construction

→ Blakers–Massey theorem (Favonia, Finster, Licata, Lumsdaine)

→ cohomology

→ Hopf invariant

→ Thom isomorphism

covering spaces (Favonia)

Seifert–van Kampen theorem (Favonia, Shulman)

quaternionic Hopf fibration (Buchholtz)

Constructivity

Martin–Löf type theory is constructive: any proof of $\exists n : \mathbb{N}, P(n)$ gives an algorithm computing such an n.

Univalent foundations is still constructive, although this is work in progress and much less understood (Coquand et al.)

The proof of $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2\mathbb{Z}$ consists of

- a proof that $\exists n : \mathbb{N}, \pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/n\mathbb{Z}$, and
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Future directions

- Formalize the proof of $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2\mathbb{Z}$ in a proof assistant
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- Do more invariant homotopy theory, e.g., Bott periodicity, K-theory, spectral sequences, etc.

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Thank you for your attention