Program Testing with Polymorphic Types

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Polymorphism

Same program for different kinds of data

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map : (a -> b) -> list(a) -> list(b)
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Same program for different kinds of data

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map : (a -> b) -> list(a) -> list(b)
```

```
map (*2) [1,2,3] = [2,4,6]
```

Polymorphism

Same program for different kinds of data

```
map : (a -> b) -> list(a) -> list(b)
```

```
map (not) [true,false] = [false,true]
```

Polymorphism

Same program for different kinds of data

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map : (a -> b) -> list(a) -> list(b)
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Question: how can we test this function?

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proj : (a * a) -> a
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$$\mathtt{proj}_1\ (x, y) = x$$

$$\mathtt{proj}_2\ (x, y) = y$$

```
proj : (a * a) -> a
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There are only two possibilities!

$$\mathtt{proj}_1\ (\mathtt{x},\mathtt{y}) = \mathtt{x}$$

$$\mathtt{proj}_2\ (\mathtt{x},\mathtt{y}) = \mathtt{y}$$

```
proj (true,false) = ?
```

```
proj : (a * a * a) -> a
```

There are only three possibilities!

```
proj₁ (x,y) = x
proj₂ (x,y) = y
proj (true,false) = ?
```

```
proj : (a * a * a) -> a
```

There are only three possibilities!

$$\mathrm{proj}_1\ (x,y,z) = x$$
$$\mathrm{proj}_2\ (x,y,z) = y$$
$$\mathrm{proj}_3\ (x,y,z) = z$$

```
proj : (a * a * a) -> a
```

There are only three possibilities!

```
proj₁ (x,y,z) = x
proj₂ (x,y,z) = y
proj₃ (x,y,z) = z
```

```
proj (0,1,2) = ?
```

```
proj : (a * a * ... * a) -> a
```

There are exactly n possibilities!

```
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```

There are exactly n possibilities!

```
proj (0,1,2,...,n-1) = ?
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```
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There are exactly n possibilities!

```
proj (0,1,2,...,n-1) = ?
```

Any type with n distinct values works

```
proj : (a * a * ... * a) -> a
```

There are exactly n possibilities!

$$\log_a(a * a * \ldots * a) = \log_a(a^n) = n$$

```
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```

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counting elements of type a in the input

```
f : α(a) -> H(a)
```

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one of the best types for testing*

$$\log_a \alpha(a)$$

* see our paper for caveats

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one of the best types for testing*

$$\log_a \alpha(a)$$

testing is still complete**

* see our paper for caveats

** the empty type might need to be separately tested

2018 I joined the U

Mentioned this project in my job talk

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2019 Zhuyang became my student

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2021 Paper published at POPL 2022
"Logarithm and Program Testing"

2010 *"Testing Polymorphic Properties"*

by Bernardy, Jansson, and Claessen

```
f : (F(a) -> a) * (G(a) -> K) -> H(a)
```

(other cases manually massaged into this form)

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2017 **Liyao Xia wrote a Haskell library roughly based on the above paper**

(with a logarithm-like operator in its codebase)

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2017 Liyao Xia wrote a Haskell library roughly based on the above paper

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Logarithm-like operations have been discovered repeatedly in the literature but no one connected logarithm to testing

e.g., [Abbott et al. 2003; Altenkirch et al. 2015]

```
f : (a -> a) * a -> a
```

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Suppose the input is the pair (s, x)
The output must be s(s(...s(x)...))

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Suppose the input is the pair (s, x)
The output must be s(s(...s(x)...))

```
f ((+1), 0) = ?
```

this reveals the number of s
in its output s(s(...s(x)...))

```
f : (a -> a) * a -> a
```

```
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```

$$
\begin{aligned}
&\log_a(a^a * a) \\
&= \log_a(a^a) + \log_a(a) \\
&= a * \log_a(a) + 1 \\
&= a * 1 + 1 \\
&\simeq a + 1
\end{aligned}
$$

```
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\begin{aligned}
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Problem: a appears in the logarithm of $a^a * a$ indicating recursion

```
f : (a -> a) * a -> a
```

$$
\begin{aligned}
& \log_a(a^a * a) \\
= {} & \log_a(a^a) + \log_a(a) \\
= {} & a * \log_a(a) + 1 \\
= {} & a * 1 + 1 \\
\simeq {} & a + 1
\end{aligned}
$$

Problem: a appears in the logarithm of $a^a * a$ indicating recursion

Solution: recursive types! $\mu a.a+1$ is the naturals $\mathbb{N}$

```
f : (a -> a) * a -> a
```

$$\begin{aligned} & \log_a(a^a * a) \\ = {} & \log_a(a^a) + \log_a(a) \\ = {} & a * \log_a(a) + 1 \\ = {} & a * 1 + 1 \\ \simeq {} & a + 1 \end{aligned}$$

Problem: a appears in the logarithm of $a^a * a$ indicating recursion

Solution: recursive types!
$\mu a.a+1$ is the naturals $\mathbb{N}$

Sufficient for `f ((+1), 0) = ?`

```
f : α(a) -> H(a)
```

one of the best types for testing*

$$\mu a.\log_a \alpha(a)$$

a sufficiently large type to describe all possible ways to generate an a-element

* see our paper for remaining caveats

Theorem*

For any two functions $f, g : \alpha(a) \to H(a)$, if they agree on all inputs when type a is instantiated with $\mu a.\log_a \alpha(a)$ and with the empty type**, then they are exactly the same function!

* see our paper for omitted conditions

** see our paper for why the empty type is needed

Logarithm for All

$$\log_a(a) = 1$$
$$\log_a(1) = 0$$
$$\log_a(\alpha_1(a) * \alpha_2(a)) = \log_a(\alpha_1(a)) + \log_a(\alpha_2(a))$$
$$\log_a(\alpha_2(a)^{\alpha_1(a)}) = \alpha_1(a) * \log_a(\alpha_2(a))$$

Logarithm for All

$\log_a(a) = 1$

$\log_a(1) = 0$

$\log_a(\alpha_1(a) * \alpha_2(a)) = \log_a(\alpha_1(a)) + \log_a(\alpha_2(a))$

$\log_a(\alpha_2(a)^{\alpha_1(a)}) = \alpha_1(a) * \log_a(\alpha_2(a))$

$\log_a(b) = \log_a(0) = 0$

$\log_a(\alpha_1(a) + \alpha_2(a)) = \log_a(\alpha_1(a)) + \log_a(\alpha_2(a))$

$\log_a(\mu b ...) = ...$

Use over-approximation to cover more types

```
proj : (a * a) -> a
```

Theorem: instantiate a with $\mu a.\log_a(a * a) = \mu a.2 \simeq 2$

```
proj : (a * a) -> a
```

Theorem: instantiate a with $\mu a.\log_a(a*a) = \mu a.2 \simeq 2$

```
proj (true,true) = ?
proj (true,false) = ?
proj (false,false) = ?
proj (false,true) = ?
```

```
proj : (a * a) -> a
```

Theorem: instantiate a with $\mu a.\log_a(a * a) = \mu a.2 \simeq 2$

```
~~proj (true,true) = ?~~
proj (true,false) = ?
~~proj (false,false) = ?~~
proj (false,true) = ?
```

~~proj (true,true) = ?~~

proj (true,false) = ?

~~proj (false,false) = ?~~

proj (false,true) = ?

Recall that logarithm is large enough to index all a-elements. The best tests are those containing only distinct a-elements.

```
map : (a -> b) -> list(a) -> list(b)
```

```
map : (a -> b) -> list(a) -> list(b)
```

one of the best types and inputs

$$a = b = \mathbb{N}$$

$$\texttt{map}\ \texttt{id}_{\mathbb{N}}\ [0,1,...,n-1] = ?$$

intuition: an evil programmer can only do these three things:
duplicating, omitting, or permuting elements in the output list
the above test case detects all possible deviations
map id [1,1,1,1] in comparison is much less useful

Theorem 2.0*

For any two functions f, g : $\alpha(a) \to H(a)$, if they agree on optimal inputs with distinct a-elements** when type a is instantiated with $\mu a.\log_a \alpha(a)$ and on all inputs with the empty type***, then they are exactly the same function!

* see our paper for omitted conditions
** see our paper for the precise definition
*** see our paper for why the empty type is needed

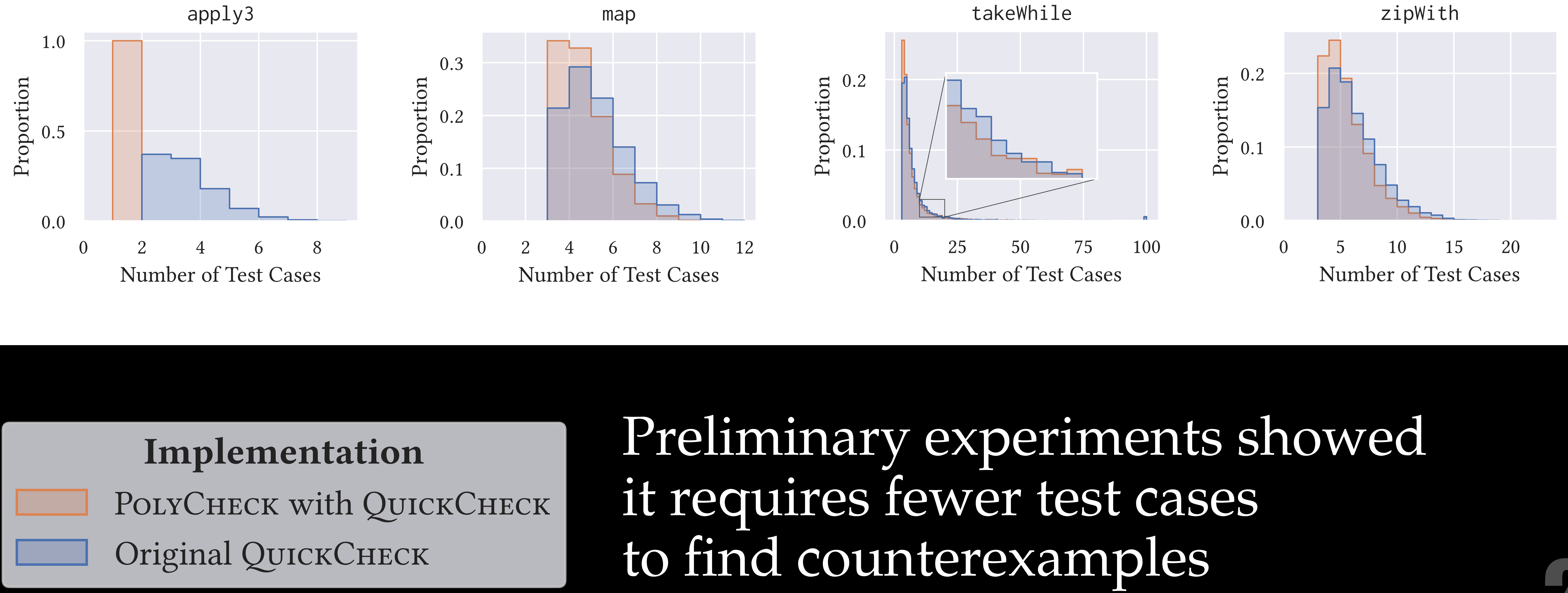
Implementation

Haskell library: `github.com/hawnzug/polycheck`

automatically specializing types and inputs

Can work with either QuickCheck or SmallCheck

Implementation



Preliminary experiments showed it requires fewer test cases to find counterexamples

Future Work

1. Incorporate information other than the typing

   e.g., `sort cmp list` expects `cmp` to form a total order

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1. Incorporate information other than the typing
   e.g., `sort cmp list` expects `cmp` to form a total order
2. Extend work to test an API, not just one function
3. Further optimize the theorem
   e.g., for `length : list(a) -> int`
   the best choice is the unit type, not natural numbers