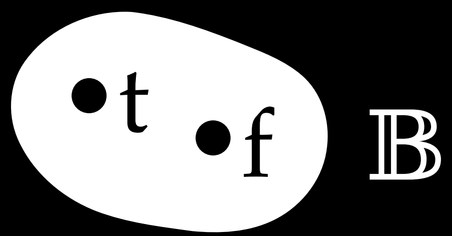
Cubical Type Theory

favonia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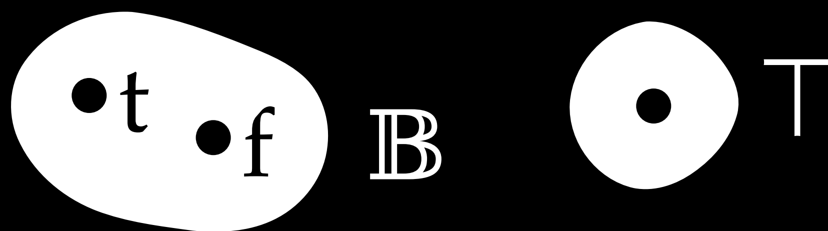
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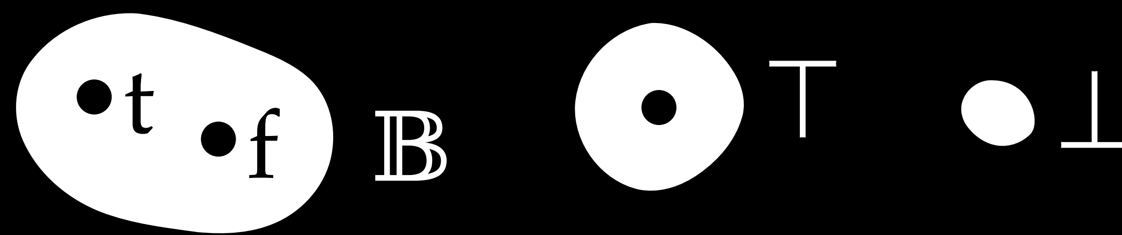
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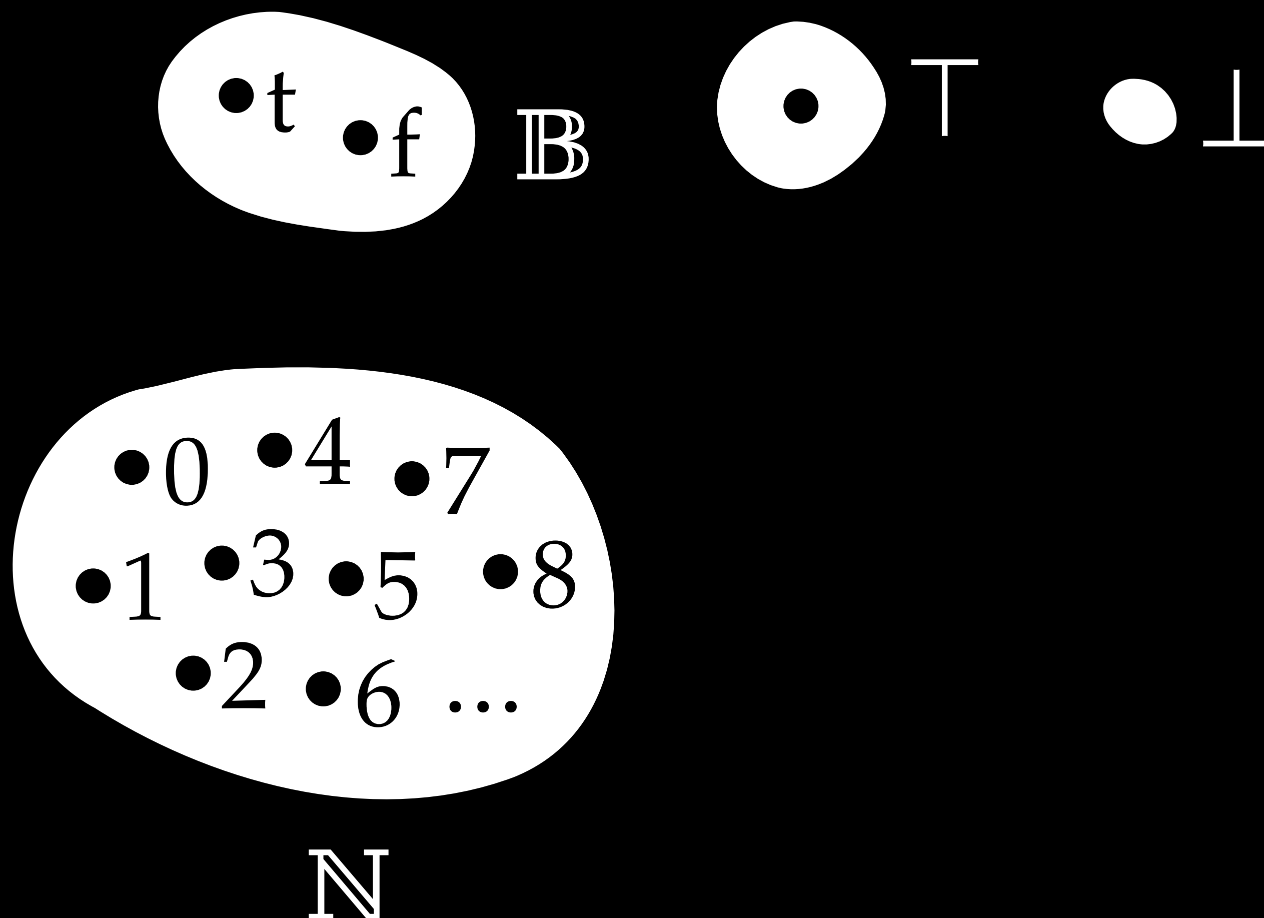
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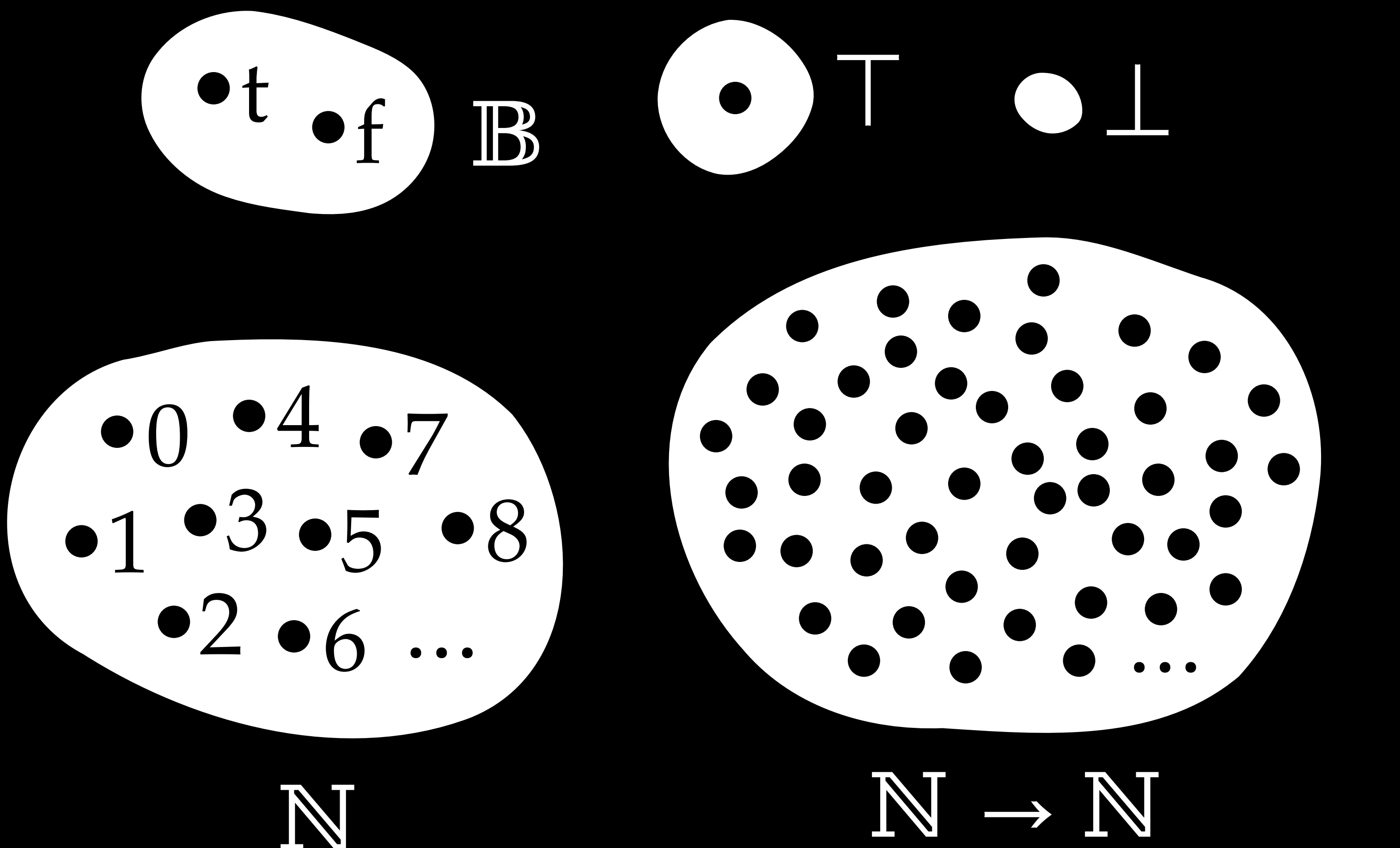
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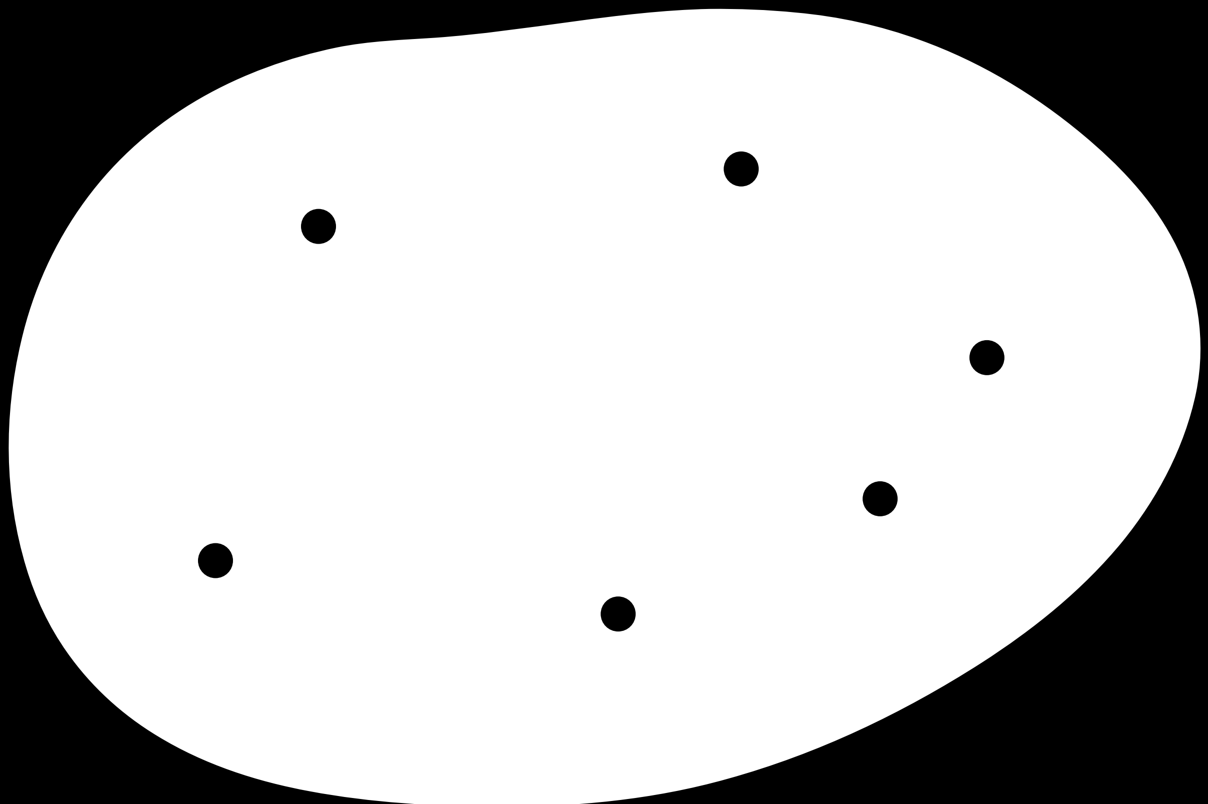

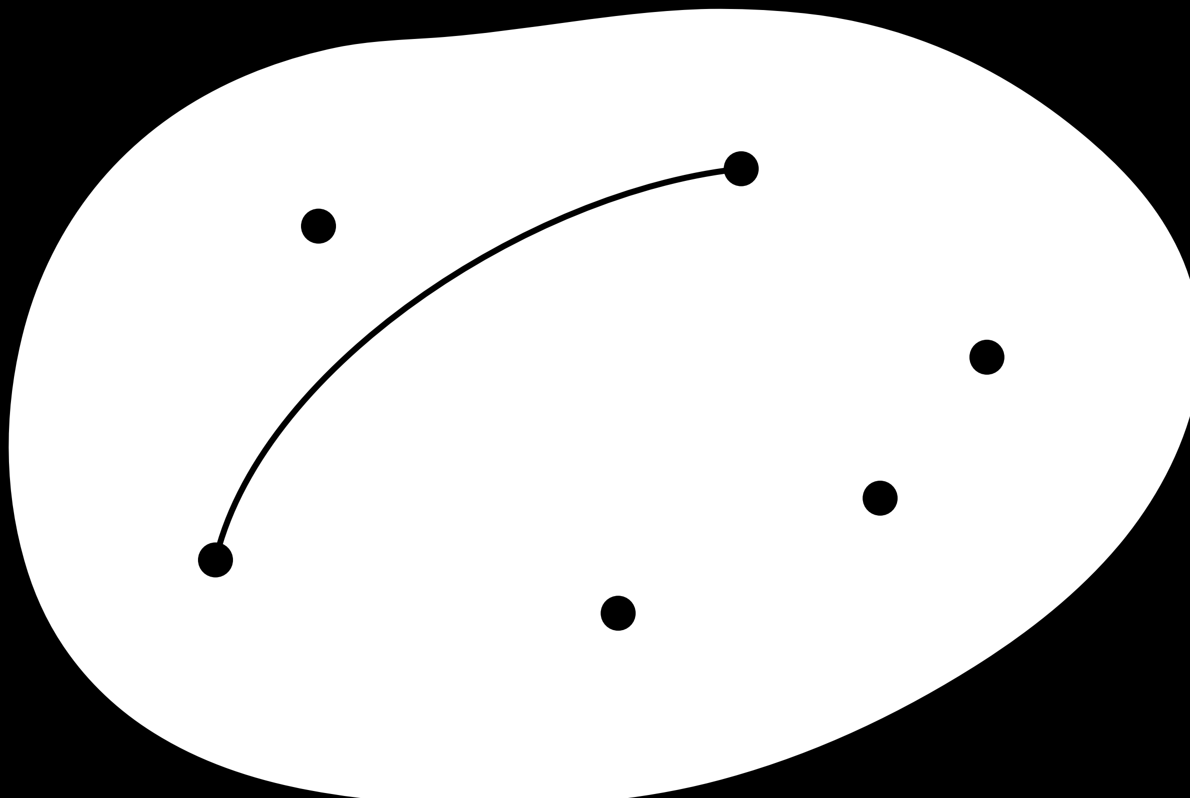

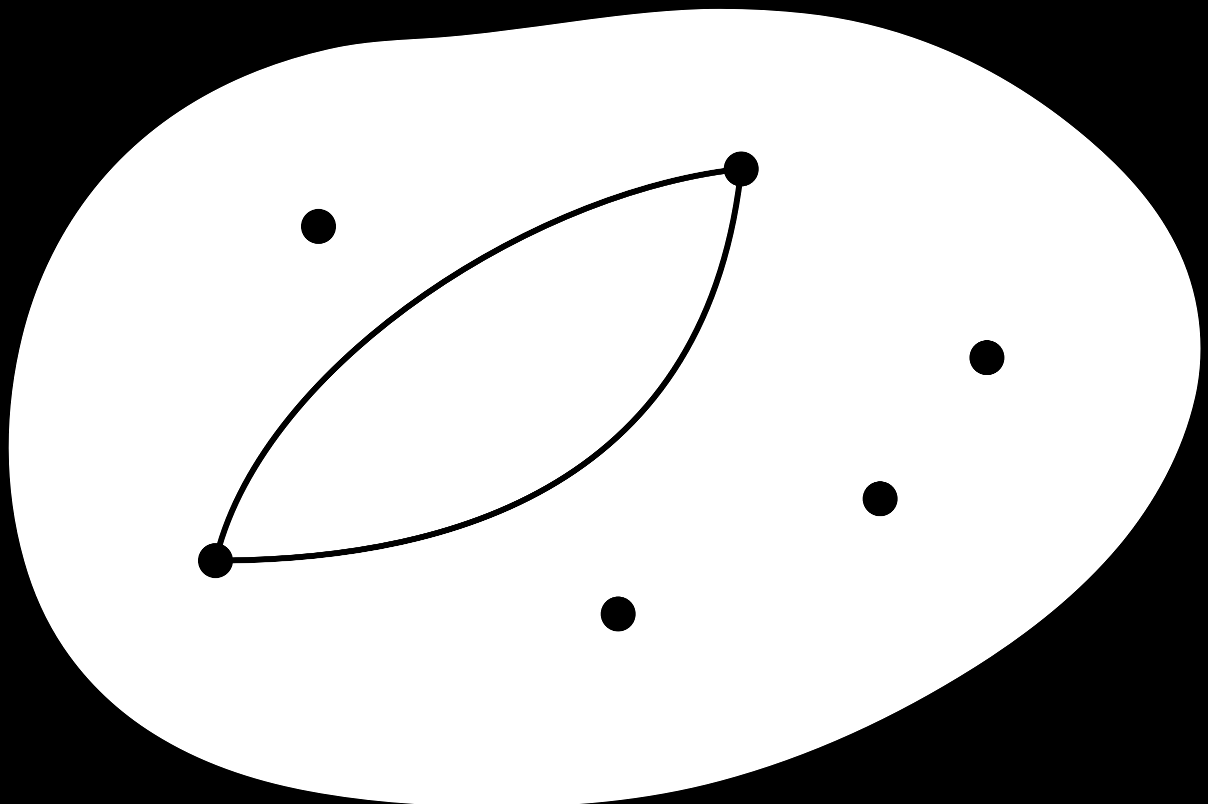

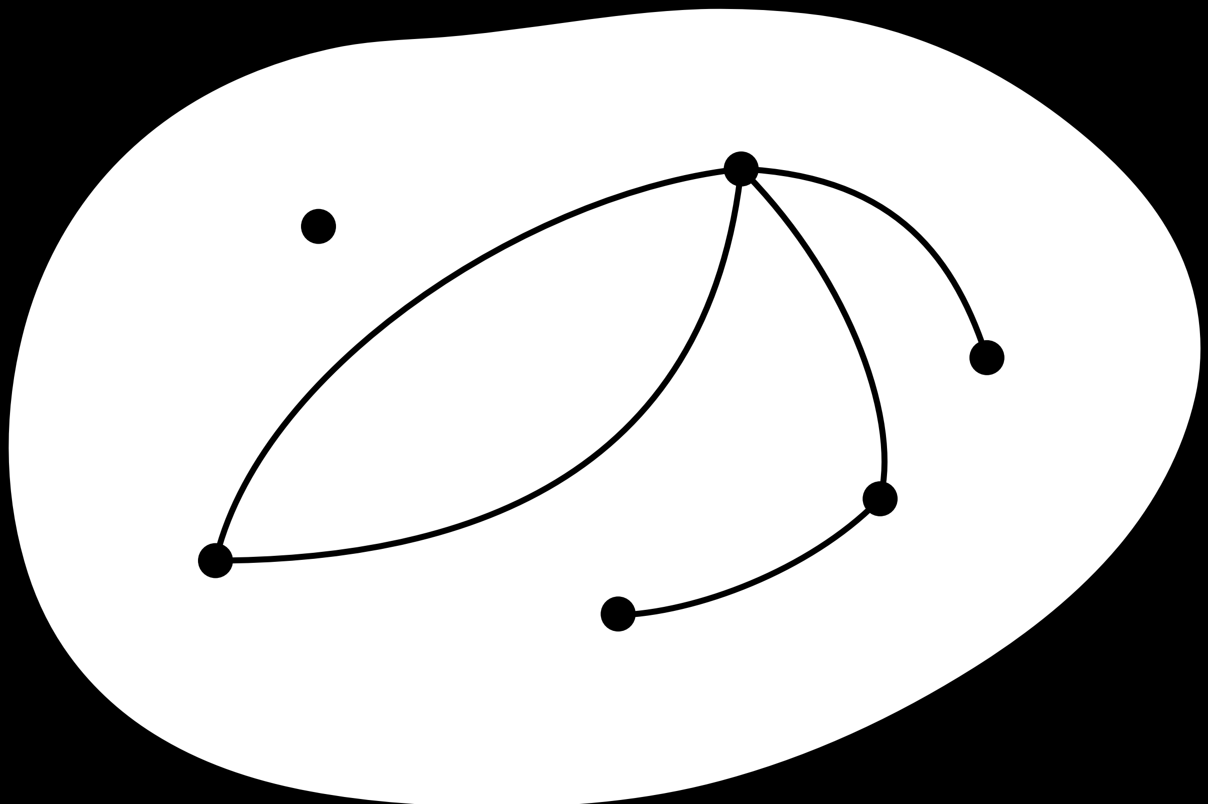

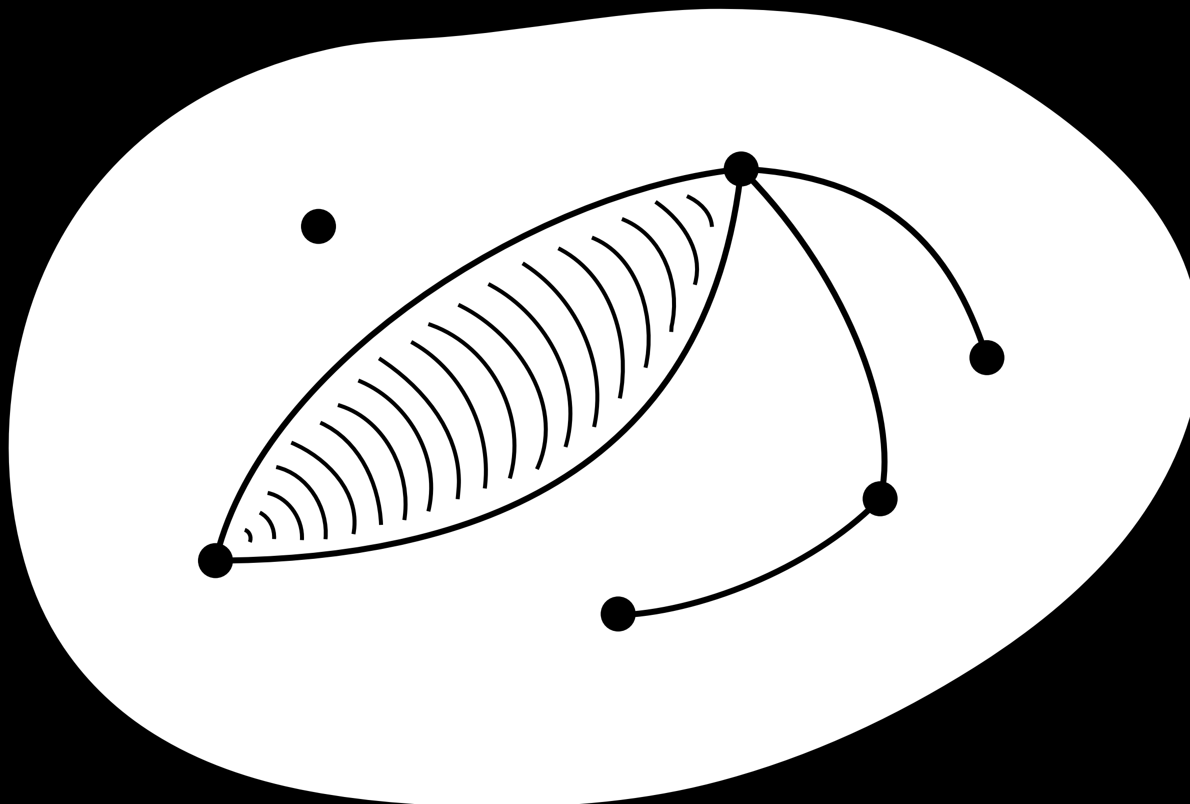

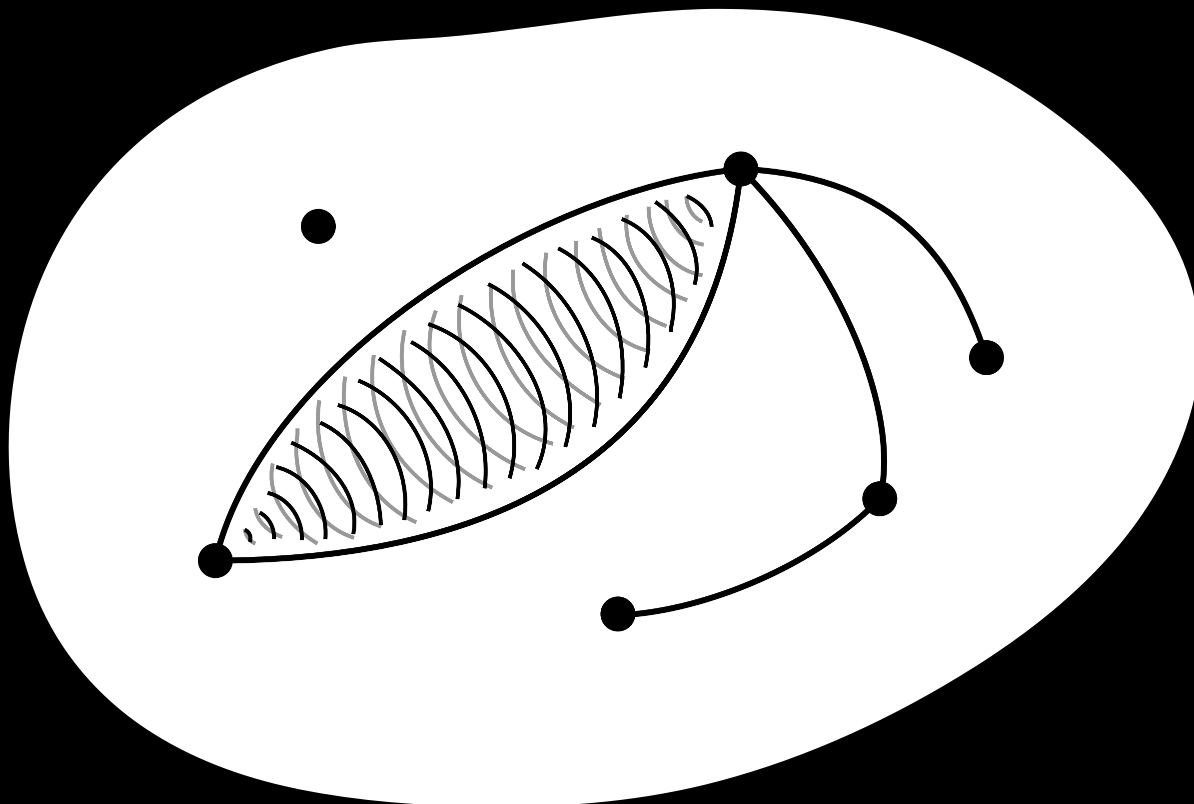

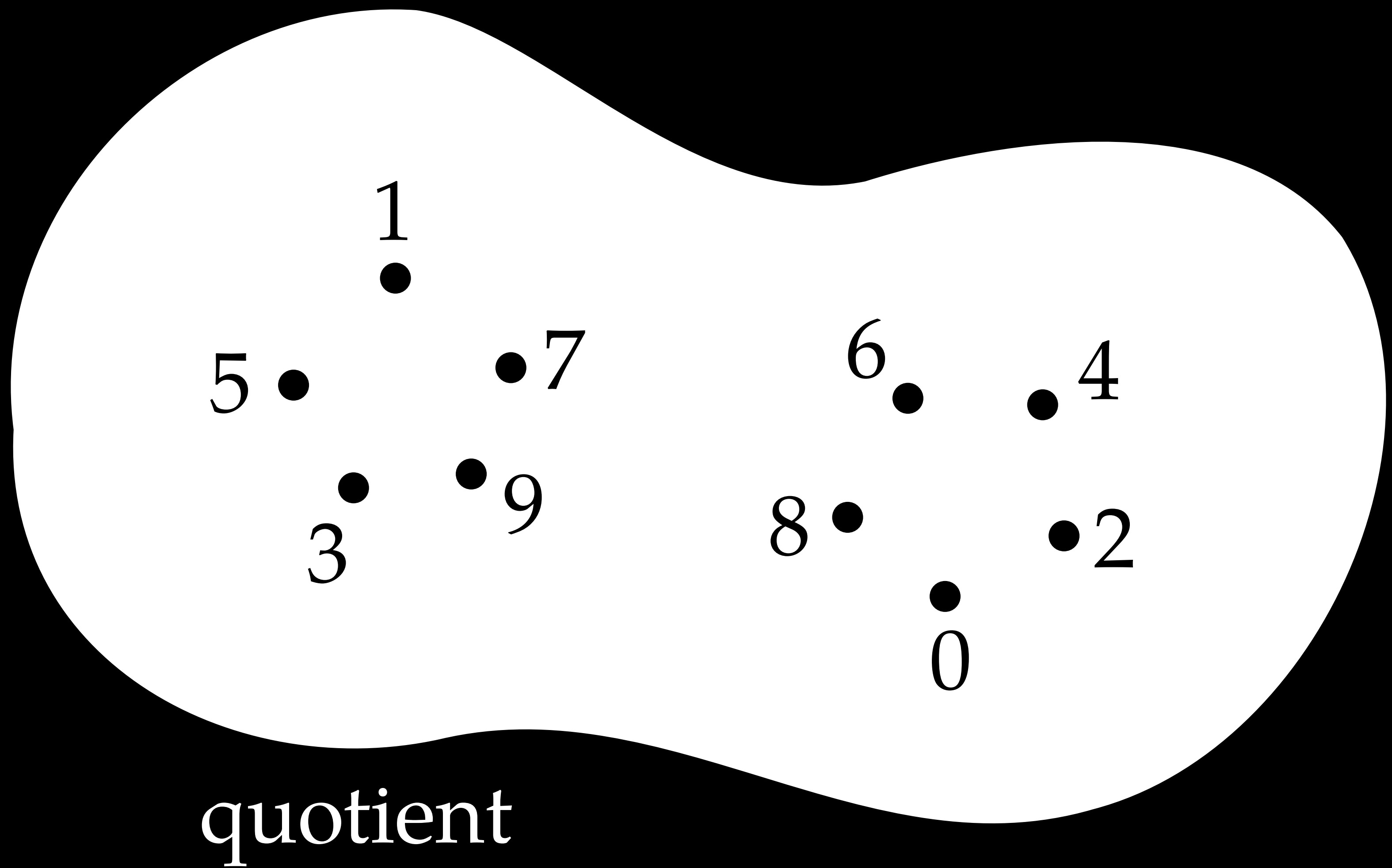
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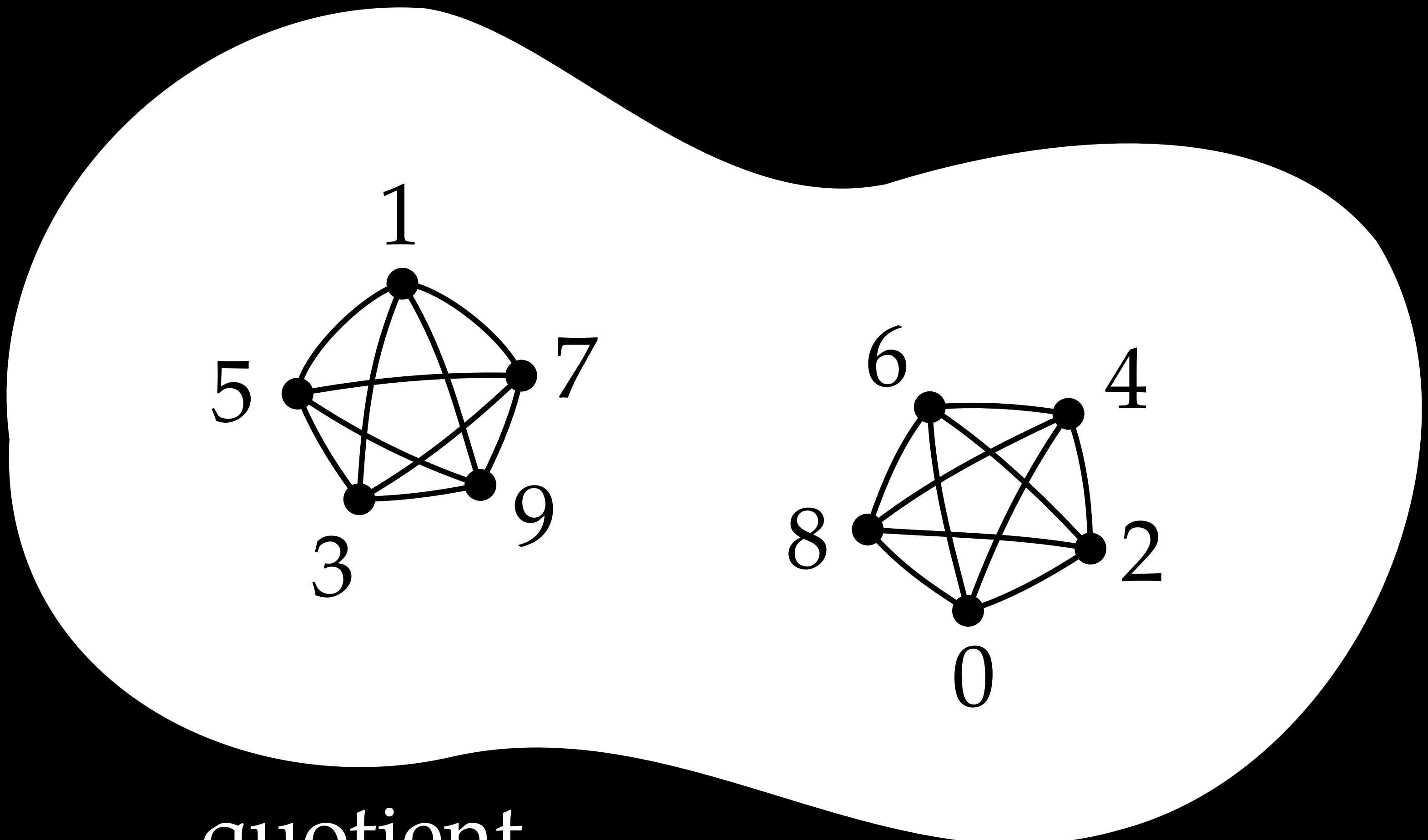
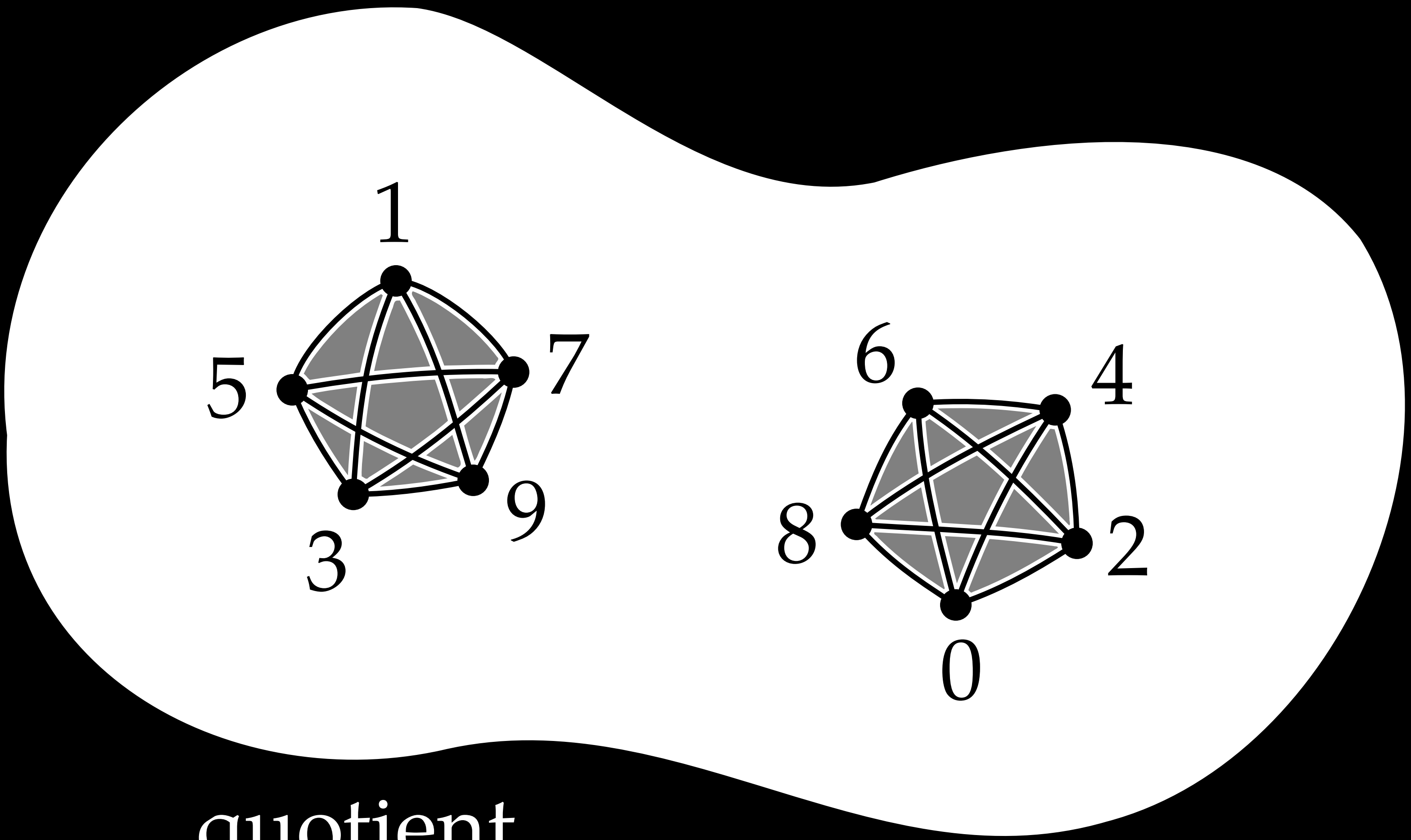
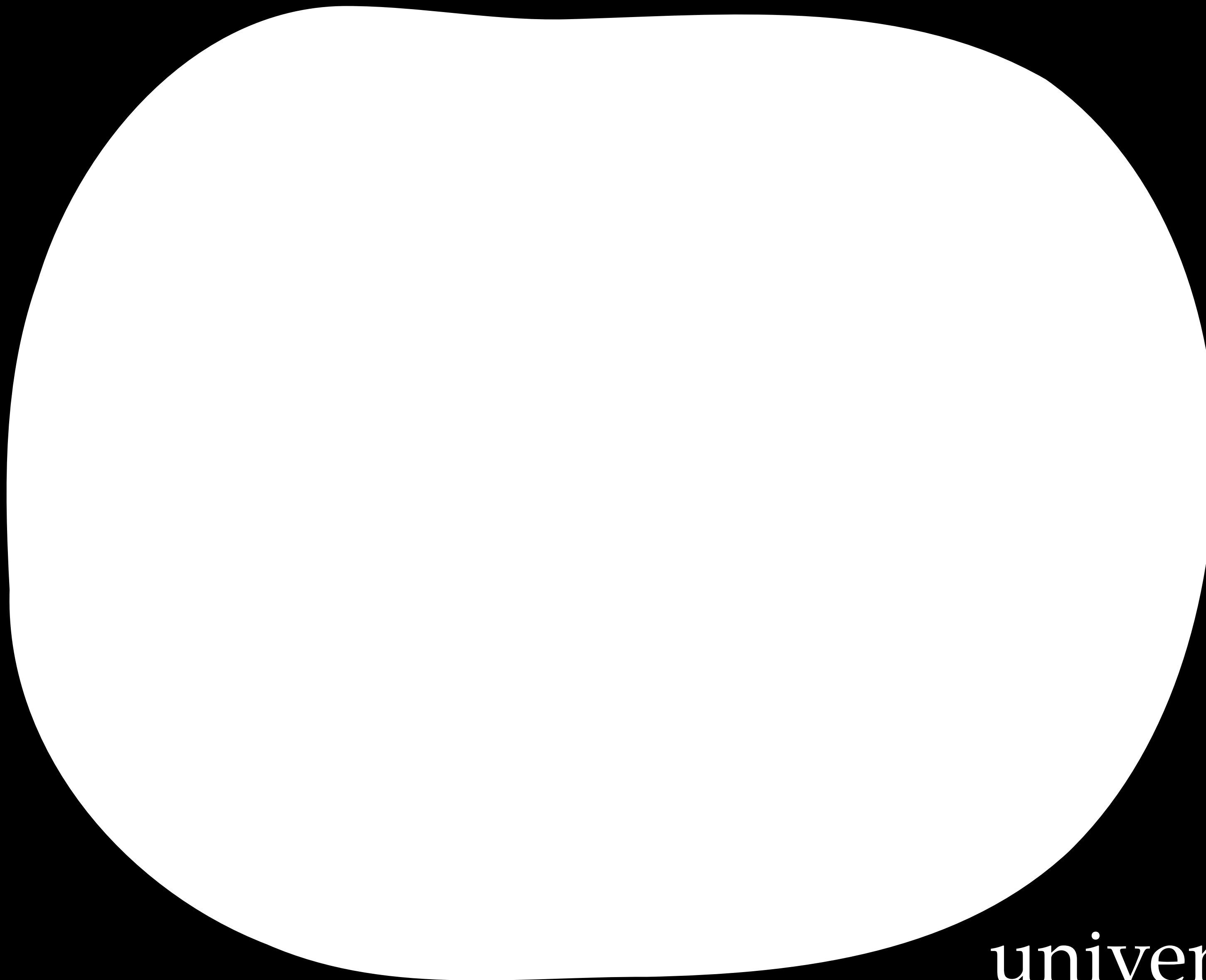


quotient



quotient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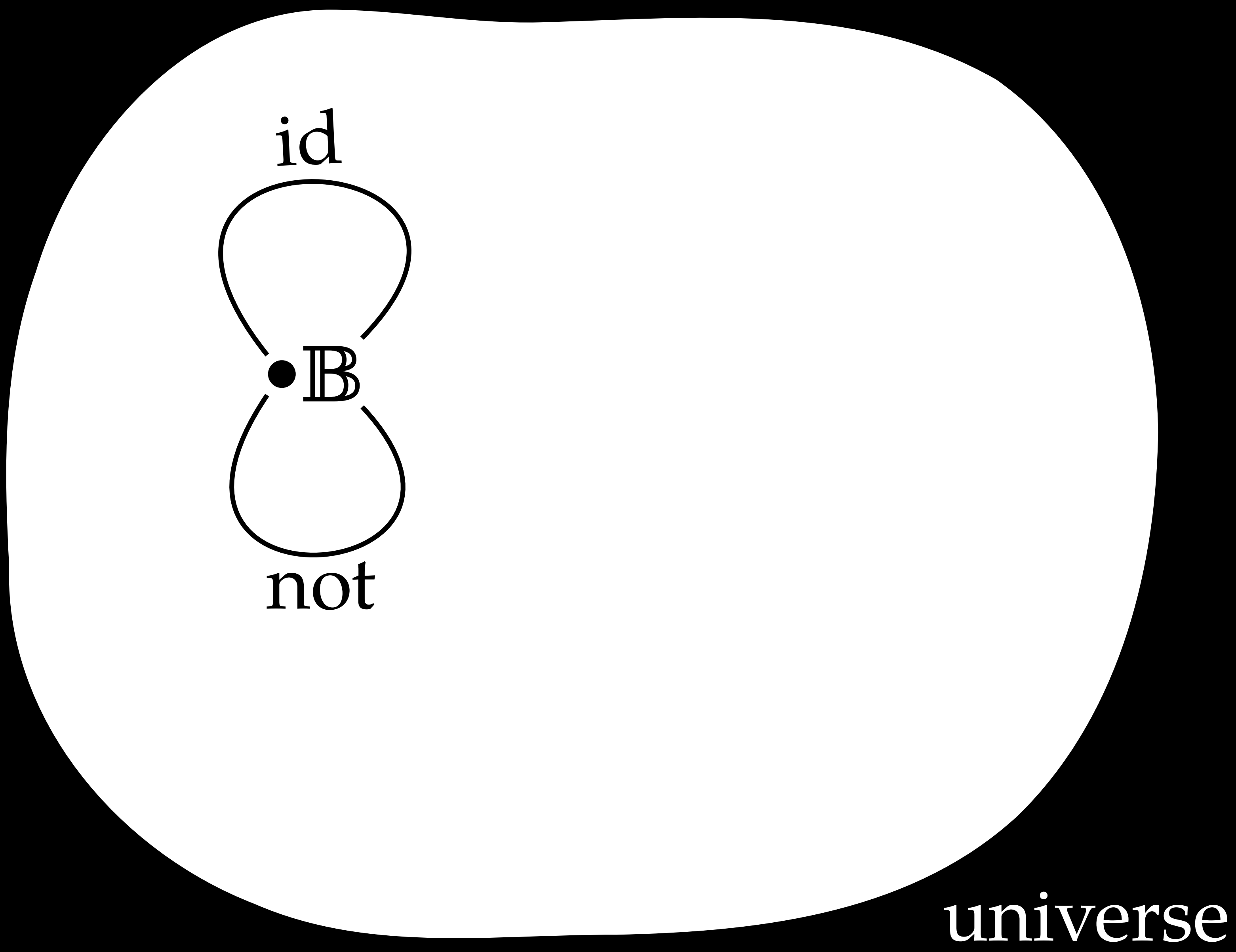
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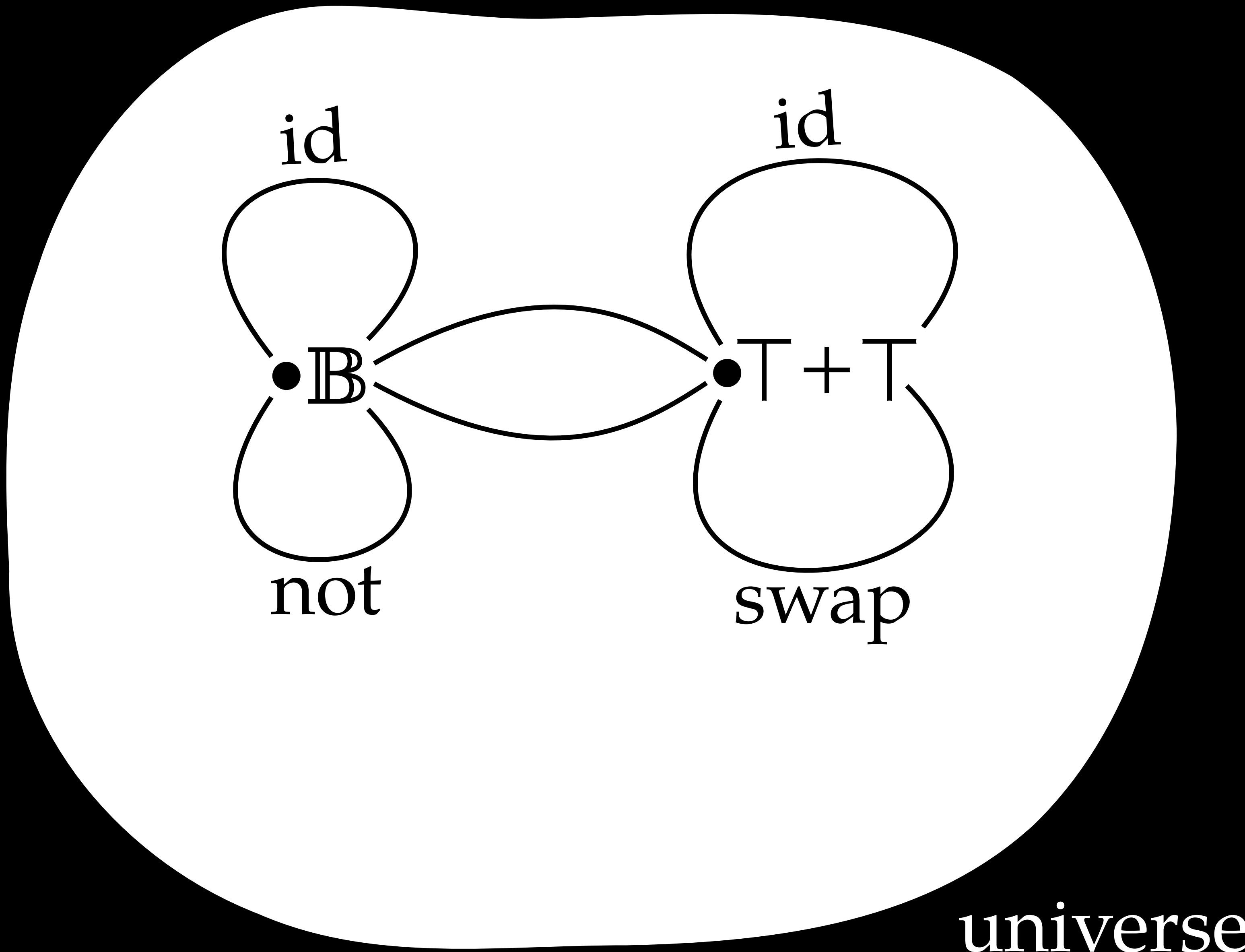
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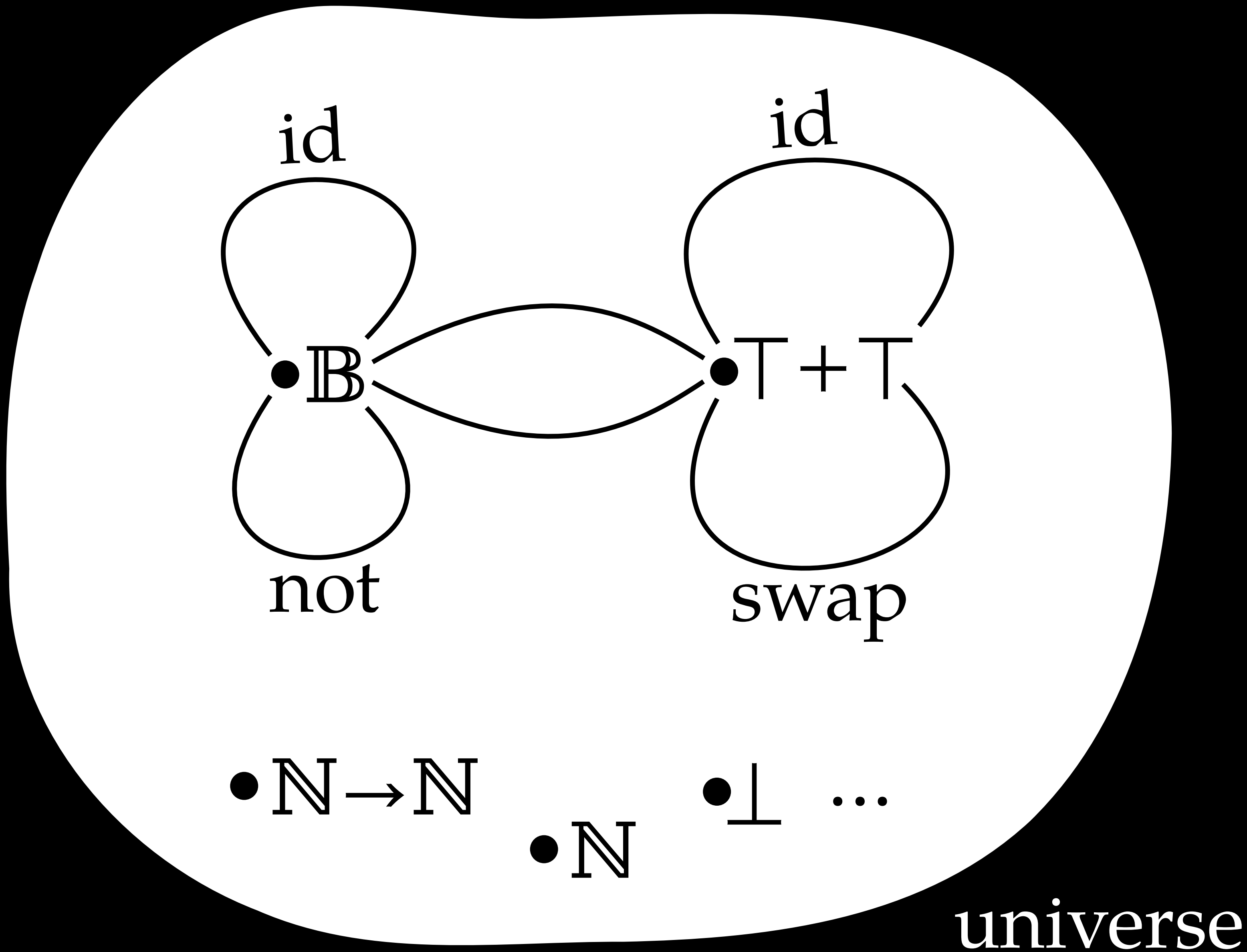
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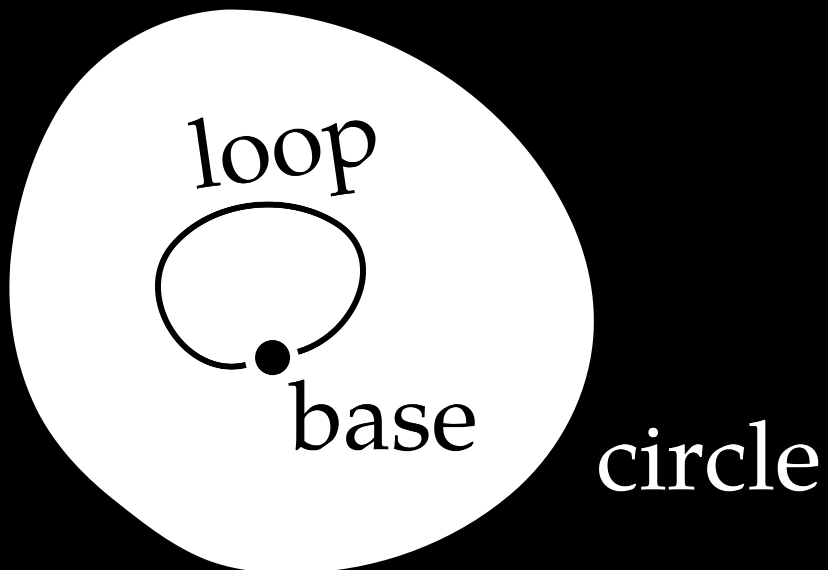
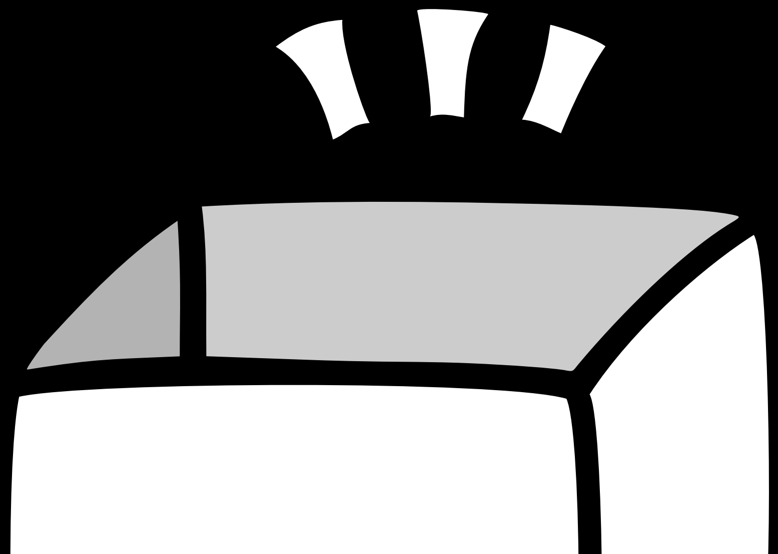
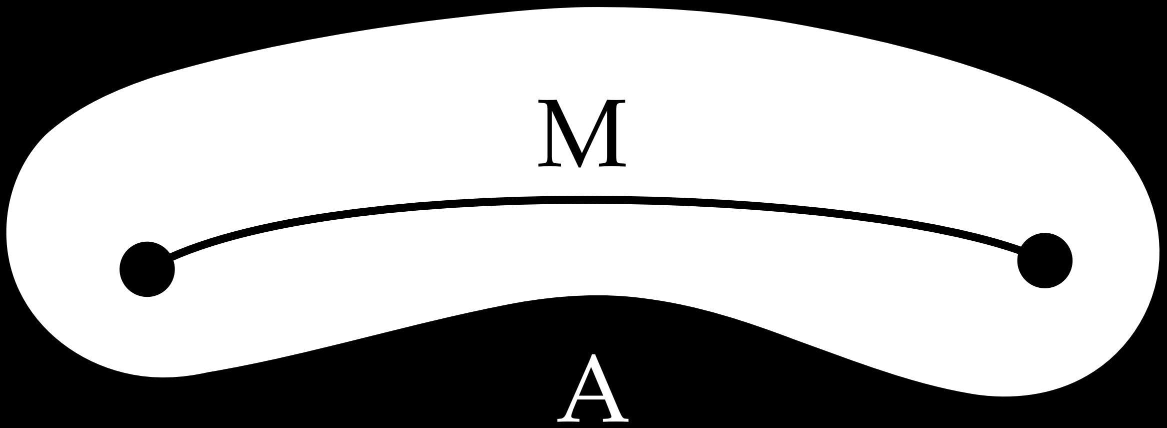
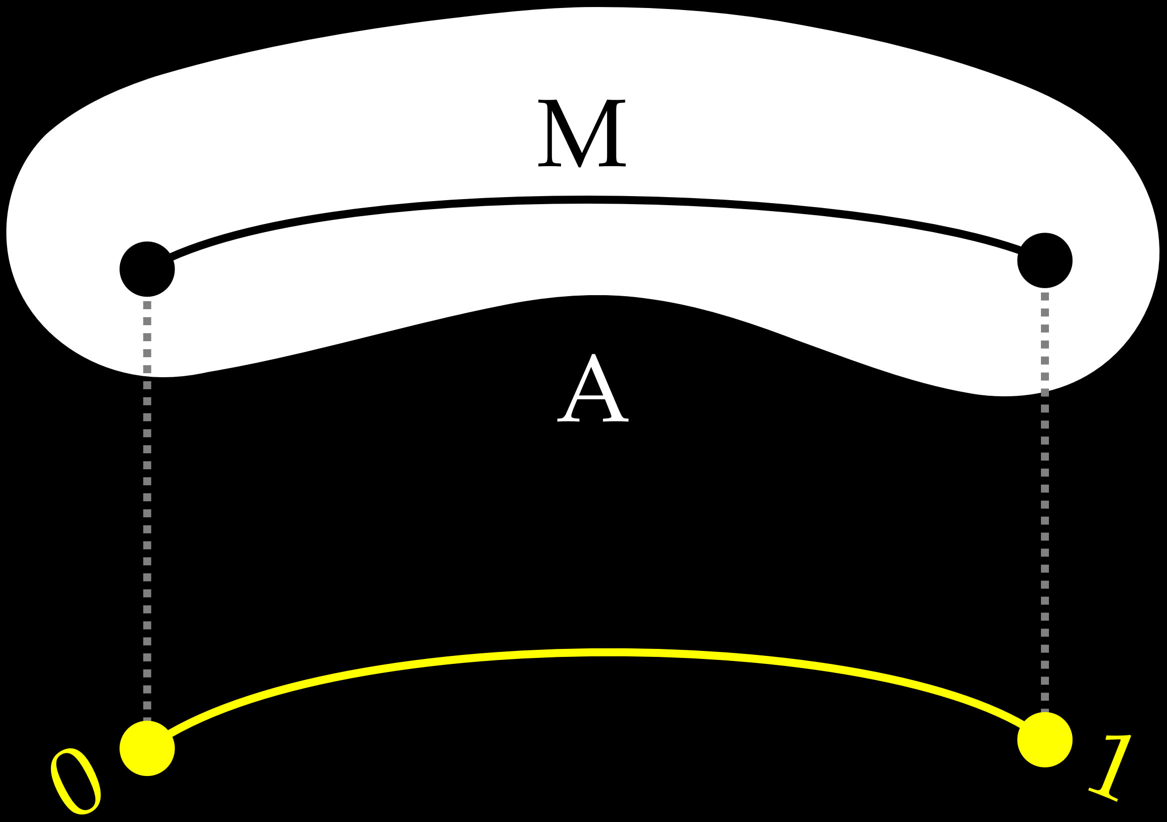


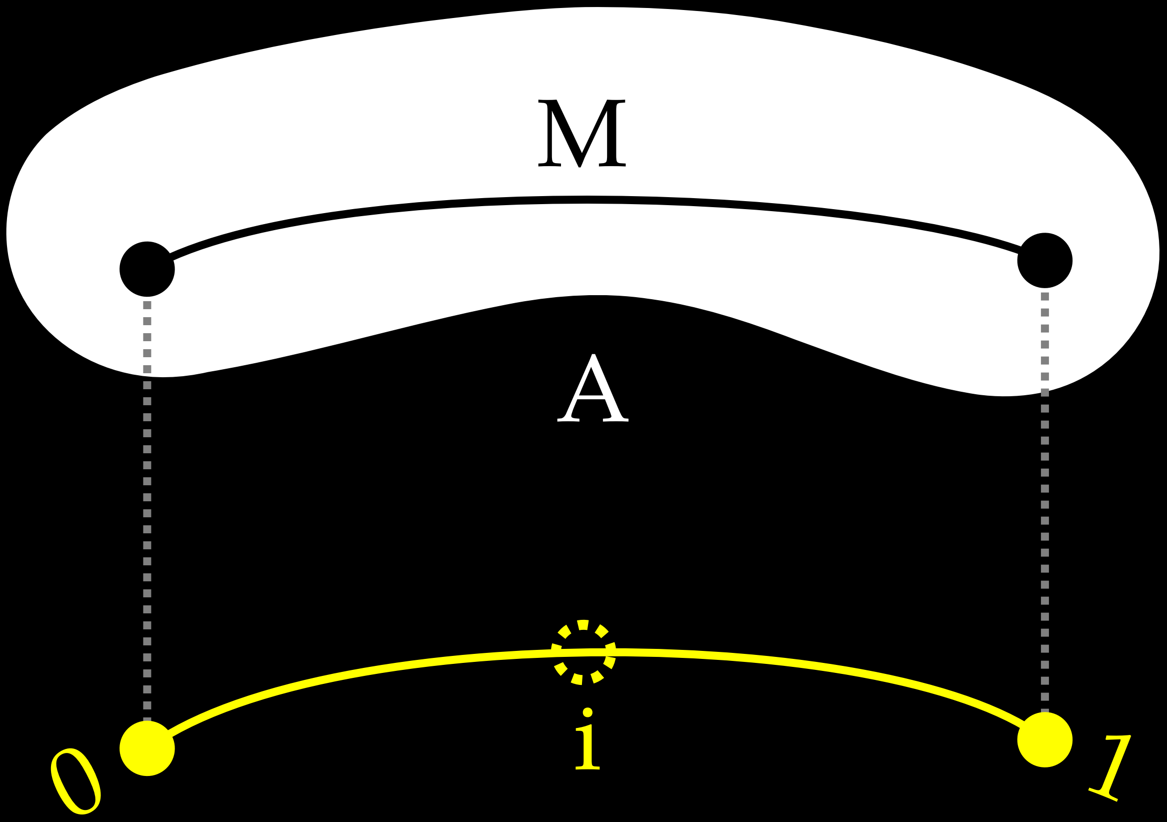
(homotopy theory)

I. Cubes

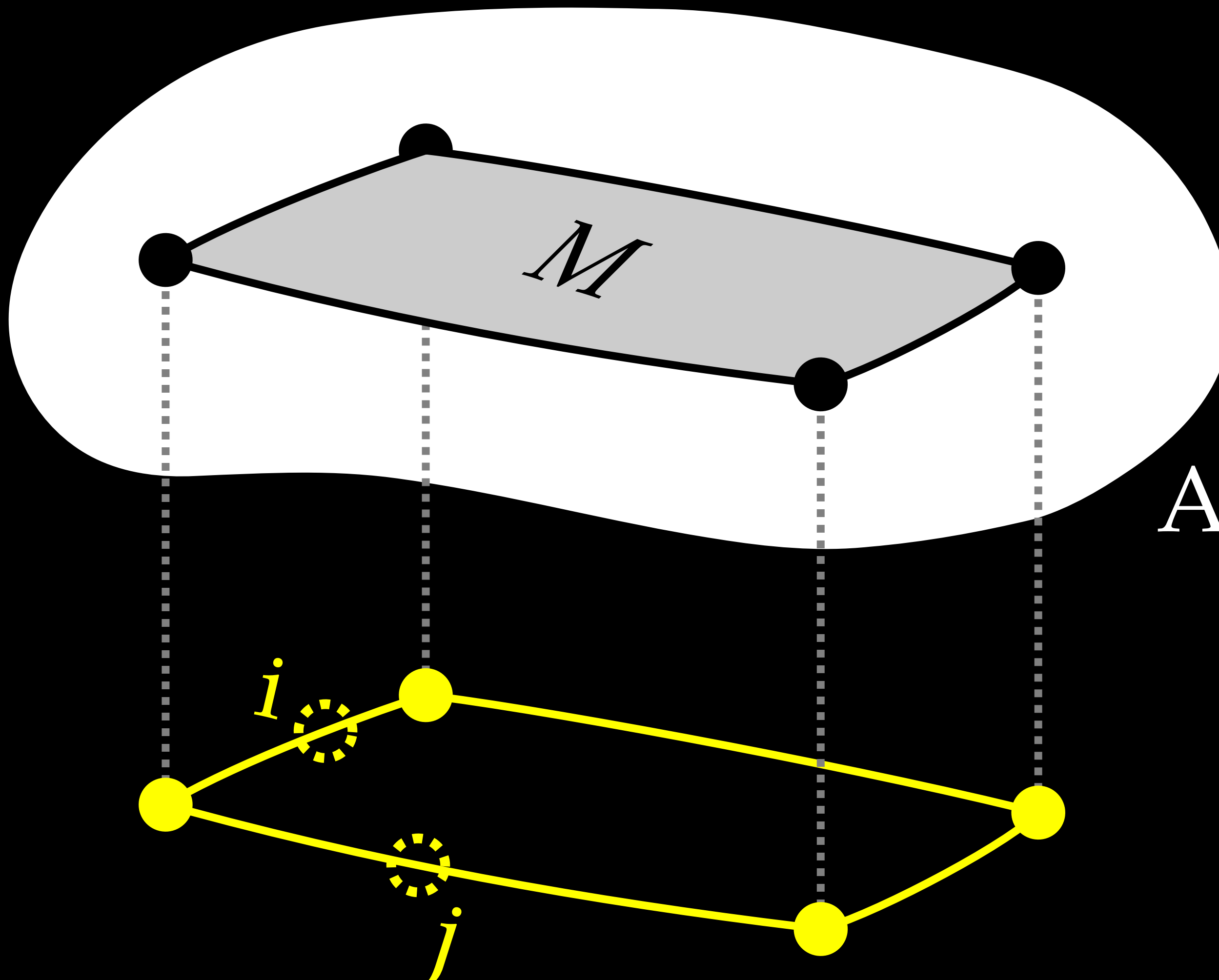



M
A

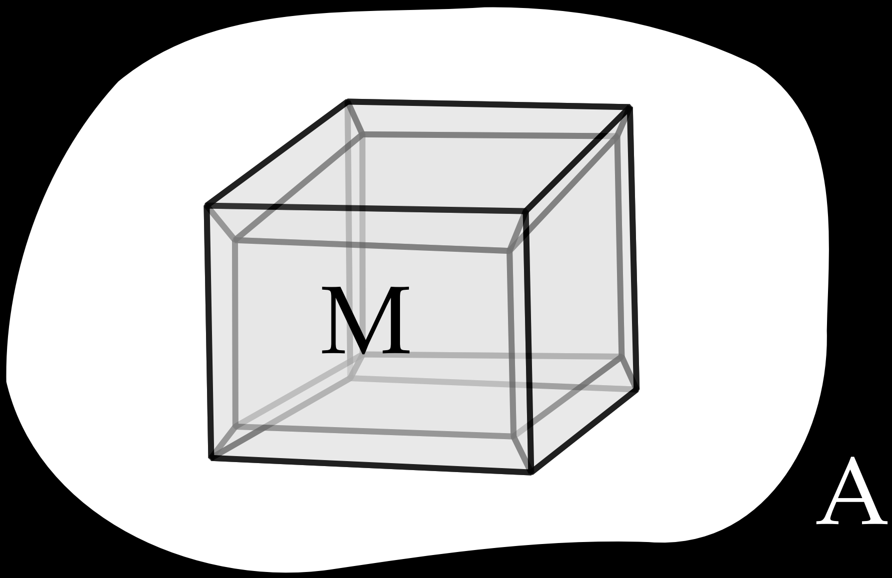

M
A
0
1



$i{:}\mathbb{I} \vdash M : A$

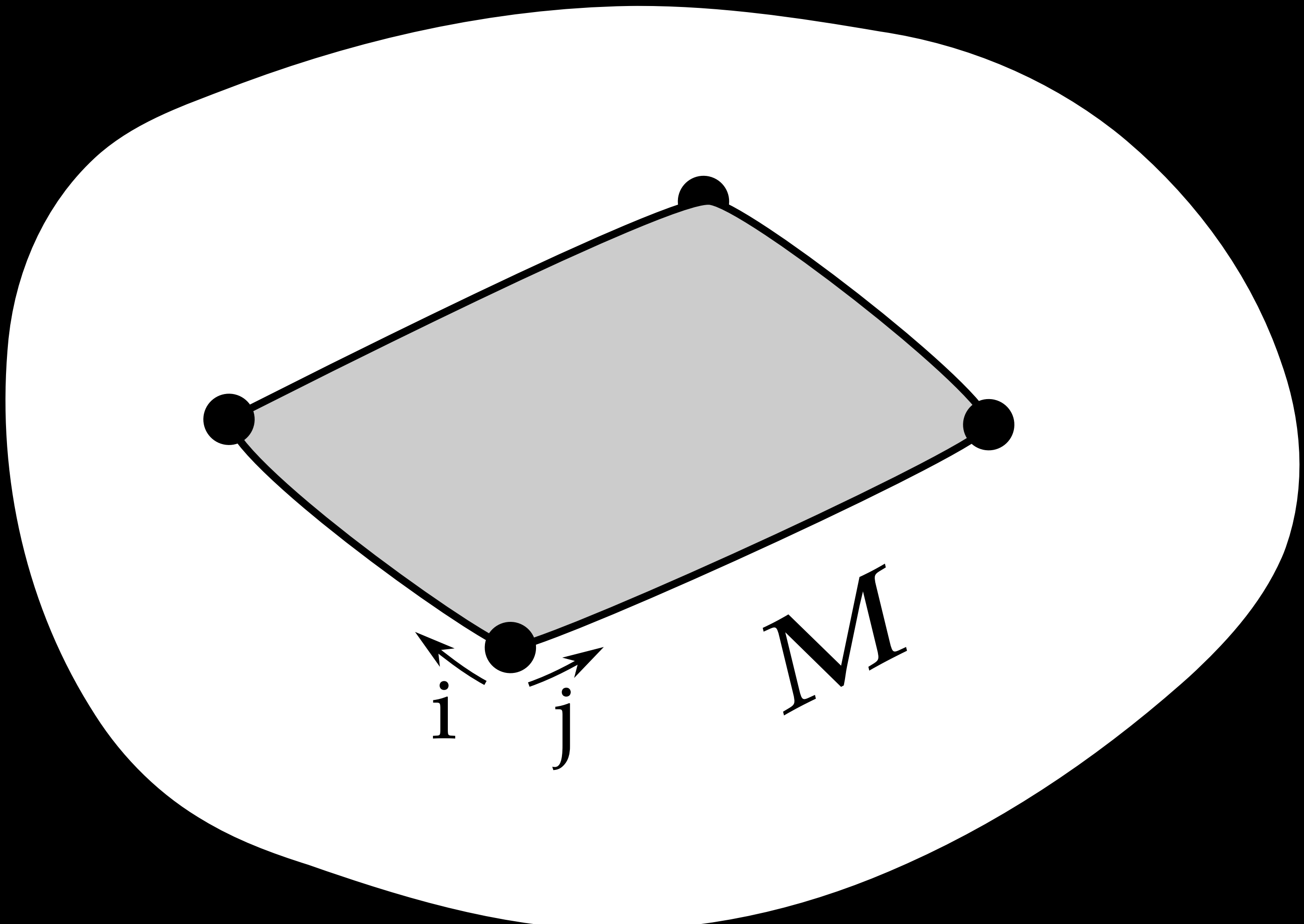


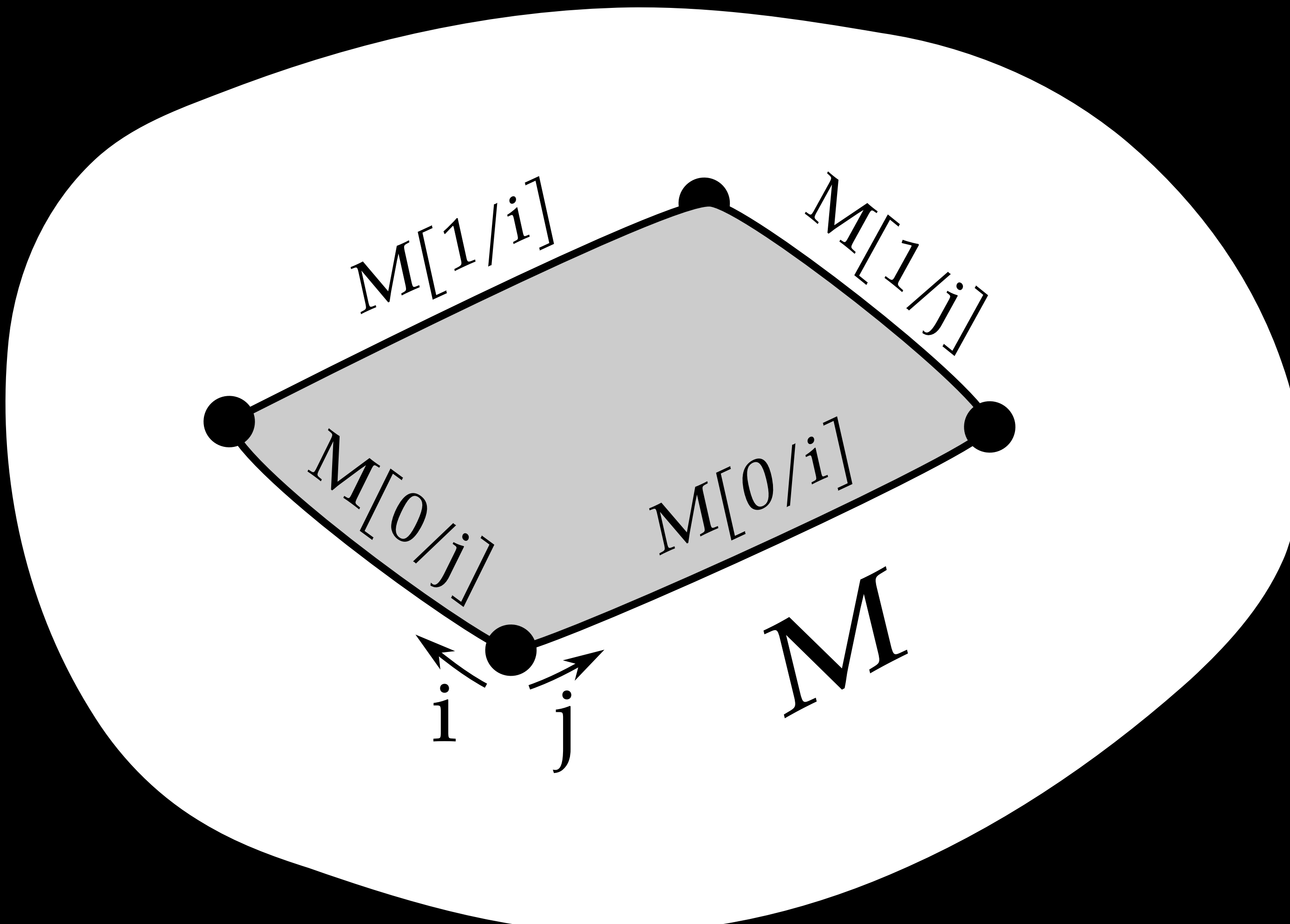
$i{:}\mathbb{I}, j{:}\mathbb{I} \vdash M : A$

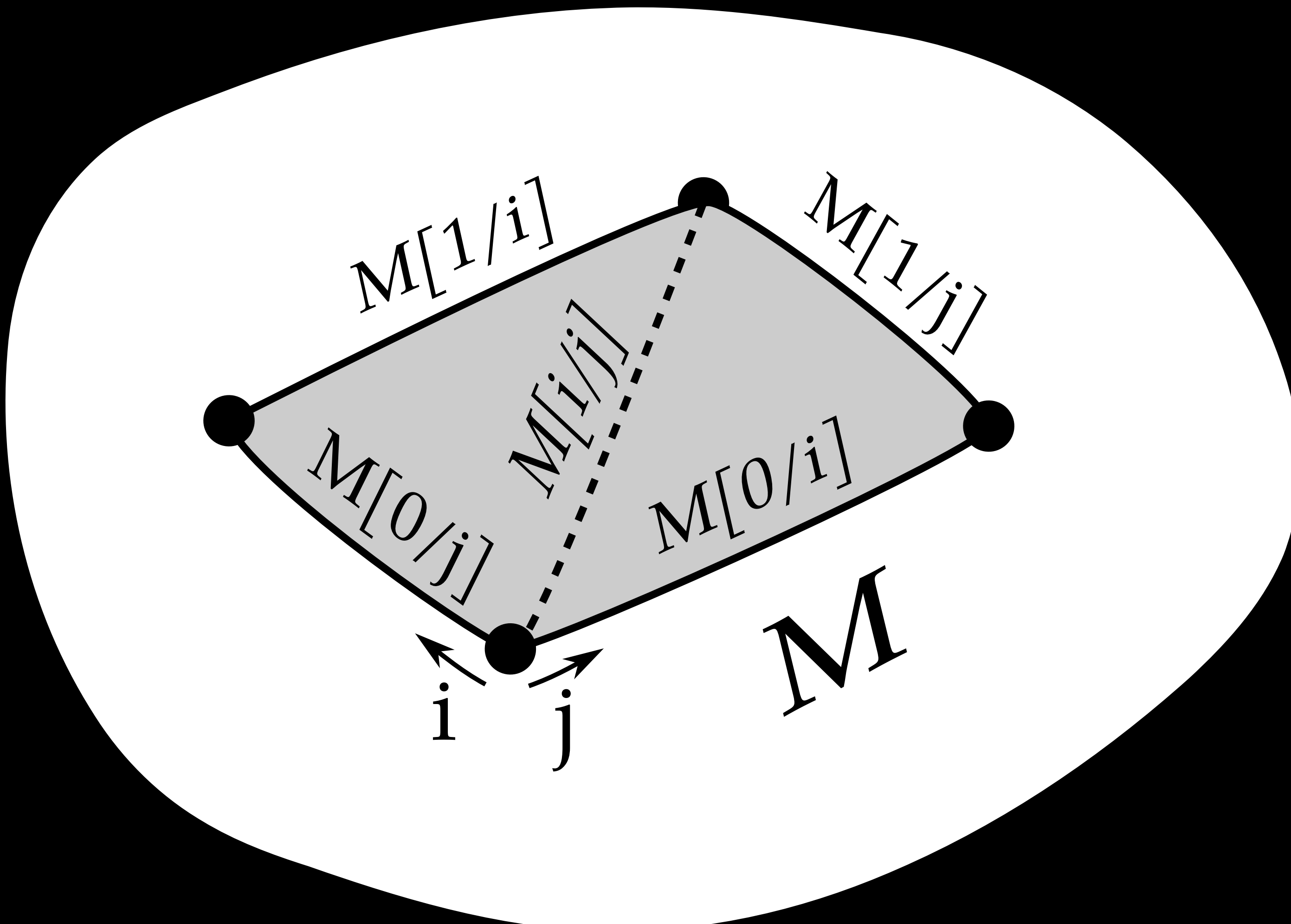


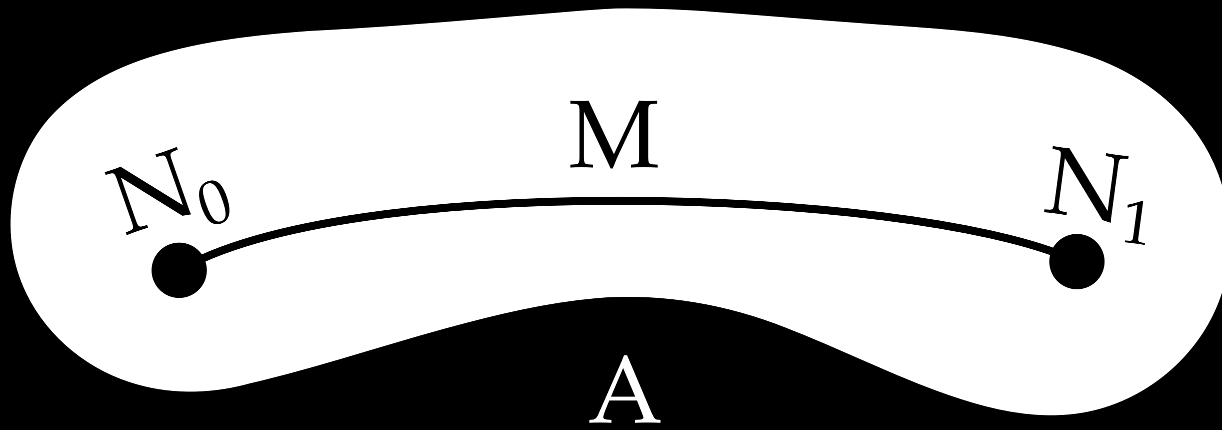
n-cube

$$i_1{:}\mathbb{I},\ i_2{:}\mathbb{I},\ \ldots,\ i_n{:}\mathbb{I} \vdash M : A$$


i
j
M

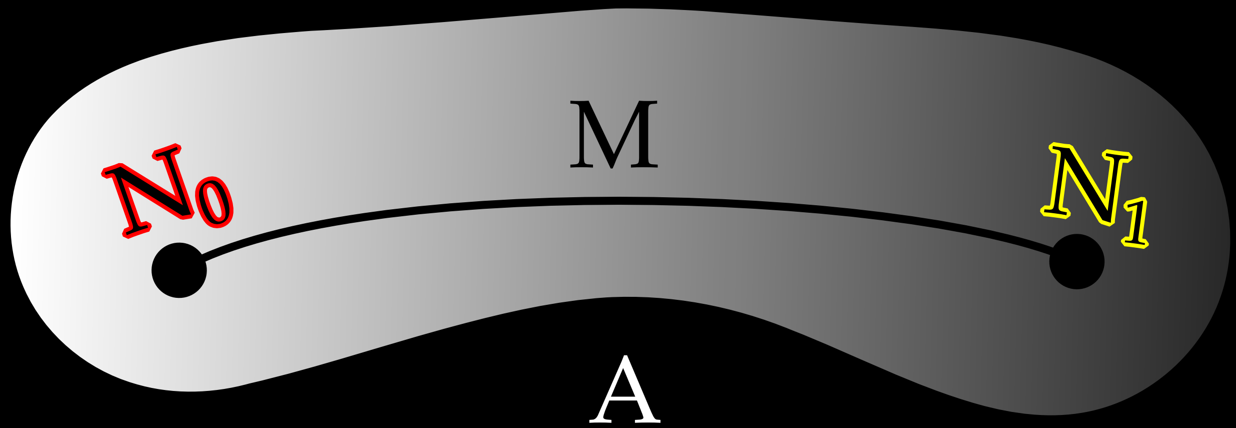

M[1/i]
M[1/j]
M[0/j]
M[0/i]
i
j
M


M[1/i]
M[1/j]
M[i/j]
M[0/j]
M[0/i]
i
j
M

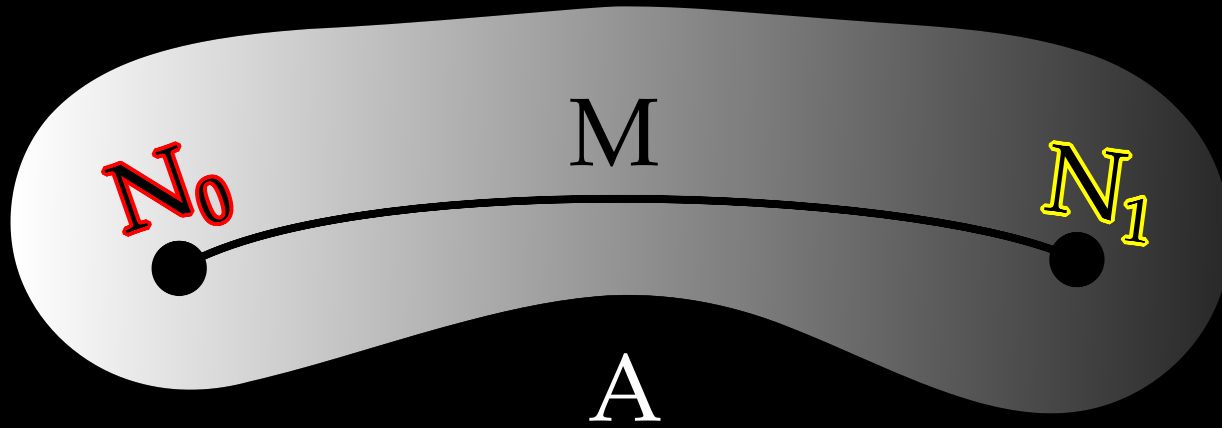


path types

functions from $\mathbb{I}$

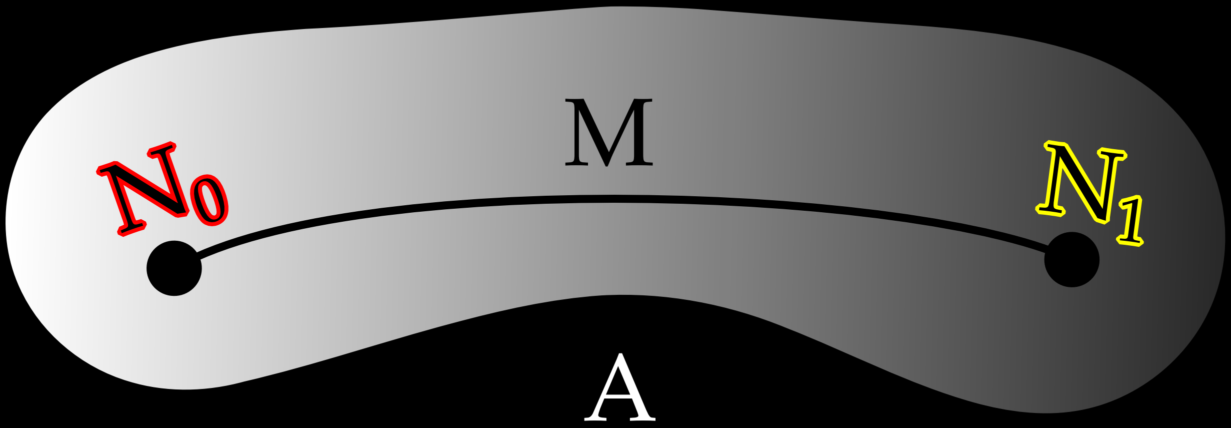


$$\frac{i{:}\mathbb{I} \vdash M : A \quad \begin{array}{l} M[0/i] \equiv N_0 : A[0/i] \\ M[1/i] \equiv N_1 : A[1/i] \end{array}}{\lambda i.M : \mathrm{Path}_{i.A}(N_0, N_1)}$$



$$\frac{i{:}\mathbb{I} \vdash M : A \qquad \begin{array}{l} M[0/i] \equiv N_0 : A[0/i] \\ M[1/i] \equiv N_1 : A[1/i] \end{array}}{\lambda i.M : \mathrm{Path}_{i.A}(N_0, N_1)}$$

$$\frac{P : \mathrm{Path}_{i.A}(N_0, N_1) \qquad r{:}\mathbb{I}}{\begin{array}{c} P@r : A[r/i] \\ P@0 \equiv N_0 : A[0/i] \\ P@1 \equiv N_1 : A[1/i] \end{array}}$$



$$\frac{i{:}\mathbb{I} \vdash M : A \quad \begin{array}{l} M[0/i] \equiv N_0 : A[0/i] \\ M[1/i] \equiv N_1 : A[1/i] \end{array}}{\lambda i.M : \mathrm{Path}_{i.A}(N_0,N_1)}$$

$$\frac{P : \mathrm{Path}_{i.A}(N_0,N_1) \quad r{:}\mathbb{I}}{\begin{array}{c} P@r : A[r/i] \\ P@0 \equiv N_0 : A[0/i] \\ P@1 \equiv N_1 : A[1/i] \end{array}}$$

$$(\lambda i.M)@r \equiv M[r/i] : A[r/i]$$

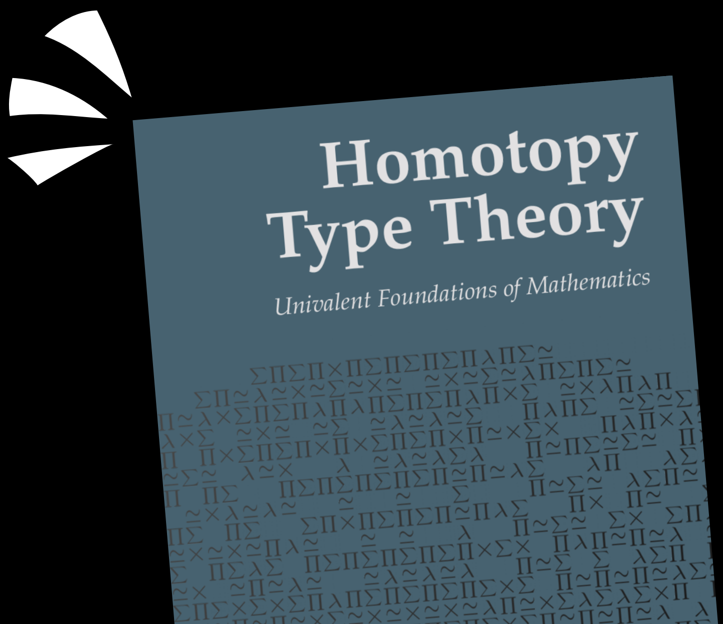
$$P \equiv \lambda i.P@i : \mathrm{Path}_{i.A}(N_0,N_1)$$

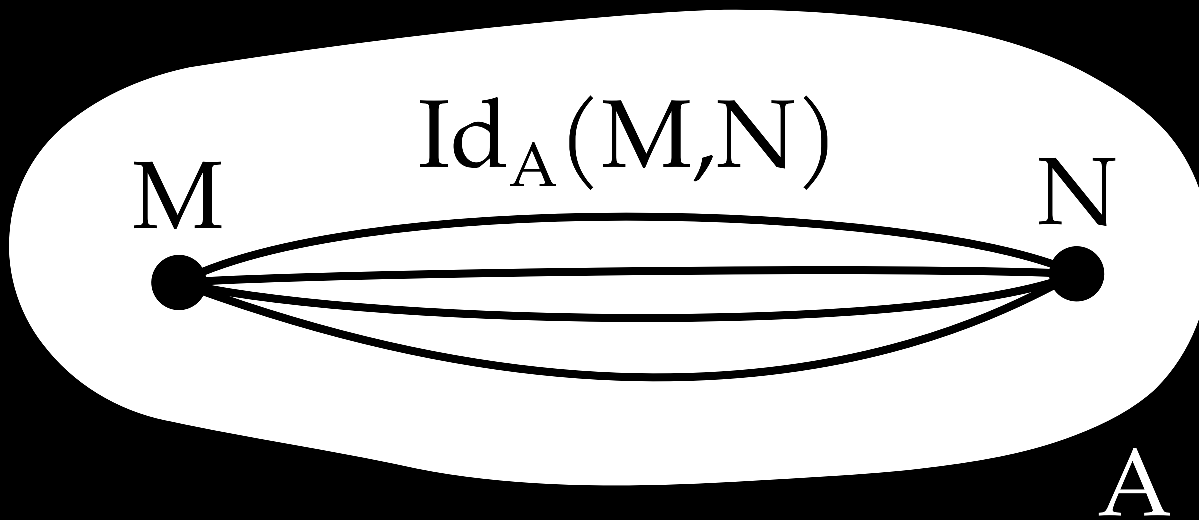
function extensionality

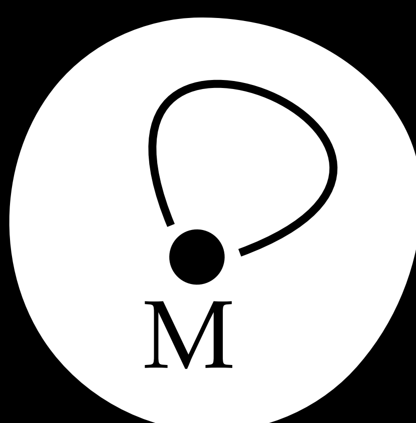
almost trivial

$$\frac{h : \Pi(x{:}A).\ \mathrm{Path}(F(x), G(x))}{\lambda i.\lambda x.h(x)@i : \mathrm{Path}(F, G)}$$

II. the Book


Homotopy
Type Theory
Univalent Foundations of Mathematics





$\mathrm{refl}_M : \mathrm{Id}_A(M,M)$

the only generator

zero ⟷ base case

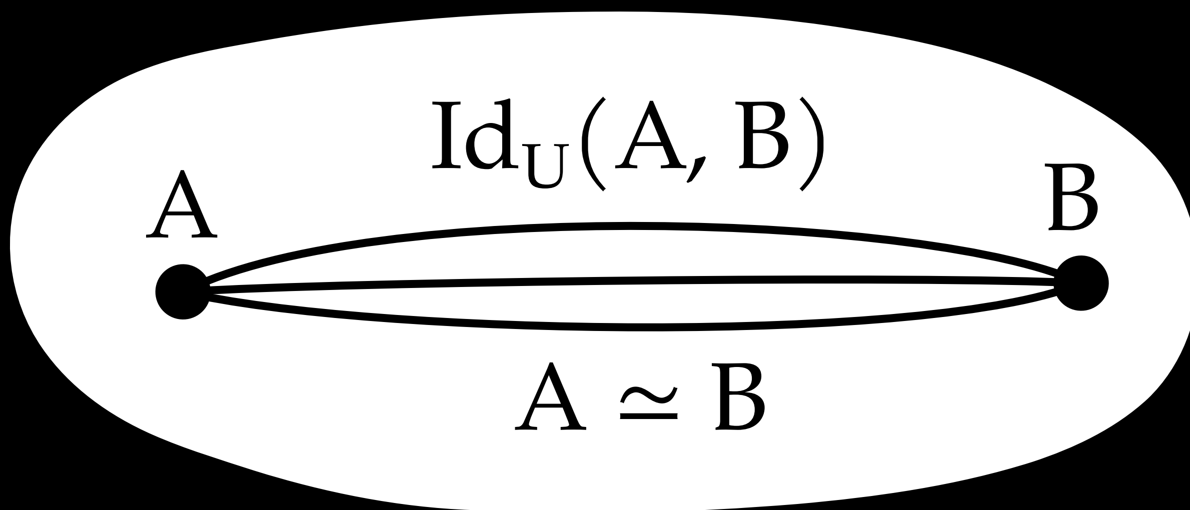
suc ⟷ induction step

mathematical induction

refl ⟷ refl case

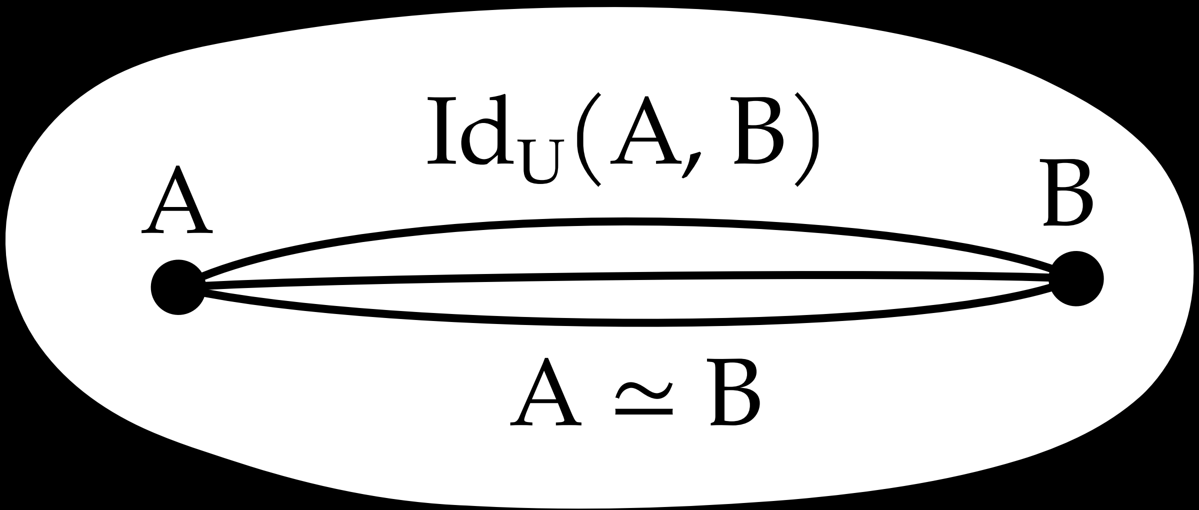
"J": induction principle for Id

univalence as axiom

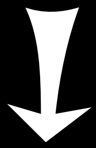


$$(A \simeq B) \simeq Id_U(A, B)$$

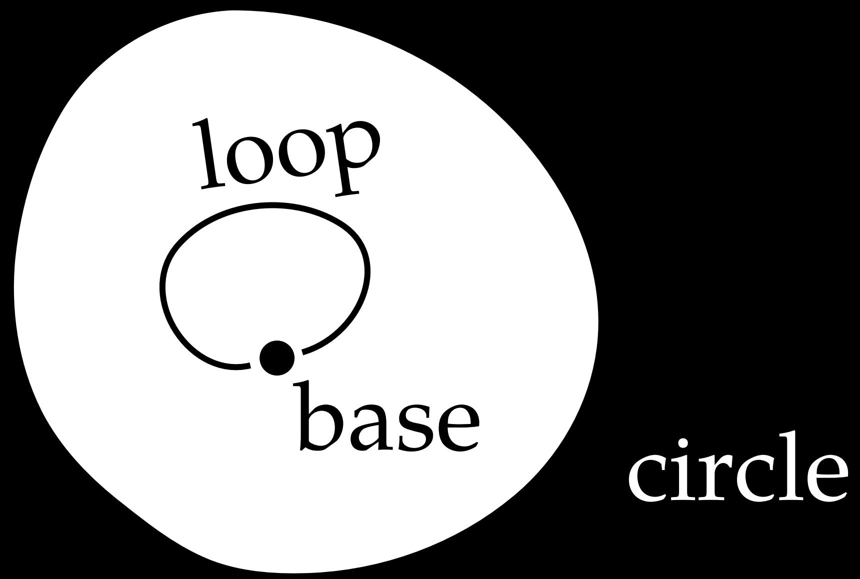
univalence as axiom



$$(A \simeq B) \simeq \mathrm{Id}_U(A, B)$$

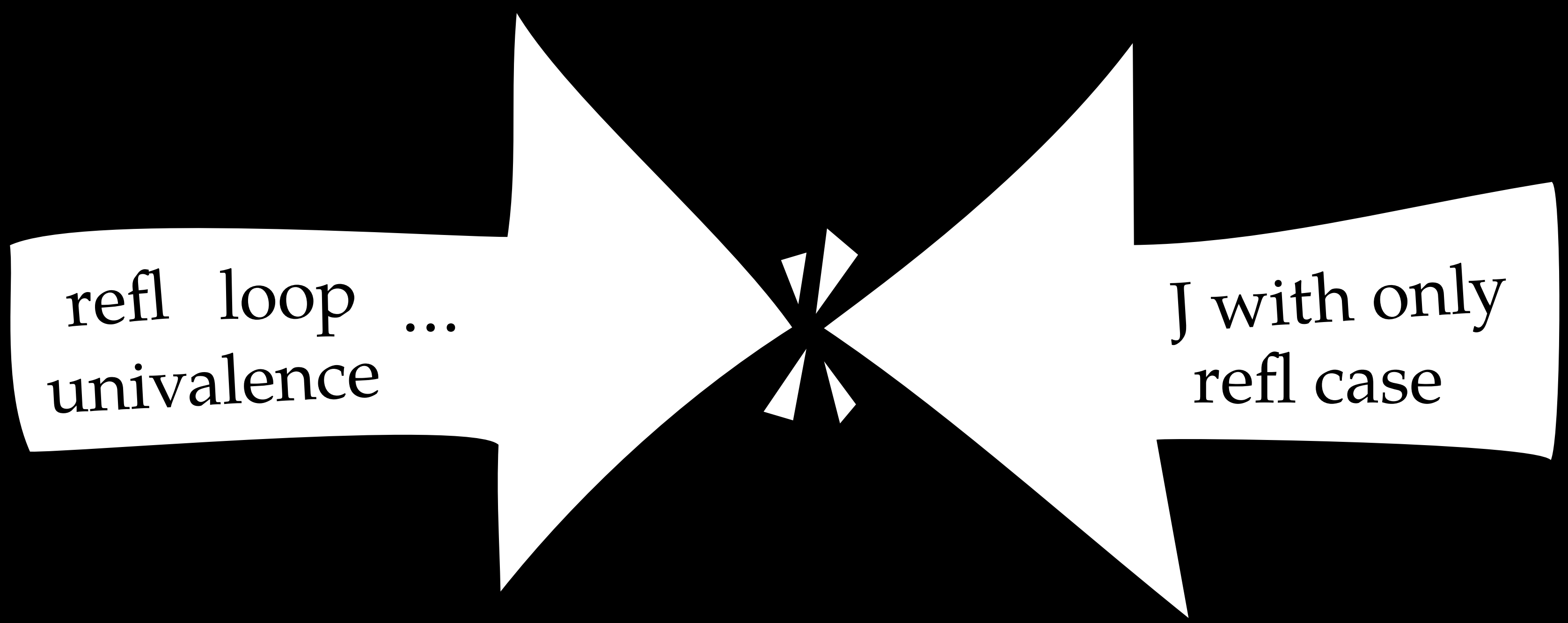


$$(A \simeq B) \to \mathrm{Id}_U(A, B)$$



base : circle

loop : $\mathrm{Id}_{\mathrm{circle}}(\mathrm{base}, \mathrm{base})$



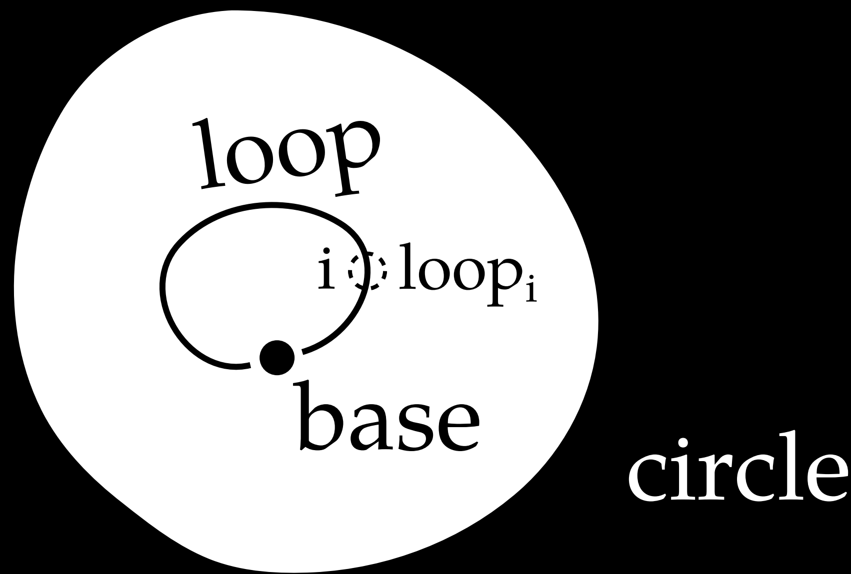
normalization in danger*

J(loop) ≡ ???

*most experts believe the normalization fails

Leave Id alone!

Id/Path should reflect existing paths, not inducing new ones



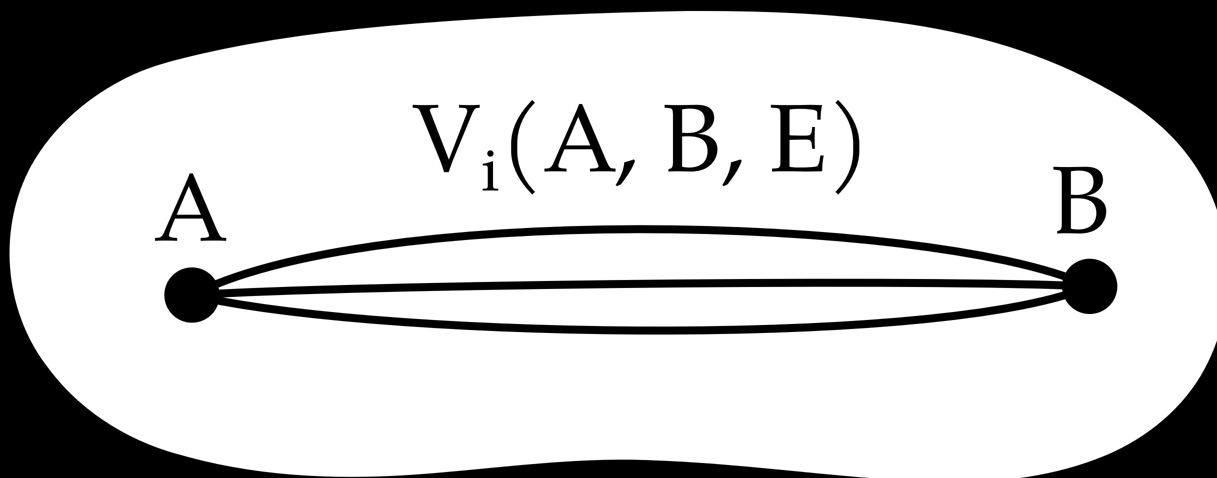
base : circle

$i{:}\mathbb{I} \vdash \mathrm{loop}_i : \mathrm{circle}$

$\mathrm{loop}_0 \equiv \mathrm{base} : \mathrm{circle}$

$\mathrm{loop}_1 \equiv \mathrm{base} : \mathrm{circle}$

univalence as a type



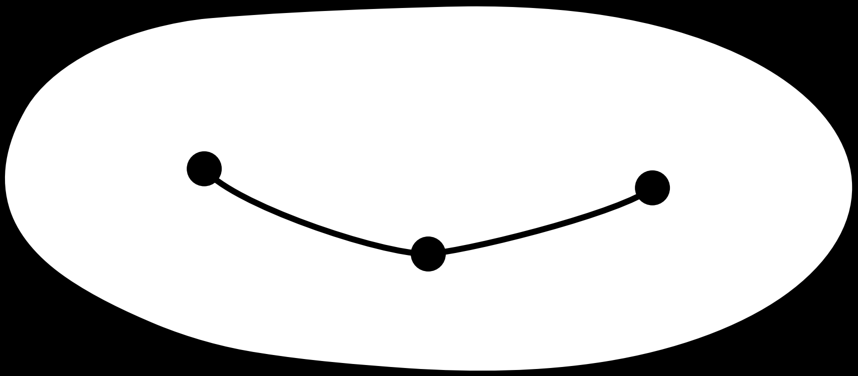
$E : A \simeq B$ (*when* $i = 0$)*

*see [AFH] (not recommended as the first paper to read)

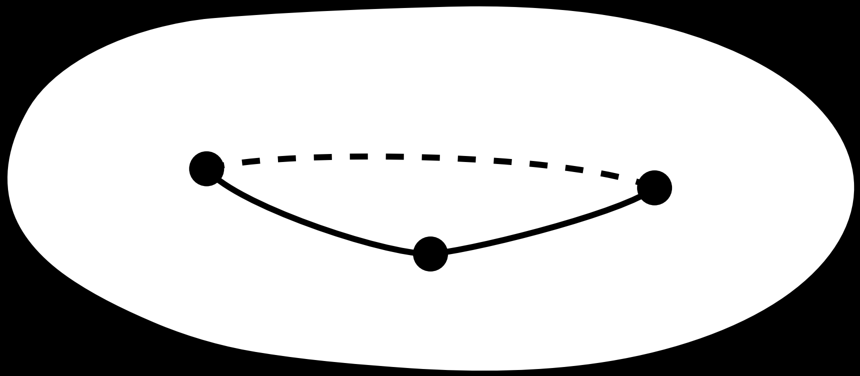
Judgmental
framework of paths

(then internalized by Path/Id)

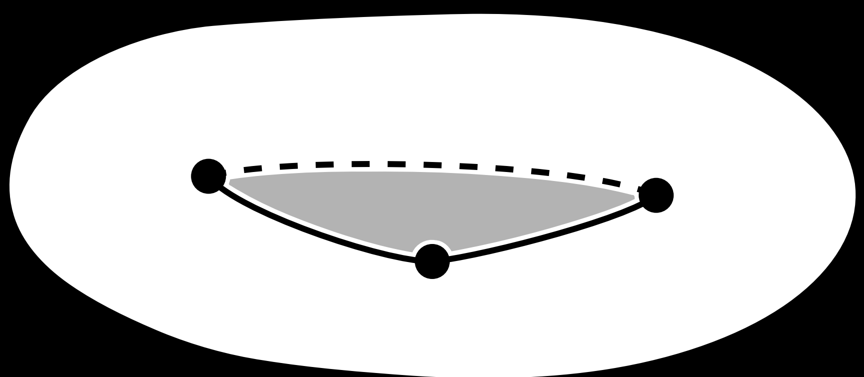
III. Compositions



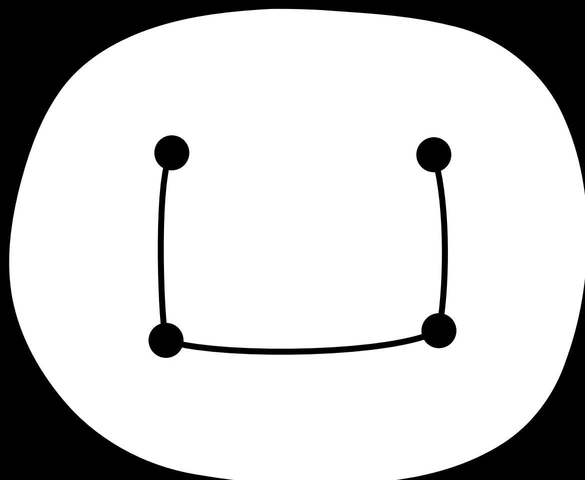
concatenation?



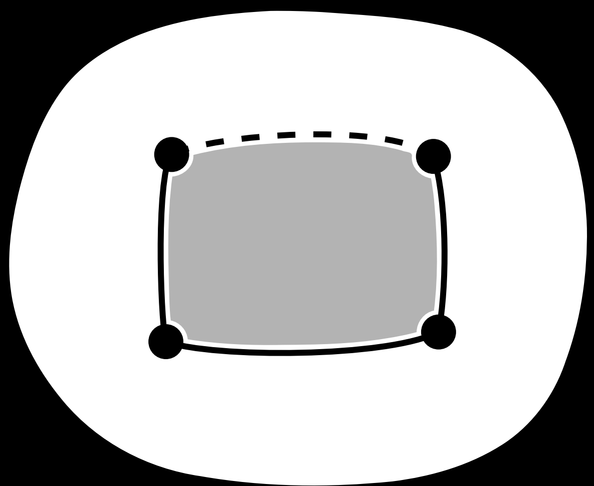
concatenation?



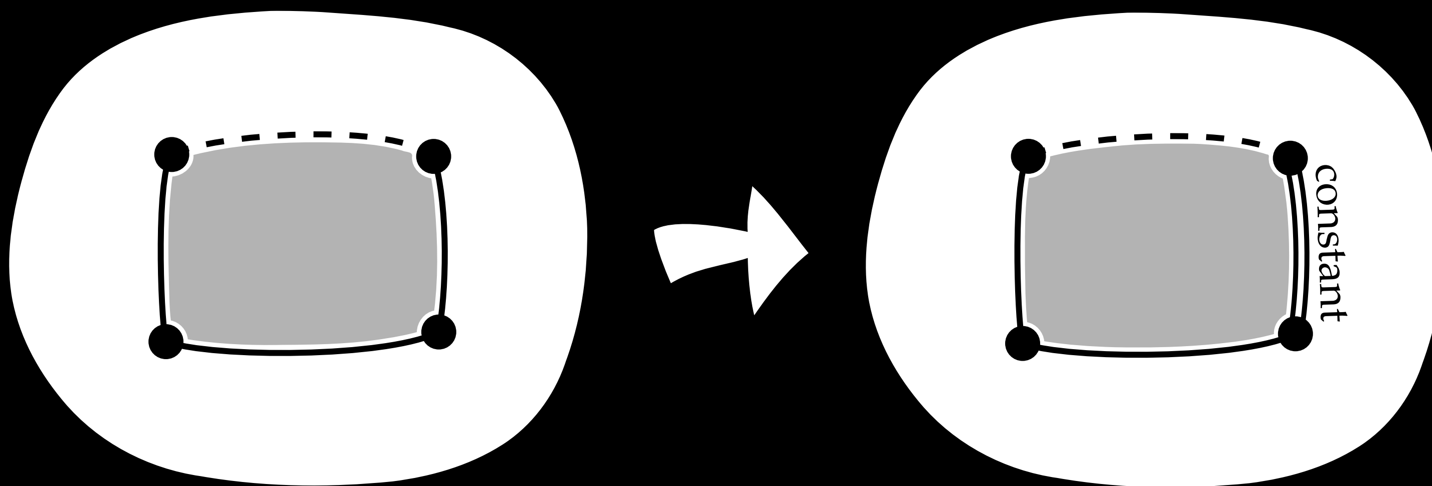
concatenation?



Kan filling
for cubes



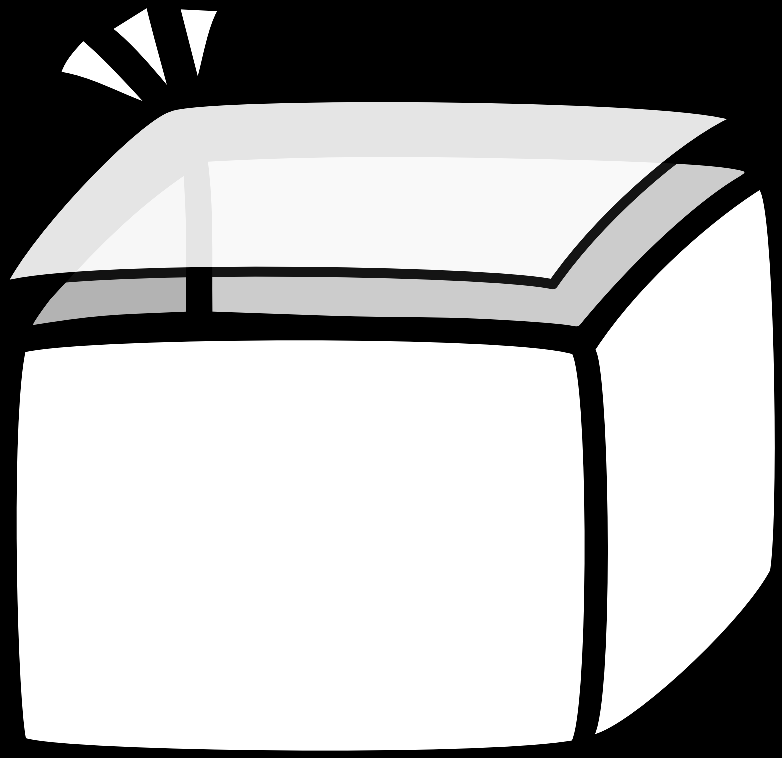
Kan filling
for cubes



Kan filling
for cubes

concatenation

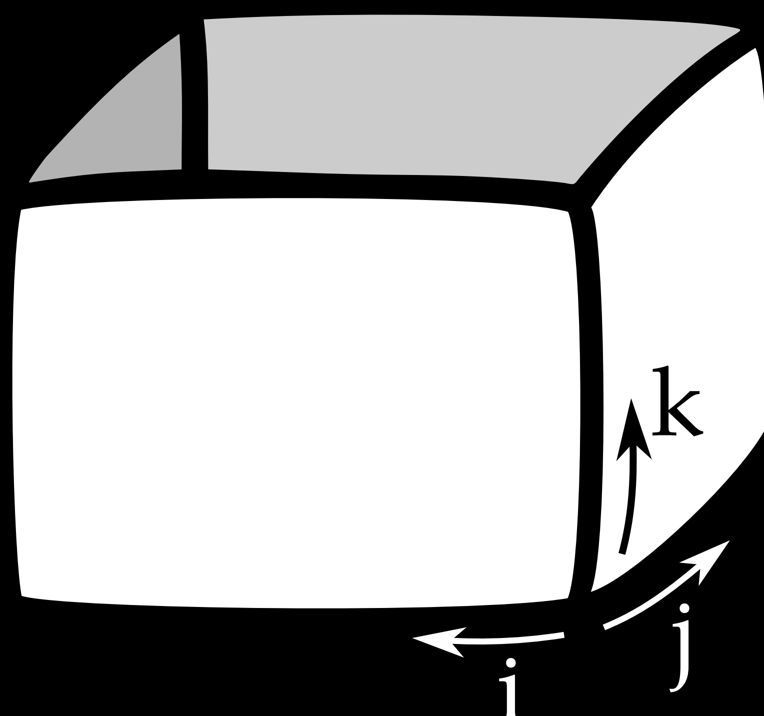
fillers can be done by higher-dimensional composition*



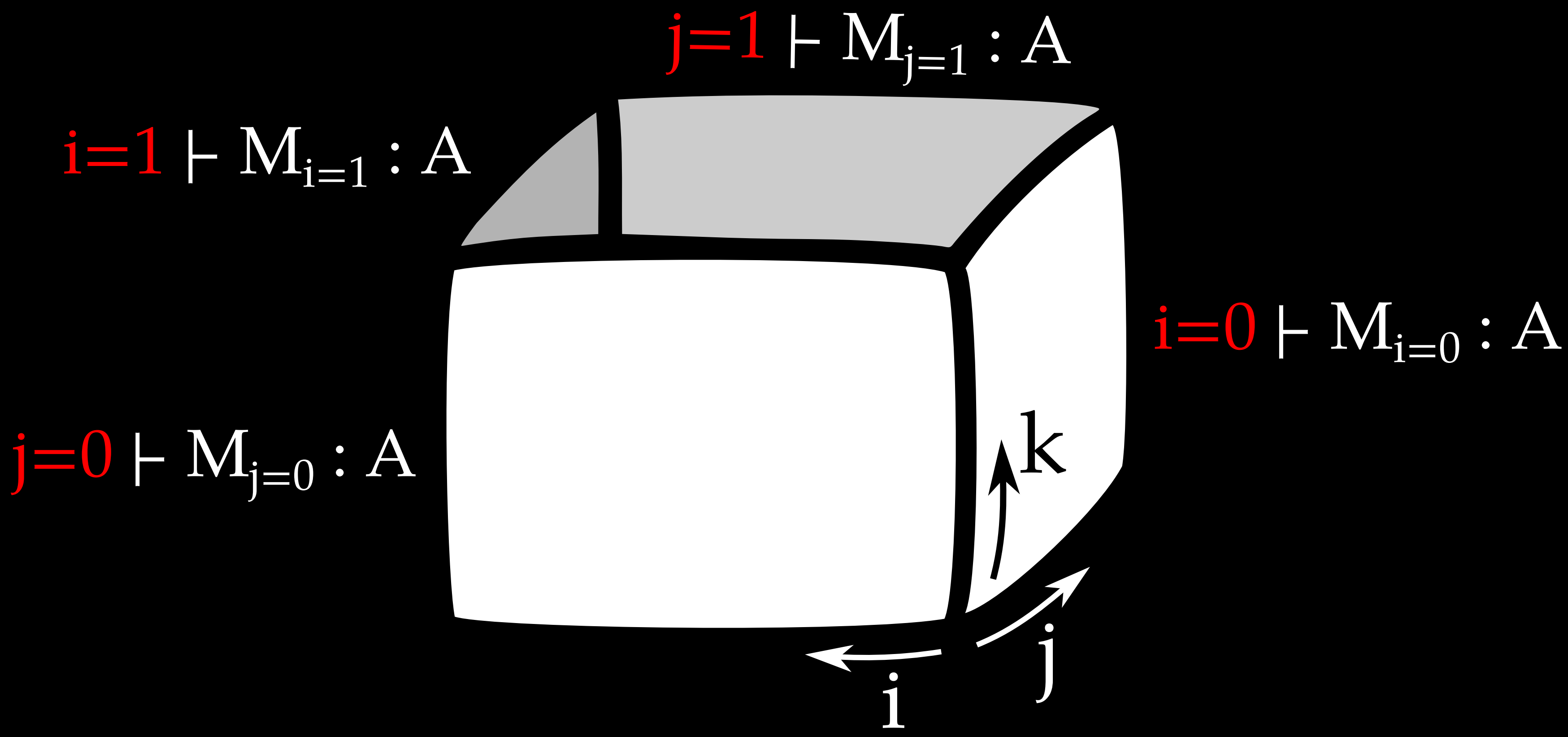
Kan composition

*technical limitations apply; see papers for real (!) math

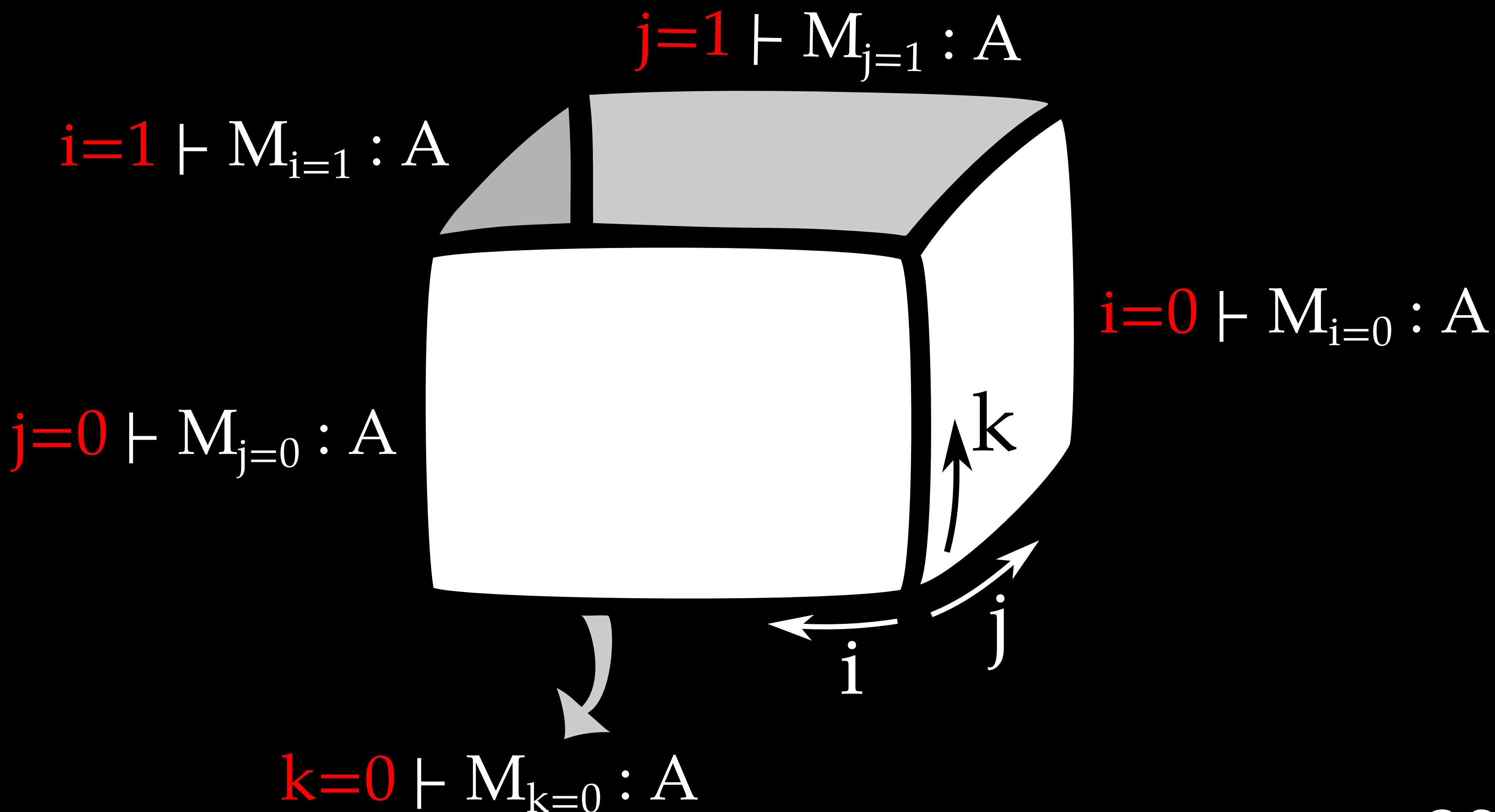
fillers can be done by
higher-dimensional composition*



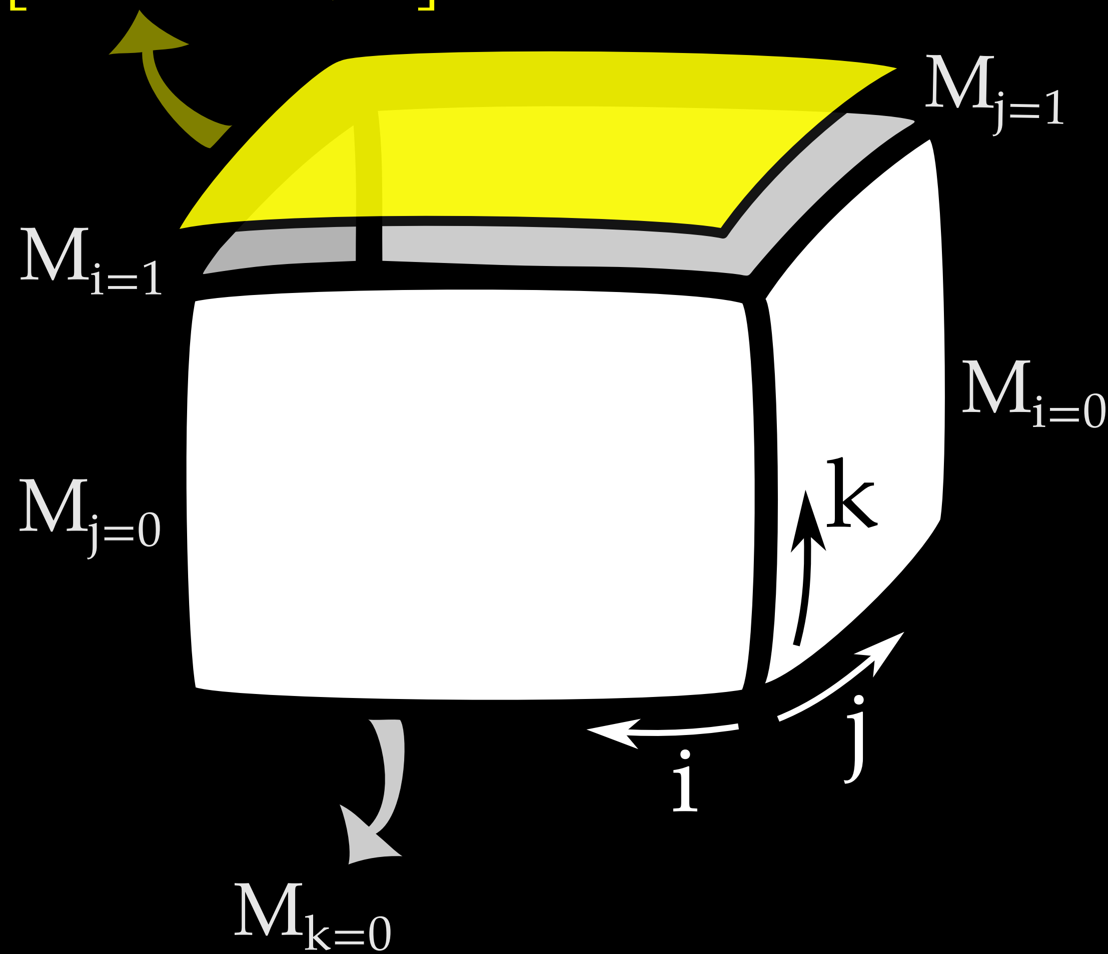
fillers can be done by higher-dimensional composition*

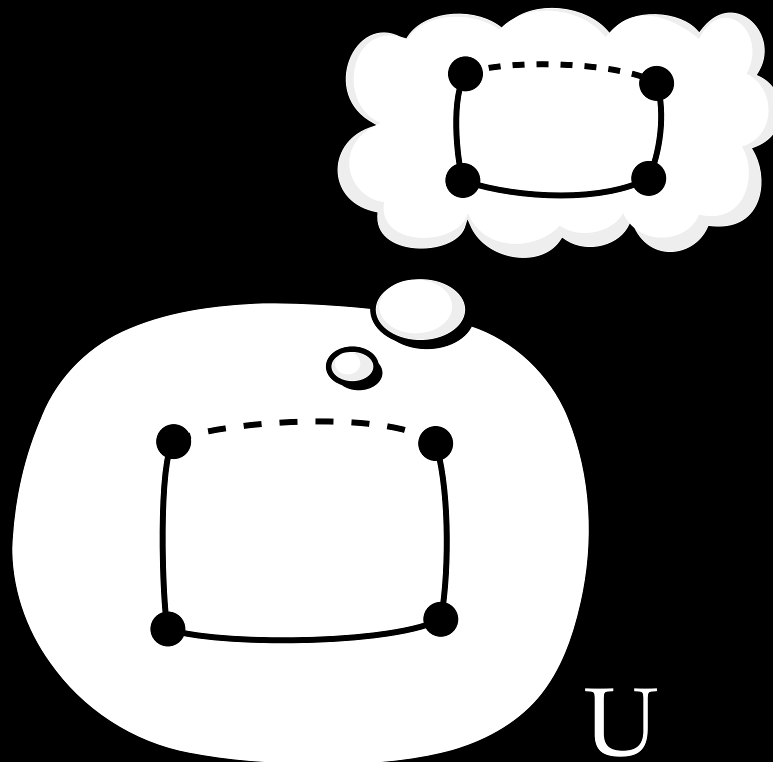


fillers can be done by higher-dimensional composition*

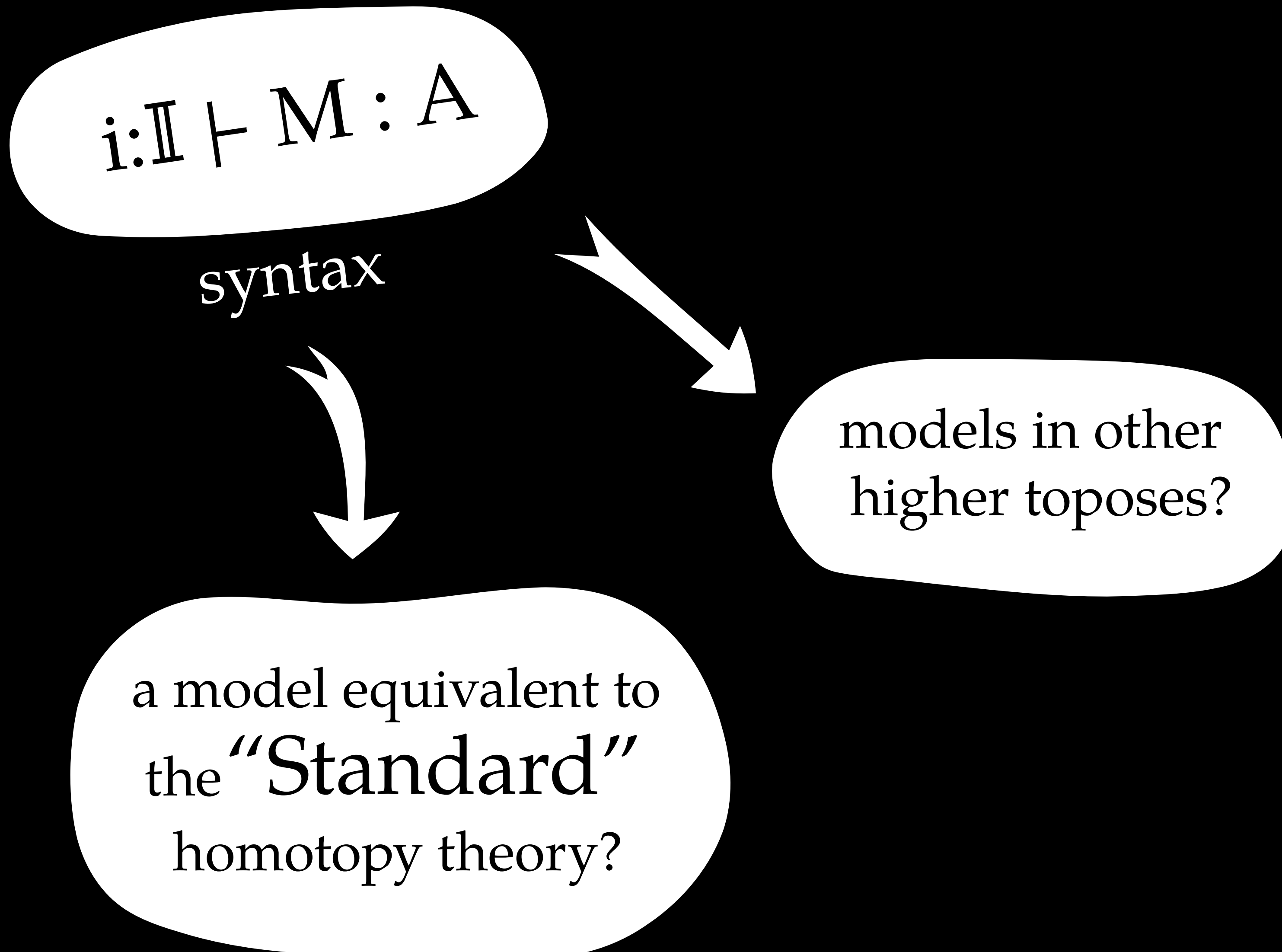


$$\mathrm{comp}_{k.A}\; M_{k=0} \begin{bmatrix} i{=}0 \hookrightarrow M_{i=0} \\ i{=}1 \hookrightarrow M_{i=1} \\ j{=}0 \hookrightarrow M_{j=0} \\ j{=}1 \hookrightarrow M_{j=1} \end{bmatrix} : A[1/k]$$





a composite in a universe is a type itself
which has its own composition operator


$i:\mathbb{I} \vdash M : A$
syntax
models in other higher toposes?
a model equivalent to the "Standard" homotopy theory?

Agda --cubical

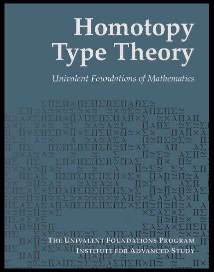
https://github.com/agda/cubical

redtt

https://github.com/RedPRL/redtt

One Path to Enlightenment

(in this order)



Homotopy Type Theory:
Univalent Foundations of Mathematics

Syntax and Models of Cartesian Cubical Type Theory [ABCFHL]

`https://github.com/dlicata335/cart-cube/blob/master/cart-cube.pdf`

Axioms for Modelling Cubical Type Theory in a Topos [OP]

(*expanded version*)

`https://arxiv.org/abs/1712.04864`

Computational Semantics of Cartesian Cubical Type Theory [A]

(*chapter 3, still changing everyday*)

`https://www.cs.cmu.edu/~cangiuli/thesis/`