Covering Spaces in Homotopy Type Theory

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Abstract. Covering spaces play an important role in classical homotopy theory, whose algebraic characteristics have deep connections with fundamental groups of underlying spaces. It is natural to ask whether these connections can be stated in homotopy type theory (HoTT), an exciting new framework coming with an interpretation in homotopy theory. This report summarizes my attempt to recover the classical results (e.g. the classification theorem) so as to explore the expressiveness of the new foundation. Some interesting techniques employed in the current proofs seem applicable to other constructions as well.

1 Introduction

Homotopy type theory (HoTT) is an exciting new interpretation of intensional type theory in terms of ∞-groupoids or topological spaces up to homotopy, which provides an abstract, *synthetic* framework for homotopy theory. [**2**–**6**, **8**–**10**] Under this interpretation, types are spaces, terms are points, sets are discrete spaces (up to homotopy), and functions are continuous maps.[1] It is natural to ask whether we can restate various homotopy-invariant concepts and theorems known in classical theories. In this report I will explore one fundamental construct: *covering spaces*. It turns out that we can express covering spaces (up to homotopy) elegantly in HoTT as follows.

Definition 1 A *covering space* of a type (space) A is a family of sets indexed by A.

That is, the type of covering spaces of A is simply $A \to \mathsf{Set}$ where Set is the type of all sets. The key insight here is that *continuity is enforced* by the framework, and thus it is sufficient to specify only the behavior on individual points. In this particular case, it is enough to say that each fiber (without mentioning neighborhoods) of the projection is a discrete set. To verify that this formulation matches the classical one, I proved in HoTT the classification theorem of covering spaces, and that homotopy equivalent classes of paths with one end fixed form a universal covering space. Let $\pi_1(A)$ be the fundamental group of a pointed space A and G-Set be the type of all G-sets, the sets equipped with an action of given group G. The classification theorem asserts the equivalence between $A \to \mathsf{Set}$ and $\pi_1(A)$-Set for any pointed, path-connected A.

It is worth emphasizing that every proof mentioned in this report has been fully mechanized [**1**] and checked by the proof assistant Agda [**7**], thanks to HoTT's ability to express many topological concepts (paths, homotopies, truncation, connectedness, circles, intervals, etc.) fairly easily from its axioms.

Another feature of HoTT is that, being a proof-relevant mathematics, it is able to capture some subtleties that are not immediately visible in the traditional framework. For example, the equivalence between $A \to \mathsf{Set}$ and $\pi_1(A)$-Set in the classification theorem will associate the $\pi_1(A)$ itself (as a $\pi_1(A)$-set) with some universal covering space.

[1] Our terminology follows the HoTT book [**8**]; in particular, *sets* means types of homotopy level zero.

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However, in general there are more than one such covering space and there is no continuous choice among them, unless we fix one point in the associated covering space. This fact can be elegantly stated in HoTT using truncation, due to the unification of logic and data where theorems themselves are also spaces and they can contain different proofs as different points.

The following sections will outline the two results of covering spaces I reproved in HoTT, namely the classification and the universality. An interesting technical device I used for the classification theorem will also be mentioned.

2 Classification

The goal in this section is to show that there is an equivalence between covering spaces of a pointed, path-connected A and $\pi_1(A)$-Set. The definitions of groups and G-sets closely follow the classical ones, with the requirement that the underlying type must be a set; A is path-connected if the 0-truncation of A (written $\|A\|_0$) is contractible.

Theorem 2 *For any path-connected, pointed type A, $(A \to \mathsf{Set}) \simeq \pi_1(A)$-$\mathsf{Set}$.*

To establish the equivalence, it is necessary to give a map from all covering spaces to all $\pi_1(A)$-sets, and an inverse map of it. The first map can be easily constructed from the lifting property of the given covering space (as a family of sets). More precisely, suppose $F : A \to \mathsf{Set}$ is a covering space of A and a is the distinguished point of A. The transport function $\mathtt{transport}_{x.F(x)}(p)$ associates an automorphism of the set $F(a)$ to each loop p at a. Because F is a family of sets, the type of automorphisms of $F(a)$ is also a set. By the universal property of truncation, an element in the 0-truncated loop space, $\pi_1(A)$, also gives rise to an automorphism of $F(a)$. We then complete the construction of a $\pi_1(A)$-set by considering the set $F(a)$ along with the above process as the action of $\pi_1(A)$ on $F(a)$.

The inverse map, from $\pi_1(A)$-sets to covering spaces of A, is more technically involved. The high-level idea is:

(1) put the given $\pi_1(A)$-set as the fiber over the distinguished point of A; and
(2) forge other fibers by introducing a formal transport; and
(3) throw in equations to mimic functoriality of `transport` (so that the formal one behaves as the real one).

We exploit higher-inductive types to achieve the final step. More formally, suppose a is the distinguished point of A. Given a $\pi_1(A)$-set X equipped with an action of type $X \to \pi_1(A) \to X$ (written $x \cdot l$ for $x : A$ and $l : \pi_1(A)$). Let $\cdot_0$ be the concatenation of two 0-truncated paths. The higher-inductive type is a family of sets `ribbon` indexed by A with the following two constructors:

$$t : \prod_{(a':A)} X \to \|a =_A a'\|_0 \to \mathtt{ribbon}(a')$$

$$\alpha : \prod_{(a':A)} \prod_{(x:X)} \prod_{(l:\pi_1(A))} \prod_{(p:\|a=_A a'\|_0)} t(a')(x \cdot l)(p) =_{\mathtt{ribbon}(a')} t(a')(x)(l \cdot_0 p).$$

The constructor t is the formal transport function to forge other fibers, and α enforces the required functoriality. Note that the formal transport t is taking a 0-truncated path of type $\|a =_A a'\|_0$ so that it goes along with the $\pi_1(A)$-action in the type of α.

Although conceptually similar to a standard argument in classical homotopy theory, the details of this proof are quite different. For example, we do not need to (explicitly) put a topology on the `ribbon` space. Because of these differences, there is only a thin layer between these high-level ideas of the classical proof and the syntactical proof in HoTT. As a consequence, computer-checking becomes practical for HoTT.

The remaining parts are the proof that the two maps are inverse to each other. This is mostly straightforward except one thing: suppose we start from a covering space $F : A \to \mathsf{Set}$. We need to show that the associated `ribbon` and F are the same. By functional extensionality and the Univalence Axiom, this reduces to a fiberwise equivalence between `ribbon` and F. The direction from $\texttt{ribbon}(a')$ to $F(a')$ is to realize the formal transport t by the real `transport`. The other direction from $F(a')$ to $\texttt{ribbon}(a')$ involves locating a point in the fiber $F(a)$ and a (truncated) path $p : \|a = a'\|_0$, as they are required by the formal transport t. However, the formal transport t needs a 0-truncated path but the path-connectedness condition only gives a (-1)-truncated path. There is still hope because we can show that the α constructor forces different choices for this path to give the same point, and thus in principle a (-1)-truncated path should suffice, which is to say that merely the existence of such path should be sufficient for our construction. The essence of this argument comes down to the following general lemma:

Lemma 3 (factorization of constant functions) *Let f be a function of type $B \to C$ where C is a set, and $|-|_{-1}$ be the projection function from B to $\|B\|_{-1}$. If*

$$\prod_{b_1, b_2 : B} f(b_1) =_C f(b_2)$$

then there is a function $g : \|B\|_{-1} \to C$ such that

$$f \equiv g \circ |-|_{-1}.$$

With $B :\equiv (a = a')$ and the required constancy condition from the α constructor, this lemma enables us to access the path in $\|a = a'\|_{-1}$ even though it is (-1)-truncated, and hence completes the main proof. The proof of this lemma depends on another high-inductive type but is beyond the scope of this report.

The final remark is that, the HoTT proof requires this factorization lemma (while the classical proof does not) because we are actually proving a stronger theorem, in the sense that the proof will associate "equivalent" equivalences to homotopically equivalent pointed spaces. Intuitively, this holds because at each step of the construction, a choice can be made in a continuous way. The factorization lemma is one of the building blocks.

3 Universality

Let A be a path-connected type with a distinguished point a. It is well-known that the homotopy equivalence classes of paths from a in A form a *universal* covering space, in the sense that it is homotopy initial in the category of pointed covering spaces of A (where the morphisms are covering projections). This particular space can be concisely written down in HoTT as follows:

$$U_A :\equiv \lambda(a' {:} A).\, \left\|a =_A a'\right\|_0$$

I reproved that every simply-connected, pointed covering space of A is equivalent to U_A, and that this covering space is indeed homotopy initial (in the category mentioned above). The proof is rather simple compared to that of the classification theorem.

The pointedness condition of covering spaces helps us pin down one equivalence between two universal covering spaces. Without it, there is no canonical choice among possibly many different equivalences, and one can only show the mere existence of such. Let F_1 and F_2 be the two covering spaces in discussion. The mere existence of such equivalence can be stated in HoTT using (-1)-truncation:

$$\left\| \prod_{a':A} F_1(a') \simeq F_2(a') \right\|_{-1}$$

The intuition is that, even though the choices made in the construction of the equivalences might not all be continuous, the type representing the *mere existence* of it is continuous in the parameters. This matches up with the classical existential quantifier, which only cares about the existence of one element. In fact, one can model much classical reasoning by truncating every theorem down to the (-1)-level. Intuitively, while the interpretation enforces continuity, we can effectively relax that condition by a suitable truncation.

4 Conclusion and Future Work

This report confirms that one can reason about covering spaces in HoTT. There are many interesting future directions; for example, (1) the correspondence between the category of covering spaces and that of $\pi_1(A)$-sets, not just the objects; and (2) the more general form $A \to n\text{-}\mathtt{Type}$ where covering spaces are the special case where $n = 0$.

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References

[1] HoTT library in Agda. https://github.com/HoTT/HoTT-Agda.

[2] Steve Awodey and Michael A Warren. Homotopy theoretic models of identity types. In *Mathematical proceedings of the Cambridge Philosophical Society*, volume 146, pages 45–56. Cambridge University Press, 2009.

[3] Nicola Gambino and Richard Garner. The identity type weak factorisation system. *Theoretical Computer Science*, 409(1):94–109, 2008.

[4] Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. *Twenty-five years of constructive type theory (Venice, 1995)*, 36:83–111, 1998.

[5] Chris Kapulkin, Peter LeFanu Lumsdaine, and Vladimir Voevodsky. The simplicial model of univalent foundations. *arXiv preprint arXiv:1211.2851*, 2012.

[6] Peter LeFanu Lumsdaine. Weak ω-categories from intensional type theory. In *Typed lambda calculi and applications*, pages 172–187. Springer, 2009.

[7] Ulf Norell. *Towards a practical programming language based on dependent type theory*. PhD thesis, Department of Computer Science and Engineering, Chalmers University of Technology, SE-412 96 Göteborg, Sweden, September 2007.

[8] The Univalent Foundations Program. *Homotopy type theory: Univalent foundations of mathematics*. 2013.

[9] Benno van den Berg and Richard Garner. Types are weak ω-groupoids. *Proceedings of the London Mathematical Society*, 102(2):370–394, 2011.

[10] Michael A Warren. *Homotopy theoretic aspects of constructive type theory*. PhD thesis, PhD thesis, Carnegie Mellon University, 2008.