Cohomology

Combinatorial
Cellular

&

Abstract
Eilenberg-Steenrod

Ulrik Buchholtz and Favonia
Cohomology Groups

\{ mappings from holes in a space \}
Cohomology Groups

{ mappings from holes in a space }

Cellular cohomology for CW complexes  |  Axiomatic Eilenberg-Steenrod cohomology
Cohomology Groups
{ mappings from holes in a space }

Cellular cohomology for CW complexes
Axiomatic Eilenberg-Steenrod cohomology

Dream: prove they are the same!
CW complexes
inductively-defined spaces
CW complexes
inductively-defined spaces

points
CW complexes
inductively-defined spaces

points
lines
CW complexes
inductively-defined spaces

points
lines
faces
CW complexes
inductively-defined spaces

points
lines
faces
(and more...)

[Diagram of CW complex structure]
CW complexes
inductively-defined spaces

points
lines
faces
(and more...)

Specification: cells and how they attach
CW complexes

Sets of cells: $A_n$
CW complexes

Sets of cells: $A_n$

Attaching: $\alpha_{n+1} : A_{n+1} \times S^n \rightarrow X_n$

$X_n$ is the construction up to dim. $n$
CW complexes

Sets of cells: $A_n$

Attaching: $\alpha_{n+1} : A_{n+1} \times S^n \to X_n$

$X_n$ is the construction up to dim. n
CW complexes

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$X_n$ is the construction up to dim. $n$

$X_0 := A_0$
CW complexes

Sets of cells: $A_n$

Attaching: $\alpha_{n+1} : A_{n+1} \times S^n \rightarrow X_n$

$X_n$ is the construction up to dim. $n$

$X_0 := A_0$

$X_{n+1} :=$

\[ A_{n+1} \times S^n \rightarrow A_{n+1} \]

\[ \alpha_{n+1} \]

\[ X_n \rightarrow X_{n+1} \]
Cellular Cohomology

\{ mappings from holes in a space \}
Cellular Cohomology
{ mappings from holes in a space }

Cellular Homology
{ holes in a space }
Cellular Cohomology
{ mappings from holes in a space }
One-Dimensional Holes*

\{ \text{elements of } \mathbb{Z}[A_1] \text{ forming cycles} \}

*Holes are cycles in the classical homology theory.
One-Dimensional Holes*

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*Holes are cycles in the classical homology theory
One-Dimensional Holes*

\{ \text{elements of } \mathbb{Z}[A_1] \text{ forming cycles} \}

*Holes are cycles in the classical homology theory

\[
\begin{align*}
\text{holes} & : \quad a + b + c \\
\text{holes} & : \quad -a - b - c
\end{align*}
\]
One-Dimensional Holes*

{ elements of $\mathbb{Z}[A_1]$ forming cycles }

*Holes are cycles in the classical homology theory
One-Dimensional Holes

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One-Dimensional Holes

\{ \text{elements of } \mathbb{Z}[A_1] \text{ forming cycles} \}

boundary function $\partial$

$\partial(\begin{array}{c} x \\ a \\ y \end{array}) = y - x$

set of holes $= \text{kernel of } \partial$
One-Dimensional Holes

\{ \text{elements of } \mathbb{Z}[A_1] \text{ forming cycles} \}

boundary function \( \partial \)

\[ \partial \left( \begin{array}{c} x \\ a \\ y \end{array} \right) = y - x \]

set of holes = kernel of \( \partial \)
One-Dimensional Holes
{ elements of $\mathbb{Z}[A_1]$ forming cycles }

boundary function $\partial$

$\partial(a+b+c) = (y - x) + (z - y) + (x - z) = 0$

set of holes = kernel of $\partial$

$x$ $a$ $y$
$c$ $b$
$z$

$\partial(a) = y - x$

$\partial(a+b+c) = (y - x) + (z - y) + (x - z) = 0$
First Homology Groups

\{ \text{unfilled one-dimensional holes} \}
First Homology Groups

\{ \text{unfilled one-dimensional holes} \}

\begin{align*}
\partial_2( & ) = a + b + c \\
2\text{-dim. boundary function } & \partial_2
\end{align*}
First Homology Groups

\{ \text{unfilled one-dimensional holes} \}

\[
\partial_2(a + b + c) = \text{filled holes} = \text{image of } \partial_2
\]
First Homology Groups
\{ \text{unfilled one-dimensional holes} \}

\[ \partial_2(\text{filled holes}) = \text{image of } \partial_2 \]

2-dim. boundary function \( \partial_2 \)
\[ \partial_2(\text{filled holes}) = a + b + c \]

\[ H_1(X) := \text{kernel of } \partial_1 / \text{image of } \partial_2 \]

\( \text{(unfilled holes)} \) \quad \text{(all holes)} \quad \text{(filled holes)} \)
Homology Groups

\{ \text{unfilled holes} \}

\[ \mathbb{C}_n := \mathbb{Z}[\mathbb{A}_n] \text{ formal sums of cells (chains)} \]
Homology Groups
\{ \text{unfilled holes} \}

\[ C_n := \mathbb{Z}[A_n] \text{ formal sums of cells (chains)} \]

\[ \cdots \rightarrow C_{n+2} \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \rightarrow \cdots \]
Homology Groups

\{ \text{unfilled holes} \}

\[ C_n := \mathbb{Z}[A_n] \text{ formal sums of cells (chains)} \]

\[ \ldots \rightarrow \partial_{n+2} C_{n+2} \rightarrow \partial_{n+1} C_{n+1} \rightarrow \partial_n C_n \rightarrow \partial_{n-1} C_{n-1} \rightarrow \partial_{n-2} C_{n-2} \rightarrow \ldots \]

\[ H_n(X) := \ker(\partial_n) / \text{image of } \partial_{n+1} \]
Cohomology Groups

\[ \cdots \rightarrow C_{n+2} \rightarrow C_{n+1} \rightarrow C_n \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \cdots \]

Dualize by \( \text{Hom}(-, G) \). Let \( C^n = \text{Hom}(C_n, G) \).

\[ \cdots \leftarrow C^{n+2} \leftarrow C^{n+1} \leftarrow C^n \leftarrow C^{n-1} \leftarrow C^{n-2} \leftarrow \cdots \]
**Cohomology Groups**

\[ ... \rightarrow C_{n+2} \rightarrow C_{n+1} \rightarrow C_n \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow ... \]

**Dualize** by Hom(—, G). Let \( C^n = \text{Hom}(C_n, G) \).

\[ ... \leftarrow C^{n+2} \leftarrow C^{n+1} \leftarrow C^n \leftarrow C^{n-1} \leftarrow C^{n-2} \leftarrow ... \]

\( H^n(X; G) := \text{kernel of } \delta_{n+1} / \text{image of } \delta_n \)
Higher-Dim. Boundary

How to compute the coefficients from $\alpha_2$?
Higher-Dim. Boundary

\[ \alpha_2(p,-) \]

identify points
squash other loops

coe\textsubscript{icient} = winding number of this map

coefficient = winding number of this map
Higher-Dim. Boundary

\[ S^n \xrightarrow{\alpha_{n+1}(p,-)} X_n \xrightarrow{\alpha_2(p,-)} X_n/X_{n-1} \cong \bigvee S^n \xrightarrow{\text{squash}} S^n \]

Coefficient = degree of this map
Higher-Dim. Boundary

\[ S^n \xrightarrow{\alpha_{n+1}(p, -)} X_n \xrightarrow{\alpha_n(p, -)} X_n/X_{n-1} \cong \bigvee S^n \xrightarrow{\text{squash}} S^n \]

coefficient = \textit{degree} of this map

- squashing needs decidable equality
- linear sum needs closure-finiteness
Higher-Dim. Boundary

\[ A_n \times S^{n-1} \rightarrow A_n \quad A_{n+1} \times S^n \rightarrow A_{n+1} \]

\[ X_{n-1} \rightarrow X_n \rightarrow X_{n+1} \]

\[ 1 \rightarrow X_n/X_{n-1} \cong \bigvee S^n \]
Cohomology Groups
\{ \text{mappings from holes in a space} \}

Cellular cohomology for CW-complexes

$H^n(X; \ G)$

Axiomatic Eilenberg-Steenrod cohomology

Dream: prove they are the same!
Eilenberg-Steenrod* cohomology

A family of functors $h^n(\_\_)$:
pointed spaces $\rightarrow$ abelian groups

*Ordinary, reduced cohomology theory
Eilenberg-Steenrod* cohomology

A family of functors $h^n(\_\_\_\_)$:
pointed spaces $\rightarrow$ abelian groups

1. $h^{n+1}(\text{susp}(X)) \cong h^n(X)$, natural in $X$

*Ordinary, reduced cohomology theory
Eilenberg-Steenrod\* cohomology

A family of functors $h^n(\_\_)$:

pointed spaces $\rightarrow$ abelian groups

1. $h^{n+1}(\text{susp}(X)) \simeq h^n(X)$, natural in $X$

2. $\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\Gamma} & \text{Cof}_f \end{array}$

\* Ordinary, reduced cohomology theory
Eilenberg-Steenrod* cohomology

A family of functors $h^n(\_)$:
pointed spaces $\rightarrow$ abelian groups

1. $h^{n+1}(\text{susp}(X)) \cong h^n(X)$, natural in $X$

2. 

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
1 & \xrightarrow{} & \text{Cof}_f
\end{array}
\begin{array}{l}
h^n(A) \\
h^n(B)
\end{array}
\xrightarrow{\text{exact!}}

\begin{array}{ccc}
& & \\
& \uparrow & \\
& \text{Cof}_f & \xrightarrow{} \\
\end{array}
\begin{array}{c}
h^n(\text{Cof}_f)
\end{array}

*Ordinary, reduced cohomology theory
Eilenberg-Steenrod* cohomology

A family of functors $h^n(\_\_)$:
pointed spaces $\to$ abelian groups

1. $h^{n+1}(\text{susp}(X)) \cong h^n(X)$, natural in $X$

2. $\xymatrix{ h^n(A) \ar[d] \ar[r]_-f & h^n(B) \ar[d] \ar[l]_{\text{exact!}} \\
1 \ar[r] & \text{Cof}_f \ar[r] & h^n(\text{Cof}_f) }$

3. $h^n(\bigvee_i X_i) \cong \prod_i h^n(X_i)$ if the index type satisfies set-level AC

* Ordinary, reduced cohomology theory
Eilenberg-Steenrod* cohomology

A family of functors $h^n(\_\_)$:
pointed spaces $\rightarrow$ abelian groups

1. $h^{n+1}(\text{susp}(X)) \cong h^n(X)$, natural in $X$

2. $h^n(A) \xleftarrow{f} h^n(B)$

3. $h^n(\bigvee_i X_i) \cong \prod_i h^n(X_i)$
   if the index type satisfies set-level AC

4. $h^n(2)$ trivial for $n \neq 0$

*Ordinary, reduced cohomology theory
Cohomology Groups
{ mappings from holes in a space }

Cellular cohomology for CW-complexes
Axiomatic Eilenberg-Steenrod cohomology

\[ H^n(X; G) \rightarrow h^n(X) \]

Dream: prove they are the same!
Our Dream

\[ h^n(X) \approx H^n(X; h^0(2)) \]
Our Dream

\[ h^n(X) \cong H^n(X; h^0(2)) \]

\[ \ker(\delta_{n+1})/\text{im}(\delta_n) \]
Our Dream

\[ h^n(X) \cong H^n(X; h^0(2)) \]

\[ \ker(\delta'_{n+1})/\text{im}(\delta'_n) \cong \ker(\delta_{n+1})/\text{im}(\delta_n) \]
Our Dream

\[ h^n(X) \cong H^n(X; h^0(2)) \]

\[ \text{Ker}(\delta_{n+1})/\text{im}(\delta'_n) \cong \text{Ker}(\delta_{n+1})/\text{im}(\delta_n) \]

1. Find \( \delta' \) such that \( h^n(X) \cong \text{Ker}(\delta'_{n+1})/\text{im}(\delta'_n) \)

   done and fully mechanized in Agda

2. Show \( \delta \) and \( \delta' \) are equivalent

   domains and codomains are isomorphic

   commutativity in progress
Our Dream: Step 1 (done!)

For any pointed CW-complex $X$ where

1. all cell sets $A_n$ satisfy set-level AC and
2. the point of $A_0$ is separable (pt = $x$ is decidable)

there exist homomorphisms $\delta'$

$$\delta'_{n+2} \preceq \delta'_{n+1} \preceq \delta'_n \preceq \delta'_{n-1} \preceq \cdots$$

such that

$$h^n(X) \simeq \text{kernel of } \delta'_{n+1} / \text{image of } \delta'_n$$
Important Lemmas for Step 1

Long exact sequences

\[ h^n(A) \xleftarrow{f} h^n(B) \]

A \xrightarrow{f} B

\[ n^{++} \]

1 \xrightarrow{} \text{Cof}_f \xrightarrow{} h^n(\text{Cof}_f)
Important Lemmas for Step 1

Long exact sequenes

\[ h^n(A) \xleftarrow{f} h^n(B) \]

\[ A \xrightarrow{f} B \]

\[ A \xrightarrow{n} B \]

\[ 1 \rightarrow \text{Cof}_f h^n(\text{Cof}_f) \]

Wedges of cells

\[ h^m(X_n/X_{n-1}) \cong \text{hom}(\mathbb{Z}[A_n], h^0(2)) \]
when \( m = n \) or trivial otherwise

\[ h^m(X_0) \cong \text{hom}(\mathbb{Z}[A_0\setminus\{pt\}], h^0(2)) \]
when \( m = 0 \) or trivial otherwise
Important Lemmas for Step 1

Long exact sequences

\[ h^n(A) \xleftarrow{f} h^n(B) \]

A \xrightarrow{n++} f \xrightarrow{} B

1 \xrightarrow{} \text{Cof}_f \xrightarrow{} h^n(\text{Cof}_f)

Wedges of cells

\[ h^m(X_n/X_{n-1}) \approx \text{hom}(\mathbb{Z}[A_n], \, h^0(2)) \]

when \( m = n \) or trivial otherwise

\[ h^m(X_0) \approx \text{hom}(\mathbb{Z}[A_0 \setminus \{\text{pt}\}], \, h^0(2)) \]

when \( m = 0 \) or trivial otherwise

trivial if \( m \neq n \)
Ultimate Cofiber Diagram

\[ X_0 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow X_n \rightarrow X_{n+1} \rightarrow \cdots \]

\[ 1 \rightarrow \cdots \rightarrow X_{n-1/0} \rightarrow X_{n/0} \rightarrow X_{n+1/0} \rightarrow \cdots \]

\[ X_{n/m} := \frac{X_n}{X_m} \]

\[ 1 \rightarrow X_{n/n-1} \rightarrow X_{n+1/n-1} \rightarrow \cdots \]

\[ 1 \rightarrow X_{n+1/n} \rightarrow \cdots \]
Ultimate Cofiber Diagram

\[ X_0 \to \cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots \]

\[ 1 \to \cdots \to X_{n-1/0} \to X_{n/0} \to X_{n+1/0} \to \cdots \]

\[ X_{n/m} := X_n/X_m \]

Plan:
Obtain long exact sequences and use group-theoretic magic
Obtain long exact sequences and use group-theoretic magic

**Plan:**

$X_{n/m} := X_n/X_m$
$X_{n/n-1} \rightarrow X_{n+1/n-1}$

$1 \rightarrow X_{n+1/n}$
$X_{n/n-1} \rightarrow X_{n+1/n-1}$

$hn(X_{n+1/n}) \rightarrow hn(X_{n+1/n-1}) \rightarrow hn(X_{n/n-1})$

$h^{n+1}(X_{n+1/n}) \rightarrow h^{n+1}(X_{n+1/n-1}) \rightarrow h^{n+1}(X_{n/n-1})$
our choice of $\delta'$

$h^n(X_{n+1/n}) \rightarrow h^n(X_{n+1/n-1}) \rightarrow h^n(X_{n/n-1})$

$h^{n+1}(X_{n+1/n}) \rightarrow h^{n+1}(X_{n+1/n-1}) \rightarrow h^{n+1}(X_{n/n-1})$
Our choice of $\delta'$ is trivial.

\[
\begin{align*}
\text{trivial} \\
\tau^n(X_{n+1/n}) &\longrightarrow \tau^n(X_{n+1/n-1}) &\longrightarrow \tau^n(X_{n/n-1}) \\
\text{our choice of } \delta' \\
\tau^{n+1}(X_{n+1/n}) &\longrightarrow \tau^{n+1}(X_{n+1/n-1}) &\longrightarrow \tau^{n+1}(X_{n/n-1}) \\
\text{trivial}
\end{align*}
\]
\[ X_{n/n-1} \rightarrow X_{n+1/n-1} \]

\[ 1 \rightarrow X_{n+1/n} \]

\[ \ker(\delta') \]

\[ \text{trivial} \]

\[ h^n(X_{n+1/n}) \xrightarrow{} h^n(X_{n+1/n-1}) \xrightarrow{} h^n(X_{n/n-1}) \]

\[ h^{n+1}(X_{n+1/n}) \xrightarrow{\text{surj}} h^{n+1}(X_{n+1/n-1}) \xrightarrow{\text{inj}} h^{n+1}(X_{n/n-1}) \]

\[ \text{our choice of } \delta' \]

\[ \text{coker}(\delta') \]

\[ \approx \approx \]
\[ X_m \rightarrow X_{m+1} \]

\[ 1 \rightarrow X_{m+1/m} \]
\[ h^n(X_{m+1/m}) \rightarrow h^n(X_{m+1}) \rightarrow h^n(X_m) \rightarrow h^{n+1}(X_{m+1/m}) \]
If \( n \neq m, m+1 \), both ends trivial, \( h^n(X_{m+1}) \approx h^n(X_m) \).
If \( n \neq m, m+1 \), both ends trivial, \( h^n(X_{m+1}) \approx h^n(X_m) \)

three possible values

\[
\begin{align*}
&h^n(X_{n-1}) \approx h^n(X_{n-2}) \approx \cdots \approx h^n(X_0), \text{ trivial} \\
&h^n(X_n) \\
&h^n(X_{n+1}) \approx h^n(X_{n+2}) \approx \cdots \approx h^n(X)
\end{align*}
\]
\[
\text{coker}(\delta'_{n}) \cong
\begin{align*}
&h^{n}(X_{n/n-2}) \leftrightarrow h^{n}(X_{n+1/n-2}) \\
&\cong h^{n}(X) \\
&\cong \ker(\delta'_{n+1})
\end{align*}
\]
\[ \text{coker}(\delta'_n) \leftarrow h^n(X) \]

\[ \text{eq. class} \]

\[ h^n(X_{n/n-1}) \leftarrow \ker(\delta'_{n+1}) \]

\[ \text{inj} \]
Using group-theoretic magic...

\[ h^n(X) \cong \ker(\delta'_n+1)/\im(\delta'_n) \]
Our Dream (updated)

\[ h^n(X) \cong H^n(X; h^0(2)) \]

\[ \ker(\delta'_{n+1})/\text{im}(\delta'_n) \quad \ker(\delta_{n+1})/\text{im}(\delta_n) \]

1. Find \( \delta' \) such that \( h^n(X) \cong \ker(\delta'_{n+1})/\text{im}(\delta'_n) \)

2. Show \( \delta \) and \( \delta' \) are equivalent
   domains and codomains are isomorphic
   commutativity in progress
Cohomology Groups

Cellular coh. for pointed CW complexes  Ordinary reduced cohomology theories

Dream: prove they give the same groups
We made an important step in proving it