


Cellular Cohomology

In Homotopy Type Theory



20180709

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Cohomology Groups

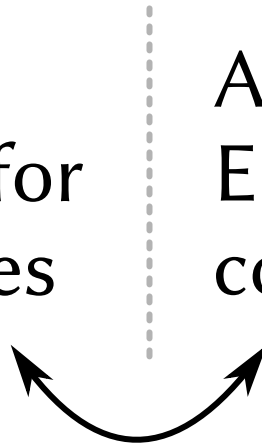
{ mappings from holes in a space }

Cohomology Groups

{ mappings from holes in a space }

Cellular
cohomology for
CW complexes

Axiomatic
Eilenberg-Steenrod
cohomology



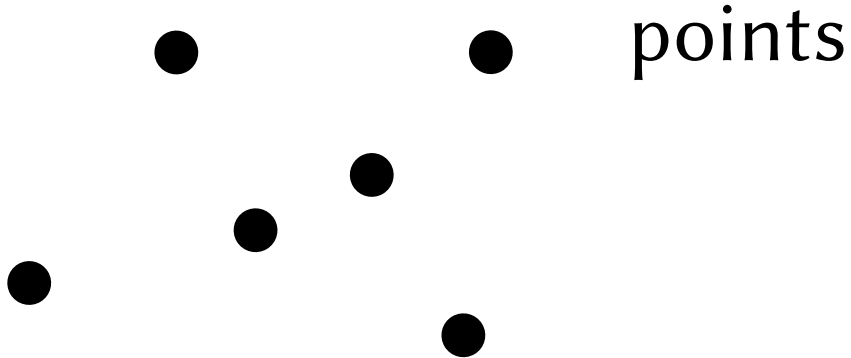
Goal: prove they are the same!

CW complexes

inductively-defined spaces

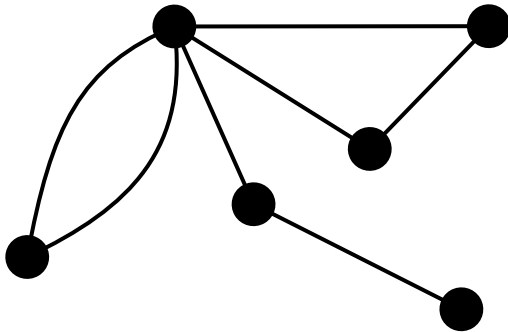
CW complexes

inductively-defined spaces



CW complexes

inductively-defined spaces

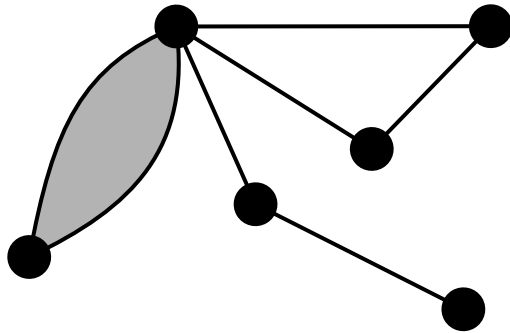


points

lines

CW complexes

inductively-defined spaces



points

lines

faces

(and more...)

Data: cells and how they attach

CW complexes

Sets of cell indices: A_n

Attaching: $\alpha_{n+1} : A_{n+1} \times S^n \rightarrow X_n$

X_n is the construction up to dim. n

$$X_0 := A_0$$

$$X_{n+1} :=$$

$$\begin{array}{ccc} A_{n+1} \times S^n & \longrightarrow & A_{n+1} \\ \alpha_{n+1} \downarrow & & \downarrow \\ X_n & \longrightarrow & X_{n+1} \end{array}$$

CW complexes

Sets of cell indices: A_n

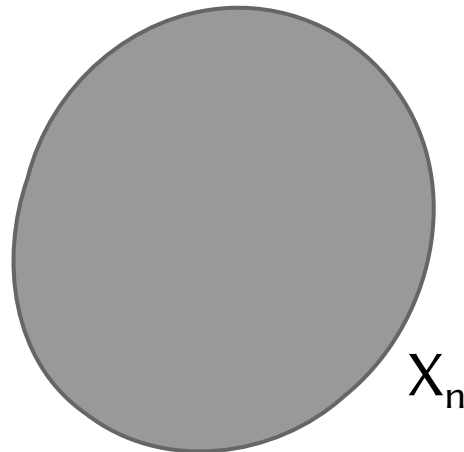
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$$\begin{array}{ccc} A_{n+1} \times S^n & \longrightarrow & A_{n+1} \\ \alpha_{n+1} \downarrow & & \downarrow \Gamma \\ X_n & \longrightarrow & X_{n+1} \end{array}$$



CW complexes

Sets of cell indices: A_n

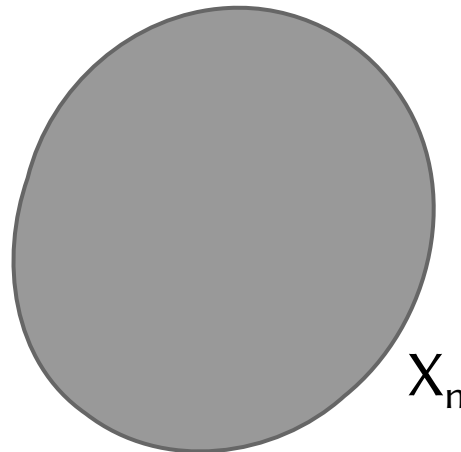
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$$\bullet a : A_{n+1}$$

CW complexes

Sets of cell indices: A_n

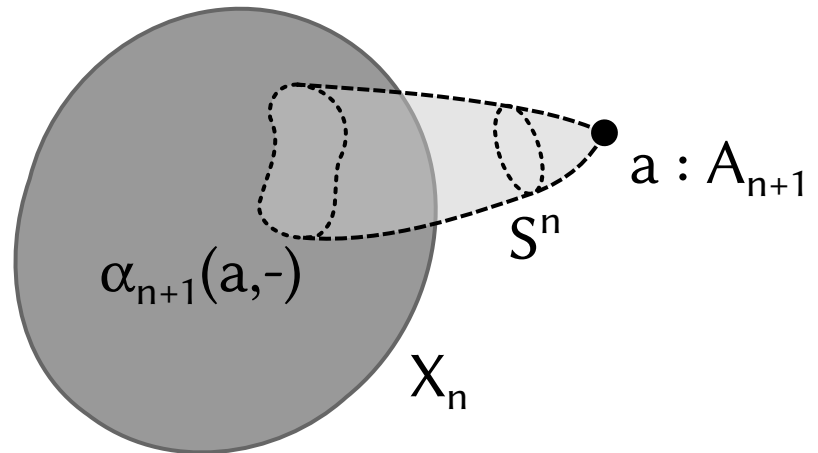
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Cellular Cohomology

{ mappings from holes in a space }



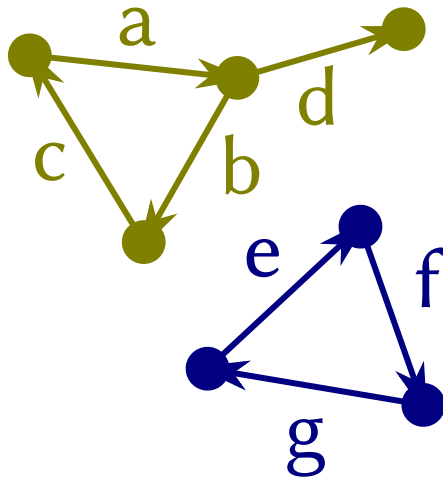
dualize

Cellular Homology

{ holes in a space }

One-Dimensional Holes*

{ elements of $Z[A_1]$ forming cycles }



holes

$$a + b + c$$

$$-a - b - c$$

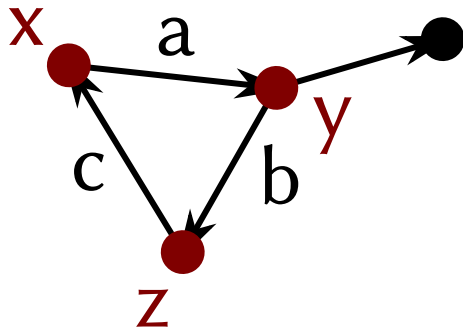
$$a + b + c + e + g + f$$

...

*Holes are cycles in the classical homology theory

One-Dimensional Holes

{ elements of $Z[A_1]$ forming cycles }



boundary function ∂

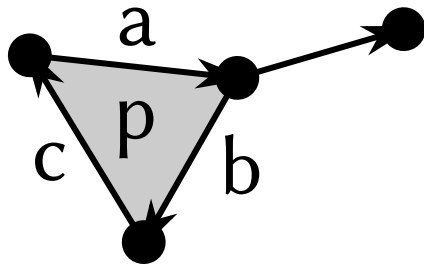
$$\partial \left(\begin{array}{c} \bullet \\ x \end{array} \xrightarrow{a} \begin{array}{c} \bullet \\ y \end{array} \right) = y - x$$

$$\begin{aligned} \partial(a+b+c) &= (y - x) + (z - y) \\ &\quad + (x - z) = 0 \end{aligned}$$

set of holes = kernel of ∂

First Homology Groups

{ unfilled one-dimensional holes }



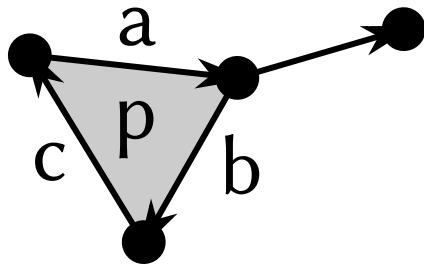
2-dim. boundary function ∂_2

$$\partial_2 \left(\begin{array}{c} \text{a} \\ \text{p} \\ \text{c} \quad \text{b} \end{array} \right) = a + b + c$$

filled holes = image of ∂_2

First Homology Groups

{ unfilled one-dimensional holes }



2-dim. boundary function ∂_2

$$\partial_2 \left(\begin{array}{c} \text{a} \\ \text{p} \\ \text{c} \quad \text{b} \end{array} \right) = a + b + c$$

A diagram showing a triangle with vertices and edges. The top edge is labeled 'a', the right edge is labeled 'b', and the left edge is labeled 'c'. A point 'p' is located inside the triangle. Red dashed lines and arrows indicate a counter-clockwise orientation: from top-left to top-right (a), from top-right to bottom (b), and from bottom to top-left (c).

filled holes = image of ∂_2

$$H_1(X) := \text{kernel of } \partial_1 \ / \ \text{image of } \partial_2$$

(unfilled holes) (all holes) (filled holes)

Homology Groups

{ unfilled holes }

$C_n := Z[A_n]$ formal sums of cells (chains)

$$\dots \rightarrow C_{n+2} \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \rightarrow \dots$$

$H_n(X) := \text{kernel of } \partial_n / \text{image of } \partial_{n+1}$

Cohomology Groups

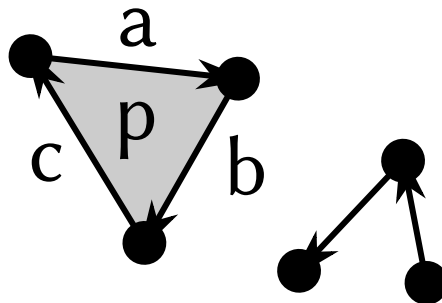
$$\dots \longrightarrow C_{n+2} \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \longrightarrow \dots$$

Dualize by $\text{Hom}(-, G)$. Let $C^n = \text{Hom}(C_n, G)$.

$$\dots \longleftarrow C^{n+2} \xleftarrow{\delta_{n+2}} C^{n+1} \xleftarrow{\delta_{n+1}} C^n \xleftarrow{\delta_n} C^{n-1} \xleftarrow{\delta_{n-1}} C^{n-2} \longleftarrow \dots$$

$H^n(X; G) := \text{kernel of } \delta_{n+1} / \text{image of } \delta_n$

2-Dimensional Boundary

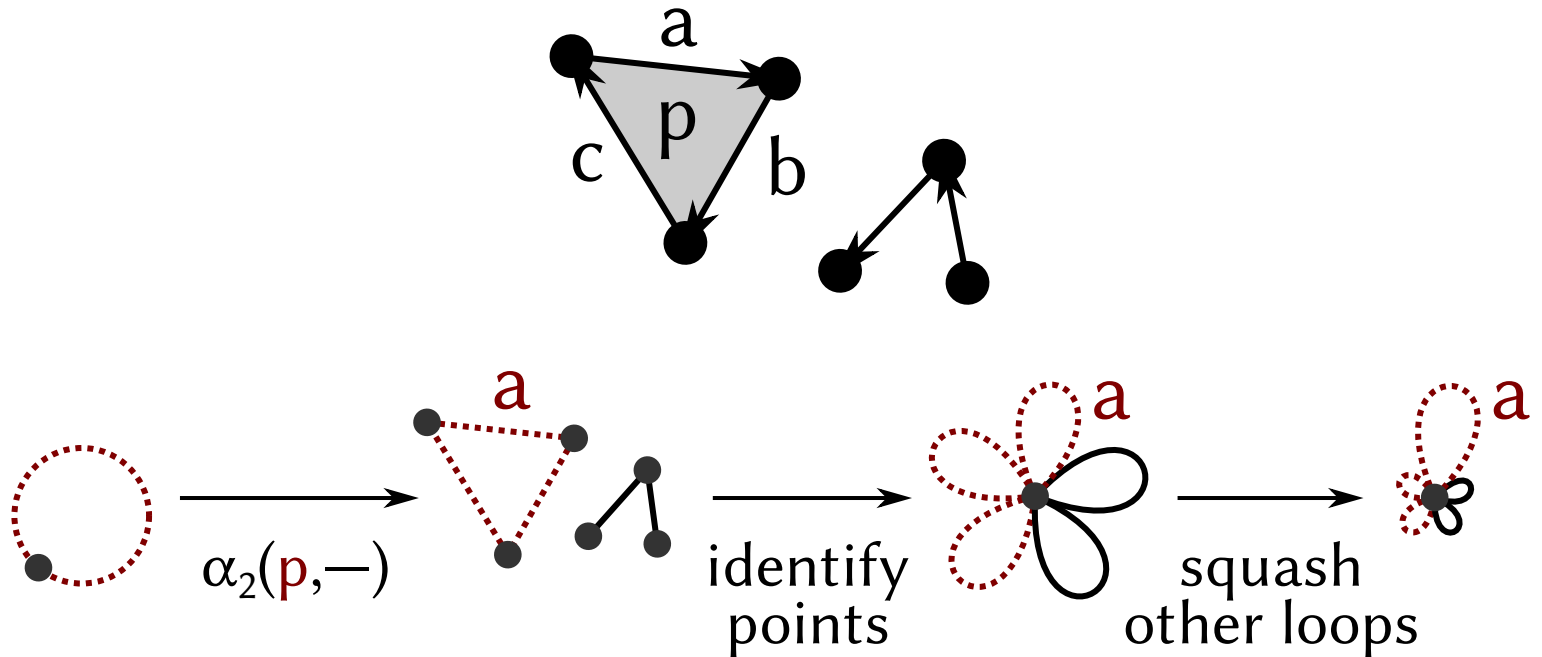


$$\partial_2(\text{triangle}) = a + b + c$$

The diagram shows a triangle with vertices marked by red dots and edges labeled 'a', 'b', and 'c' with red arrows pointing counter-clockwise. The interior is labeled 'p'. This triangle is part of the equation above, representing the boundary of a 2D region.

How to compute the coefficients from α_2 ?

2-Dimensional Boundary



coefficient = winding number of this map
(can be generalized to higher dimensions)

Cohomology Groups

{ mappings from holes in a space }

Cellular
cohomology for
CW-complexes
 $H^n(X; G)$

Axiomatic
Eilenberg-Steenrod
cohomology



Prove they are the same!

Eilenberg-Steenrod* cohomology

A family of functors $h^n(-)$:
pointed spaces \rightarrow abelian groups

*ordinary, reduced

Eilenberg-Steenrod* cohomology

A family of functors $h^n(-)$:
pointed spaces \rightarrow abelian groups

1. $h^{n+1}(\text{susp}(X)) \cong h^n(X)$, natural in X

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Eilenberg-Steenrod* cohomology

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2.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \text{Cof}_f \end{array}$$

*ordinary, reduced

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2.
$$\begin{array}{ccc}
 h^n(A) & \longleftarrow \cdots & h^n(B) \\
 \downarrow & \xrightarrow{f} & \downarrow \\
 1 & \longrightarrow & \text{Cof}_f \\
 & & \uparrow \text{exact!} \\
 & & h^n(\text{Cof}_f)
 \end{array}$$

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 \end{array}$$

3. $h^n(\bigvee_i X_i) \cong \prod_i h^n(X_i)$
if the index type is nice enough**

*ordinary, reduced

**see our paper

Eilenberg-Steenrod* cohomology

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 \end{array}$$

3. $h^n(\bigvee_i X_i) \cong \prod_i h^n(X_i)$
if the index type
is nice enough**

4. $h^n(2)$ trivial for $n \neq 0$

*ordinary, reduced

**see our paper

Cohomology Groups

{ mappings from holes in a space }

Cellular
cohomology for
CW-complexes

$$H^n(X; G)$$

Axiomatic
Eilenberg-Steenrod
cohomology

$$h^n(X)$$



Prove they are the same!

Proof Plan

$$H^n(X; h^0(2)) \stackrel{?}{\simeq} h^n(X)$$

Proof Plan

$$\begin{array}{ccc} H^n(X; h^0(2)) & \stackrel{?}{\cong} & h^n(X) \\ \text{ii} & & \text{R} \\ \ker(\delta_{n+1})/\text{im}(\delta_n) & & \ker(\delta'_{n+1})/\text{im}(\delta'_n) \end{array}$$

1. Find δ' such that $h^n(X) \cong \ker(\delta'_{n+1})/\text{im}(\delta'_n)$

Proof Plan

$$\begin{array}{ccc} H^n(X; h^0(2)) & \stackrel{?}{\cong} & h^n(X) \\ \text{ii} & & \text{R} \\ \ker(\delta_{n+1})/\text{im}(\delta_n) & & \ker(\delta'_{n+1})/\text{im}(\delta'_n) \\ & \xleftrightarrow{\delta \cong \delta'} & \end{array}$$

1. Find δ' such that $h^n(X) \cong \ker(\delta'_{n+1})/\text{im}(\delta'_n)$
2. Show δ and δ' are equivalent

Proof Plan

$$\begin{array}{ccc} H^n(X; h^0(2)) & \stackrel{?}{\cong} & h^n(X) \\ \text{ii} & & \text{R} \\ \ker(\delta_{n+1})/\text{im}(\delta_n) & & \ker(\delta'_{n+1})/\text{im}(\delta'_n) \\ & \xleftrightarrow{\delta \cong \delta'} & \end{array}$$

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2. Show δ and δ' are equivalent

As usual, fully mechanized in Agda!

Step 1: Reverse Engineering

For any theory h , finite pointed CW-complex X , there exist homomorphisms δ'

$$\dots \longleftarrow D^{n+2} \xleftarrow{\delta'_{n+2}} D^{n+1} \xleftarrow{\delta'_{n+1}} D^n \xleftarrow{\delta'_n} D^{n-1} \xleftarrow{\delta'_{n-1}} D^{n-2} \longleftarrow \dots$$

such that

$$h^n(X) \cong \text{kernel of } \delta'_{n+1} / \text{image of } \delta'_n$$

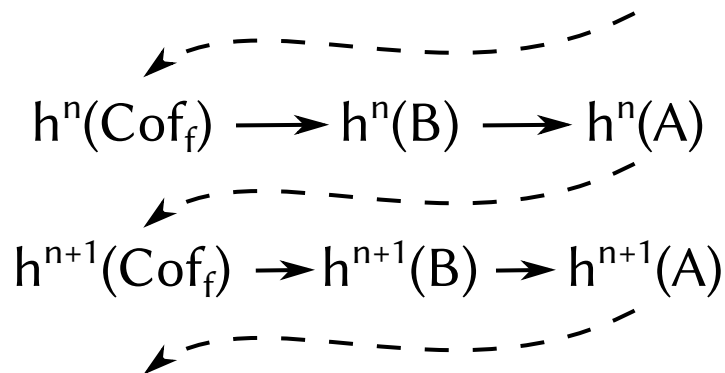
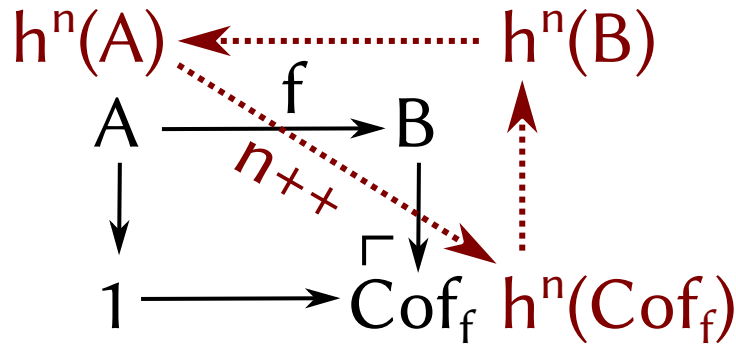
Important Lemmas for Step 1

Long exact sequences

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \text{Cof}_f \end{array}$$

Important Lemmas for Step 1

Long exact sequences



Important Lemmas for Step 1

Long exact sequences

$$\begin{array}{ccccc}
 h^n(A) & \longleftarrow & & & h^n(B) \\
 \downarrow & \nearrow f & \longrightarrow & & \downarrow \\
 A & \xrightarrow{\quad} & B & & \\
 \downarrow & \searrow n_{++} & \downarrow \Gamma & & \uparrow \\
 1 & \longrightarrow & \text{Cof}_f & & h^n(\text{Cof}_f)
 \end{array}$$

Wedges of cells

$$h^m(X_n/X_{n-1}) \cong \text{hom}(Z[A_n], h^0(2))$$

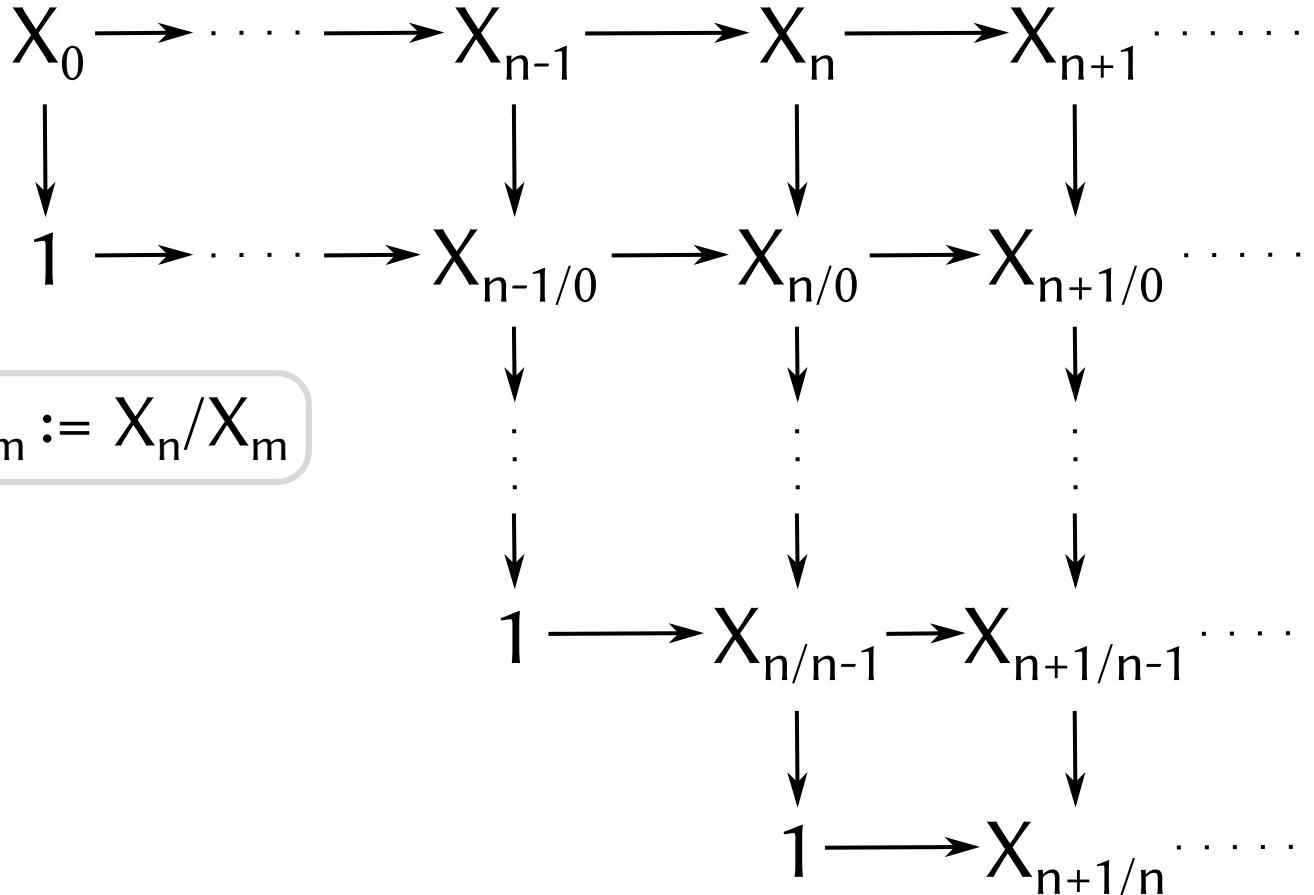
when $m = n$ or trivial otherwise

$$h^m(X_0) \cong \text{hom}(Z[A_0 \setminus \{\text{pt}\}], h^0(2))$$

when $m = 0$ or trivial otherwise

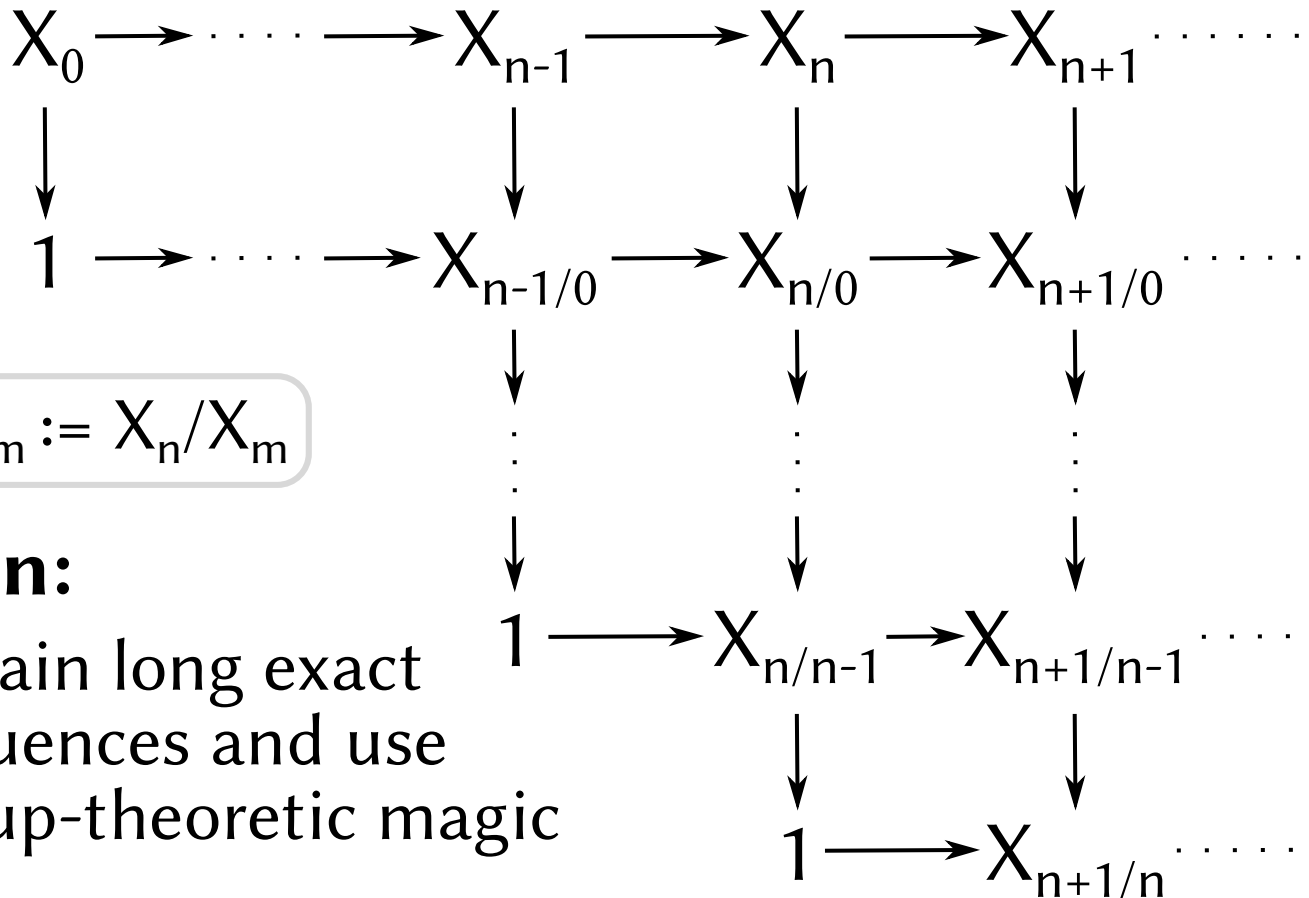
trivial if
 $m \neq n$

Ultimate Cofiber Diagram



$$X_{n/m} := X_n / X_m$$

Ultimate Cofiber Diagram

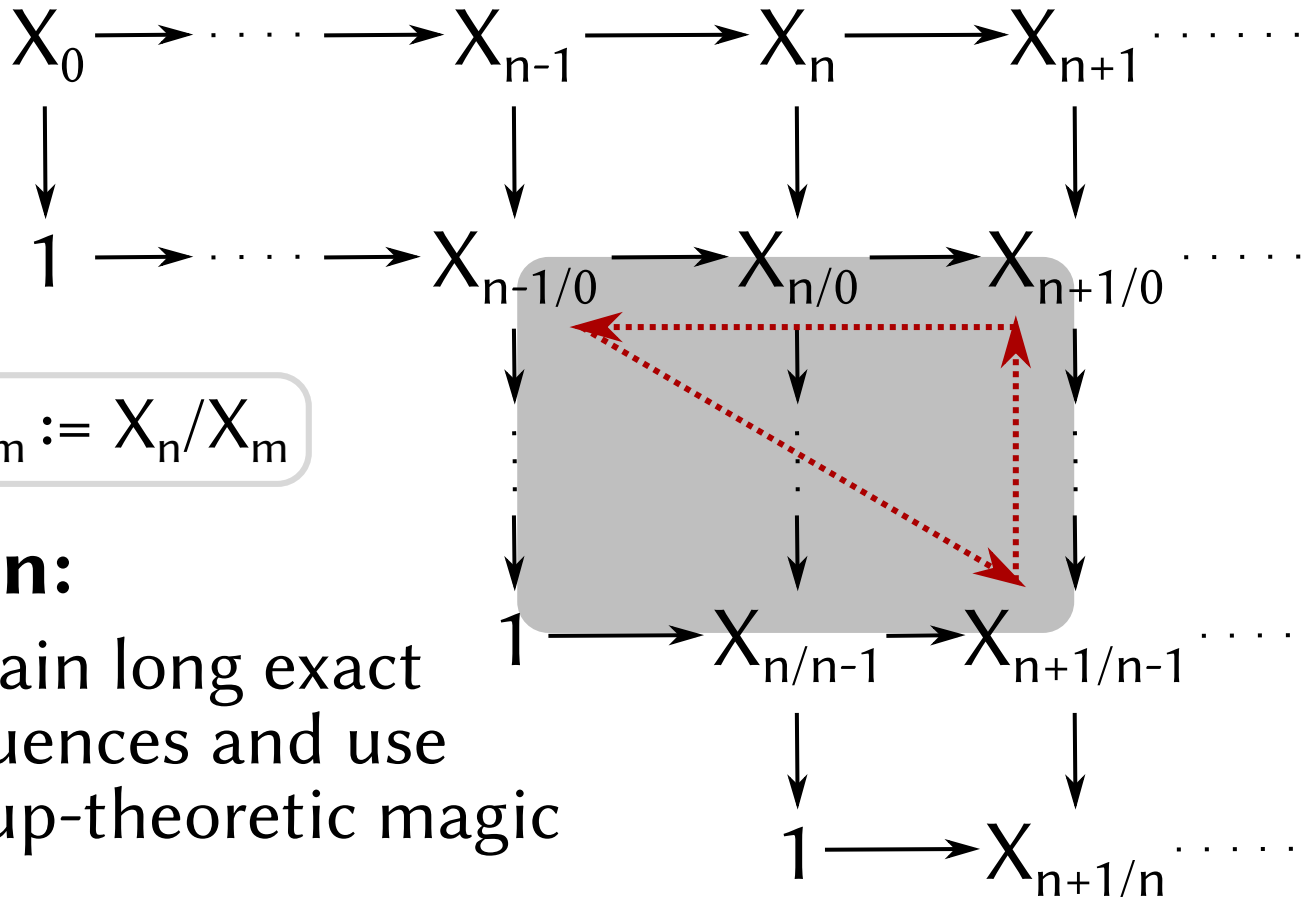


$$X_{n/m} := X_n / X_m$$

Plan:

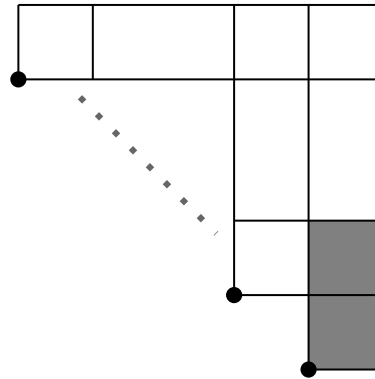
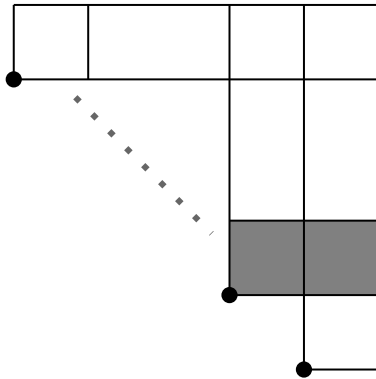
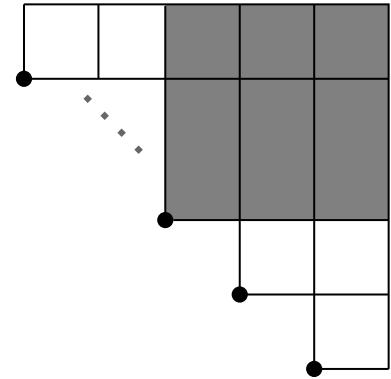
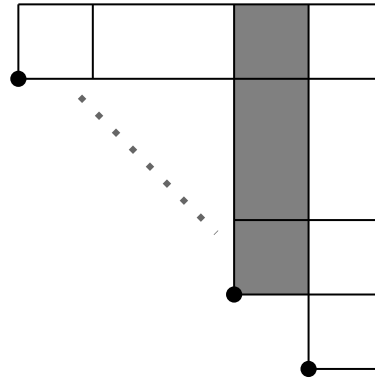
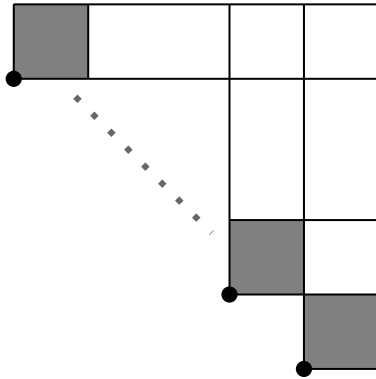
Obtain long exact sequences and use group-theoretic magic

Ultimate Cofiber Diagram



Plan:

Obtain long exact sequences and use group-theoretic magic



+ group theory

$$h^n(X) \cong \ker(\delta'_{n+1})/\text{im}(\delta'_n)$$

Proof Plan (updated)

$$\begin{array}{ccc} H^n(X; h^0(2)) & \stackrel{?}{\cong} & h^n(X) \\ \text{ii} & & \text{R} \\ \ker(\delta_{n+1})/\text{im}(\delta_n) & & \ker(\delta'_{n+1})/\text{im}(\delta'_n) \\ & \xleftrightarrow{\delta \cong \delta'} & \end{array}$$

1. Find δ' such that $h^n(X) \cong \ker(\delta'_{n+1})/\text{im}(\delta'_n)$
2. Show δ and δ' are equivalent

Step 2: Calculation

$$\begin{array}{ccccccccc}
 \dots & \leftarrow & C^{n+2} & \xleftarrow{\delta_{n+2}} & C^{n+1} & \xleftarrow{\delta_{n+1}} & C^n & \xleftarrow{\delta_n} & C^{n-1} & \xleftarrow{\delta_{n-1}} & C^{n-2} & \leftarrow & \dots \\
 & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \\
 \dots & \leftarrow & D^{n+2} & \xleftarrow{\delta'_{n+2}} & D^{n+1} & \xleftarrow{\delta'_{n+1}} & D^n & \xleftarrow{\delta'_n} & D^{n-1} & \xleftarrow{\delta'_{n-1}} & D^{n-2} & \leftarrow & \dots
 \end{array}$$

The $n=0$ case ($C^0 \cong D^0$) is interesting

Summary

Cellular
cohomology
groups \cong
(finite) Ordinary reduced
cohomology
groups

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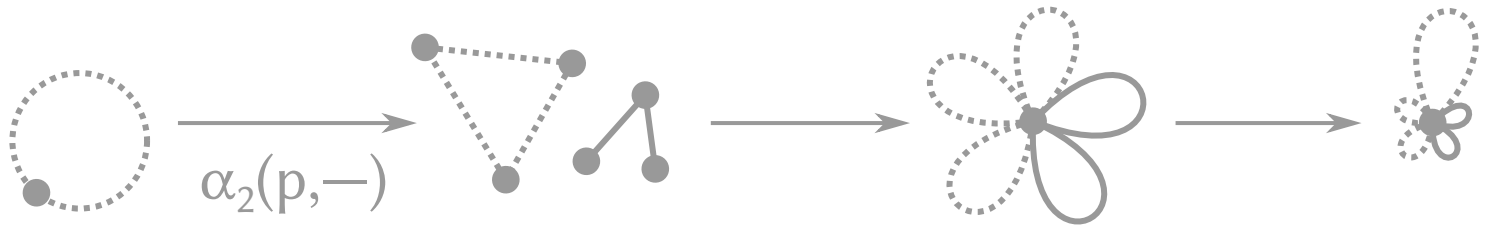
To-Do

Infinity: colimits

Homology \Rightarrow Poincaré duality, ...

Parametrization \Rightarrow non-orientability, ...

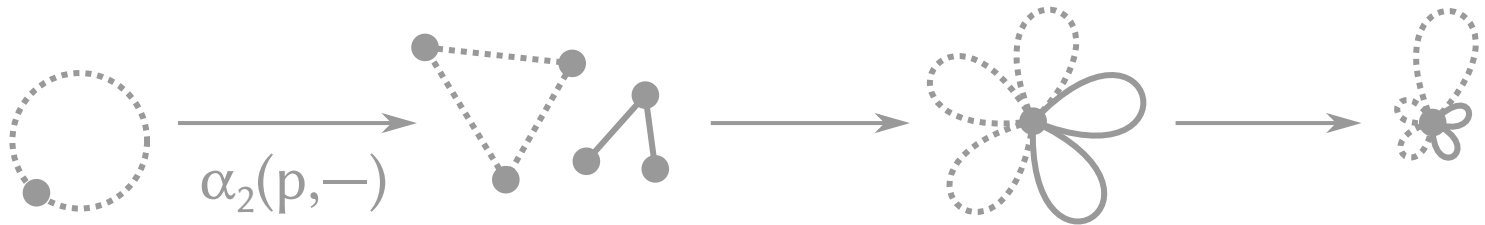
Higher-Dim. Boundary



$$\begin{array}{ccccccc}
 S^n & \longrightarrow & X_n & \longrightarrow & X_n/X_{n-1} \cong \vee S^n & \longrightarrow & S^n \\
 \alpha_{n+1}(p, -) & & & \text{identify} & & \text{squash} & \\
 & & & \text{lower structs.} & & &
 \end{array}$$

coefficient = degree of this map

Higher-Dim. Boundary



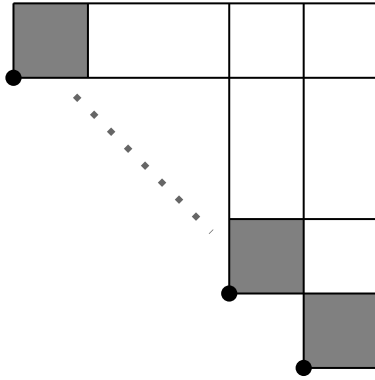
$$S^n \xrightarrow{\alpha_{n+1}(p, -)} X_n \xrightarrow[\text{lower structs.}]{\text{identify}} X_n/X_{n-1} \cong \vee S^n \xrightarrow{\text{squash}} S^n$$

coefficient = *degree* of this map

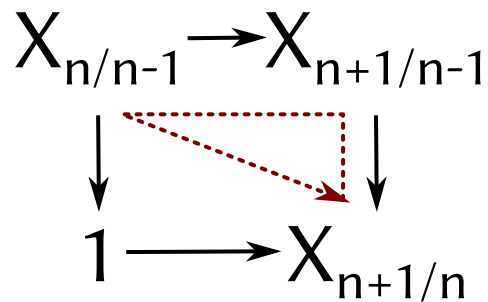
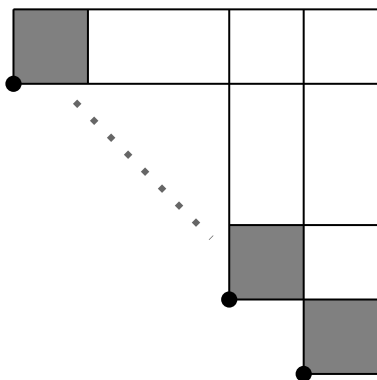
- squashing needs decidable equality
- linear sum needs closure-finiteness (free for finite cases)

Higher-Dim. Boundary

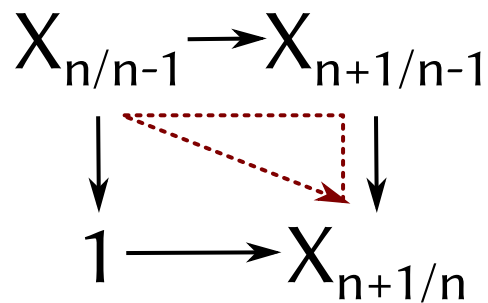
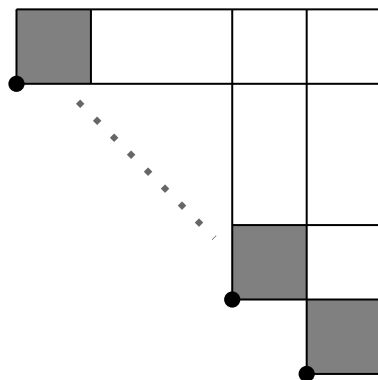
$$\begin{array}{ccccc}
 & & S^n & & \\
 & & \vdots & & \\
 A_n \times S^{n-1} & \longrightarrow & A_n & \xrightarrow{A_{n+1} \times S^n} & A_{n+1} \\
 \downarrow & & \downarrow & \nearrow & \downarrow \\
 X_{n-1} & \longrightarrow & X_n & \longrightarrow & X_{n+1} \\
 \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & X_n / X_{n-1} \cong VS^n & & \\
 & & \downarrow & & \\
 & & S^n & &
 \end{array}$$



$$\begin{array}{ccc}
 X_{n/n-1} & \longrightarrow & X_{n+1/n-1} \\
 \downarrow & & \downarrow \\
 1 & \longrightarrow & X_{n+1/n}
 \end{array}$$



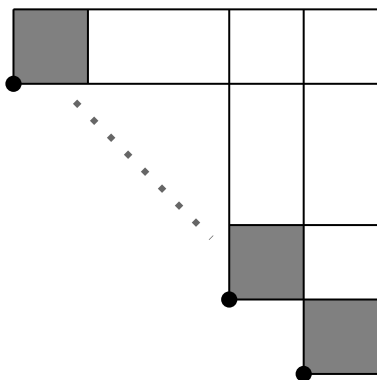
$$\begin{array}{c}
 h^n(X_{n+1/n}) \longrightarrow h^n(X_{n+1/n-1}) \longrightarrow h^n(X_{n/n-1}) \\
 \swarrow \text{dashed arrow} \\
 h^{n+1}(X_{n+1/n}) \longrightarrow h^{n+1}(X_{n+1/n-1}) \longrightarrow h^{n+1}(X_{n/n-1})
 \end{array}$$



$$h^n(X_{n+1/n}) \longrightarrow h^n(X_{n+1/n-1}) \longrightarrow h^n(X_{n/n-1})$$

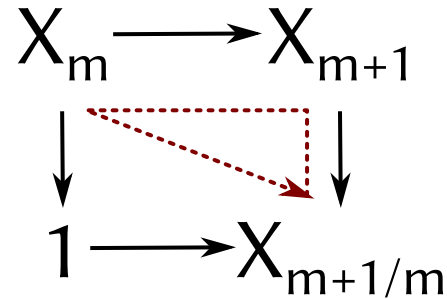
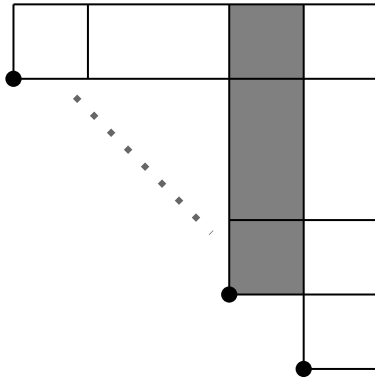
our choice of δ'

$$h^{n+1}(X_{n+1/n}) \longrightarrow h^{n+1}(X_{n+1/n-1}) \longrightarrow h^{n+1}(X_{n/n-1})$$



$$\begin{array}{ccc}
 X_{n/n-1} & \longrightarrow & X_{n+1/n-1} \\
 \downarrow & \searrow \text{dotted red arrow} & \downarrow \\
 1 & \longrightarrow & X_{n+1/n}
 \end{array}$$

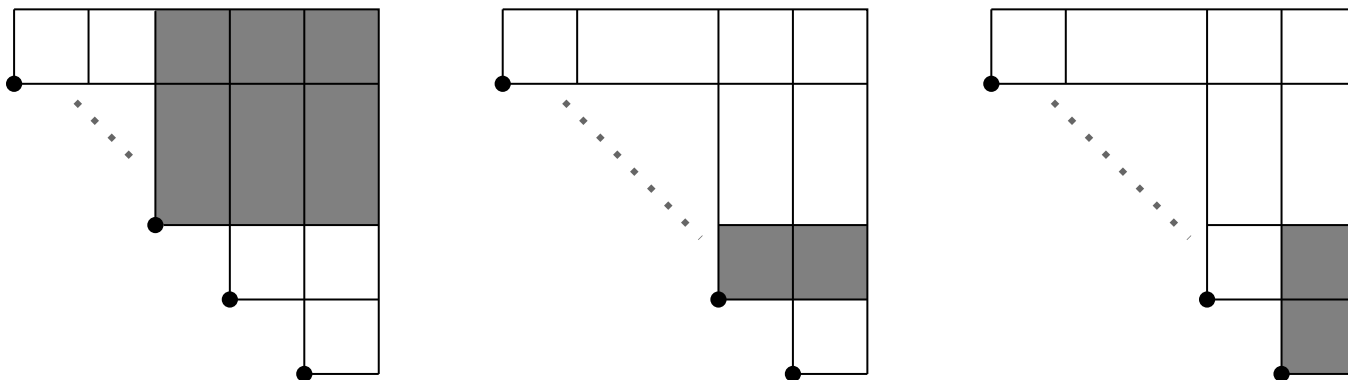
$$\begin{array}{ccccc}
 & & \text{ker}(\delta') & & \\
 & \text{trivial} & \cong & & \\
 h^n(X_{n+1/n}) & \longrightarrow & h^n(X_{n+1/n-1}) & \xrightarrow{\text{inj}} & h^n(X_{n/n-1}) \\
 & \swarrow \text{dashed arrow} & & & \\
 & \text{our choice of } \delta' & & & \\
 h^{n+1}(X_{n+1/n}) & \xrightarrow{\text{surj}} & h^{n+1}(X_{n+1/n-1}) & \longrightarrow & h^{n+1}(X_{n/n-1}) \\
 & & \cong & & \text{trivial} \\
 & & \text{coker}(\delta') & &
 \end{array}$$



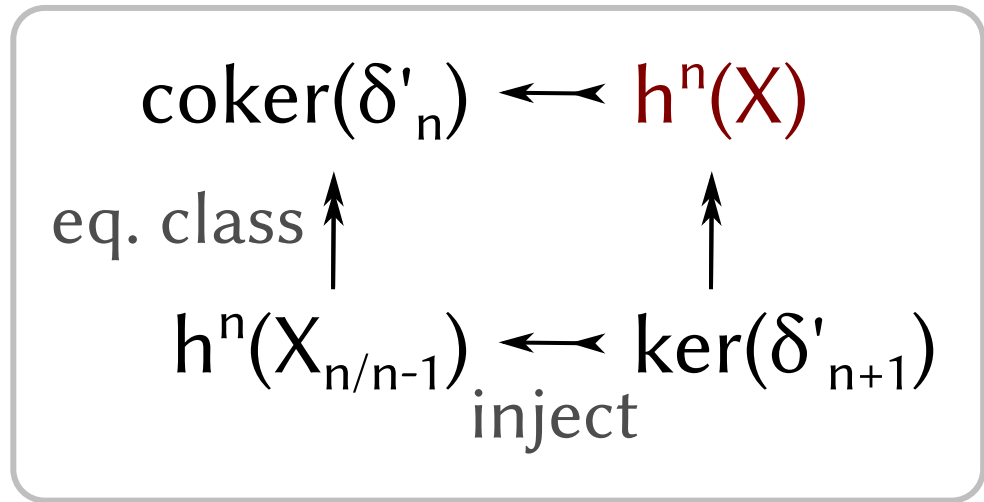
$$h^n(X_{m+1/m}) \longrightarrow h^n(X_{m+1}) \longrightarrow h^n(X_m) \longrightarrow h^{n+1}(X_{m+1/m})$$

If $n \neq m, m+1$, both ends trivial, $h^n(X_{m+1}) \cong h^n(X_m)$

three possible values $\left\{ \begin{array}{l} h^n(X_{n-1}) \cong h^n(X_{n-2}) \cong \dots \cong h^n(X_0), \text{ trivial} \\ h^n(X_n) \\ h^n(X_{n+1}) \cong h^n(X_{n+2}) \cong \dots \cong h^n(X) \end{array} \right.$



$$\begin{array}{ccc}
 \text{coker}(\delta'_n) \cong & & \\
 h^n(X_{n/n-2}) \longleftarrow h^n(X_{n+1/n-2}) & \cong h^n(X) & \\
 \text{eq. class} \uparrow & \uparrow & \\
 h^n(X_{n/n-1}) \longleftarrow h^n(X_{n+1/n-1}) & \cong \ker(\delta'_{n+1}) & \\
 \text{inject} & &
 \end{array}$$



Chasing the diagram,

$$\text{h}^n(X) \cong \text{ker}(\delta'_{n+1})/\text{im}(\delta'_n)$$