Cellular Cohomology

In Homotopy Type Theory

20180709

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Cohomology Groups

{ mappings from holes in a space }

Cohomology Groups

{ mappings from holes in a space }

Cellular cohomology for CW complexes

Axiomatic Eilenberg-Steenrod cohomology

Goal: prove they are the same!

CW complexes

inductively-defined spaces

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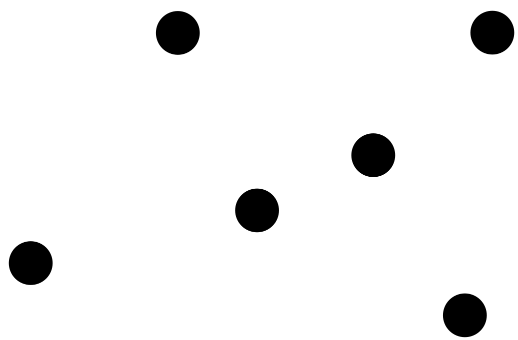
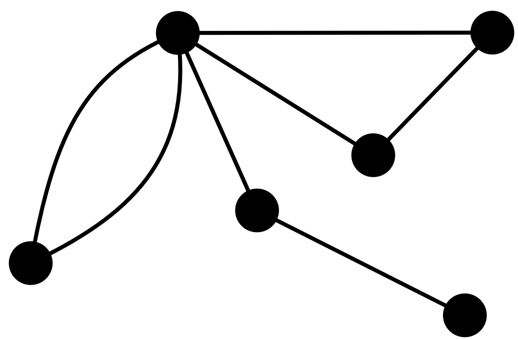
inductively-defined spaces



points

CW complexes

inductively-defined spaces

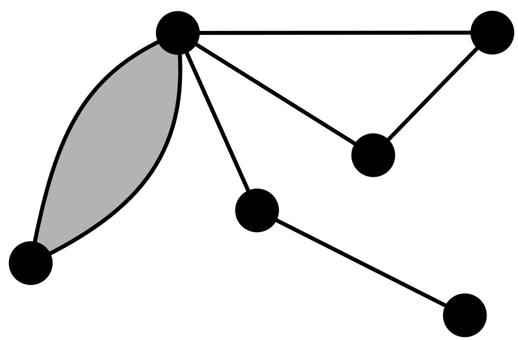


points

lines

CW complexes

inductively-defined spaces



points

lines

faces

(and more...)

Data: cells and how they attach

CW complexes

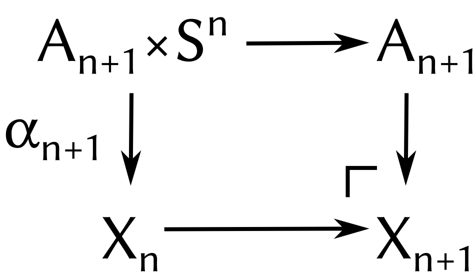
Sets of cell indices: A_n

Attaching: $\alpha_{n+1} : A_{n+1} \times S^n \longrightarrow X_n$

X_n is the construction up to dim. n

$X_0 := A_0$

$X_{n+1} :=$



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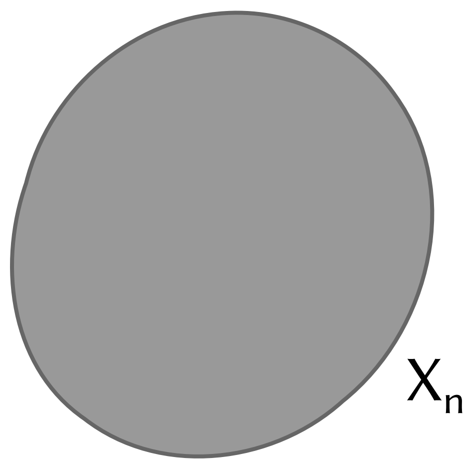

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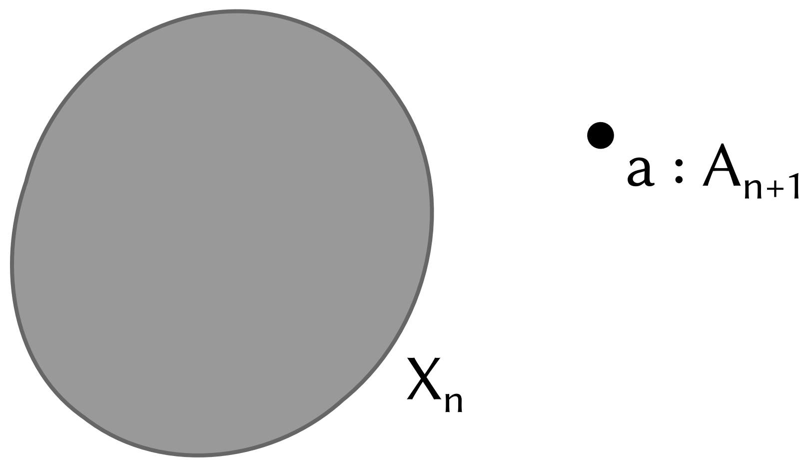

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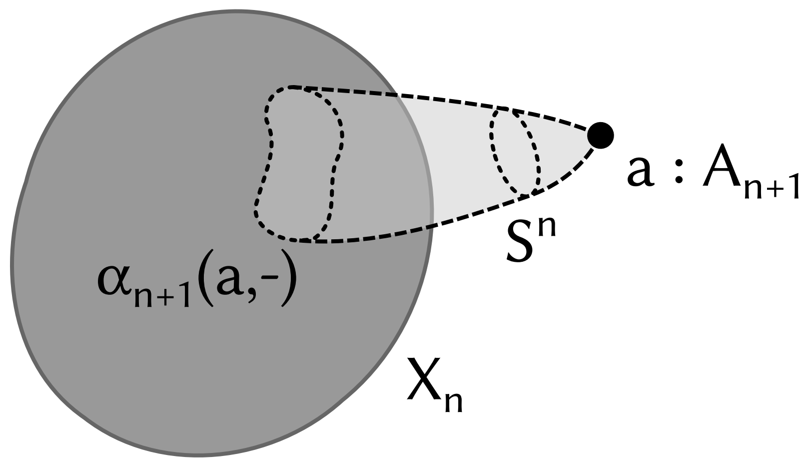


Cellular Cohomology

{ mappings from holes in a space }

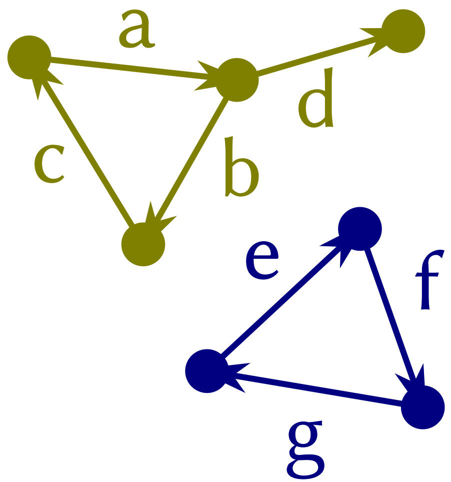
↑ dualize

Cellular Homology

{ holes in a space }

One-Dimensional Holes*

{ elements of $Z[A_1]$ forming cycles }



holes

a + b + c

- a - b - c

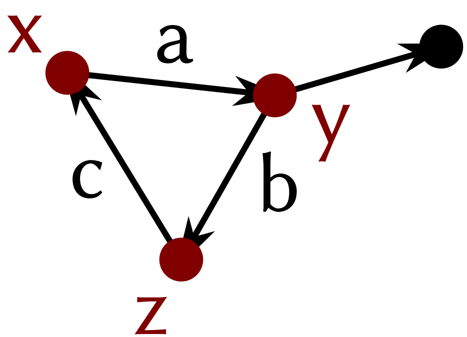
a + b + c + e + g + f

...

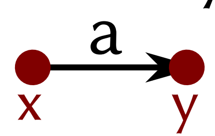
*Holes are cycles in the classical homology theory

One-Dimensional Holes

{ elements of $Z[A_1]$ forming cycles }



boundary function ∂

∂(

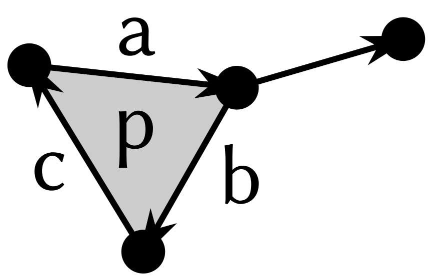
) = y - x

$$\partial(a+b+c) = (y - x) + (z - y) + (x - z) = 0$$

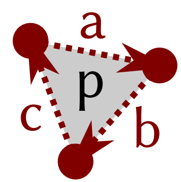
set of holes = kernel of ∂

First Homology Groups

{ unfilled one-dimensional holes }



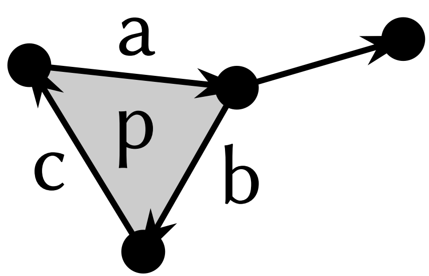
2-dim. boundary function ∂_2

$\partial_2($  $) = a + b + c$

filled holes = image of ∂_2

First Homology Groups

{ unfilled one-dimensional holes }



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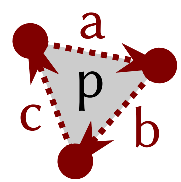
$\partial_2($ 

$) = a + b + c$

filled holes = image of ∂_2

$H_1(X) :=$ kernel of ∂_1 / image of ∂_2

(unfilled holes) (all holes) (filled holes)

Homology Groups

{ unfilled holes }

$C_n := Z[A_n]$ formal sums of cells (chains)

$$\cdots \longrightarrow C_{n+2} \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \longrightarrow \cdots$$

$H_n(X) :=$ kernel of ∂_n / image of ∂_{n+1}

Cohomology Groups

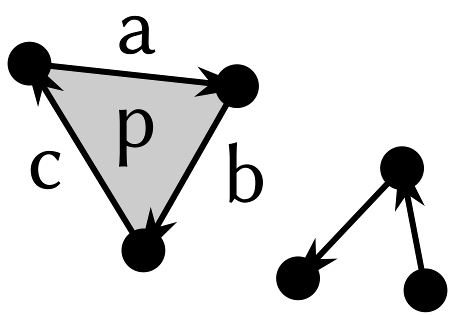
$$\cdots \longrightarrow C_{n+2} \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \longrightarrow \cdots$$

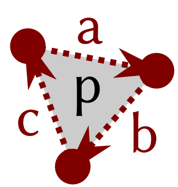
Dualize by Hom(—, G). Let C^n = Hom(C_n, G).

$$\cdots \longleftarrow C^{n+2} \xleftarrow{\delta_{n+2}} C^{n+1} \xleftarrow{\delta_{n+1}} C^n \xleftarrow{\delta_n} C^{n-1} \xleftarrow{\delta_{n-1}} C^{n-2} \longleftarrow \cdots$$

$H^n(X; G)$:= kernel of δ_{n+1} / image of δ_n

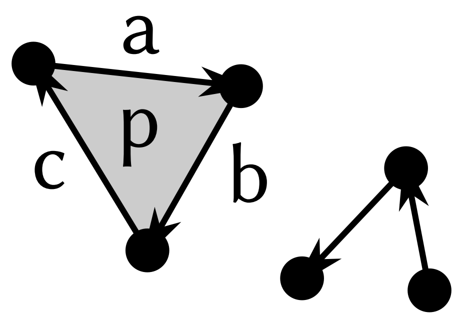
2-Dimensional Boundary

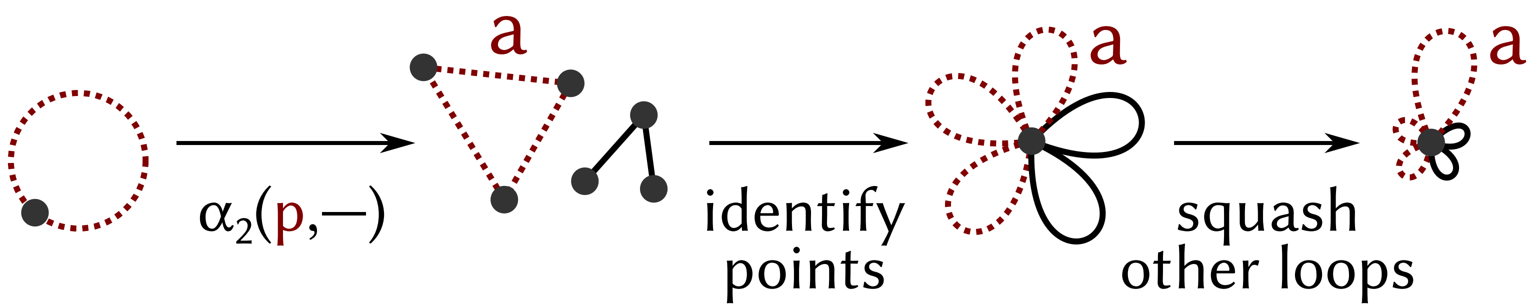


$\partial_2($  $) = a + b + c$

How to compute the coefficients from α_2?

2-Dimensional Boundary





coefficient = winding number of this map

(can be generalized to higher dimensions)

Cohomology Groups

{ mappings from holes in a space }

Cellular cohomology for CW-complexes

$$\mathbf{H^n(X; G)}$$

Axiomatic Eilenberg-Steenrod cohomology

Prove they are the same!

Eilenberg-Steenrod* cohomology

A family of functors $h^n(-)$:

pointed spaces $\longrightarrow$ abelian groups

*ordinary, reduced

Eilenberg-Steenrod* cohomology

A family of functors $h^n(-)$:
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1. $h^{n+1}(\mathrm{susp}(X)) \simeq h^n(X)$, natural in X

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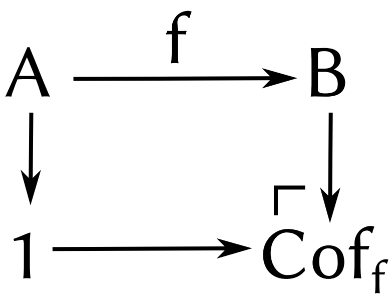

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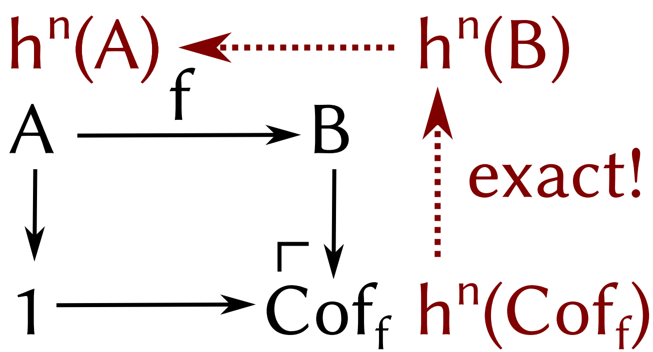

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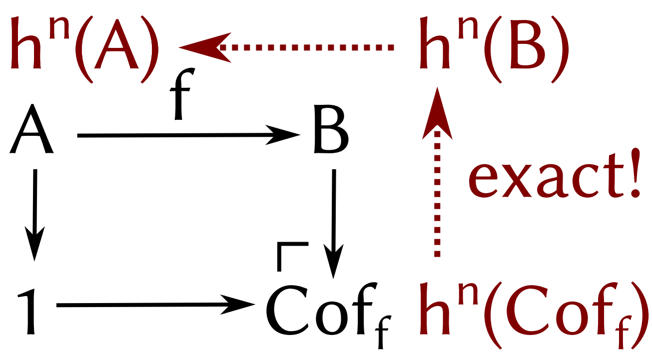
2. 

3. $h^n(\bigvee_i X_i) \simeq \prod_i h^n(X_i)$
if the index type is nice enough**

*ordinary, reduced **see our paper

Eilenberg-Steenrod* cohomology

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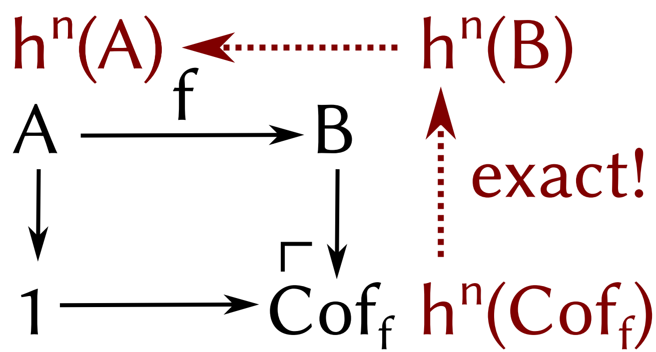
2. 

3. $h^n(\bigvee_i X_i) \simeq \prod_i h^n(X_i)$
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4. $h^n(2)$ trivial for $n \neq 0$

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Cohomology Groups

{ mappings from holes in a space }

Cellular cohomology for CW-complexes

$\mathbf{H^n(X; G)}$

Axiomatic Eilenberg-Steenrod cohomology

$\mathbf{h^n(X)}$

Prove they are the same!

Proof Plan

$$H^n(X; h^0(2)) \simeq^? h^n(X)$$

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$$\ker(\delta_{n+1})/\mathrm{im}(\delta_n) \qquad \ker(\delta'_{n+1})/\mathrm{im}(\delta'_n)$$

1. Find δ' such that $h^n(X) \simeq \ker(\delta'_{n+1})/\mathrm{im}(\delta'_n)$

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$$\delta \simeq \delta'$$

1. Find δ' such that $h^n(X) \simeq \ker(\delta'_{n+1})/\mathrm{im}(\delta'_n)$
2. Show δ and δ' are equivalent

Proof Plan

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$$\rotatebox{90}{$\coloneqq$} \qquad\qquad\qquad \wr$$

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As usual, fully mechanized in Agda!

Step 1: Reverse Engineering

For any theory h, finite pointed CW-complex X, there exist homomorphisms δ'

$$\cdots \leftarrow D^{n+2} \xleftarrow{\delta'_{n+2}} D^{n+1} \xleftarrow{\delta'_{n+1}} D^{n} \xleftarrow{\delta'_{n}} D^{n-1} \xleftarrow{\delta'_{n-1}} D^{n-2} \leftarrow \cdots$$

such that

$$h^n(X) \simeq \text{kernel of } \delta'_{n+1} \,/\, \text{image of } \delta'_n$$

Important Lemmas for Step 1

Long exact sequenses

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \ulcorner\, \mathrm{Cof}_f \end{array}$$

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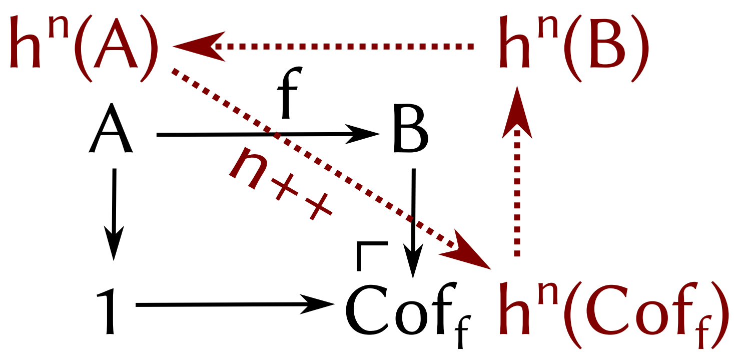

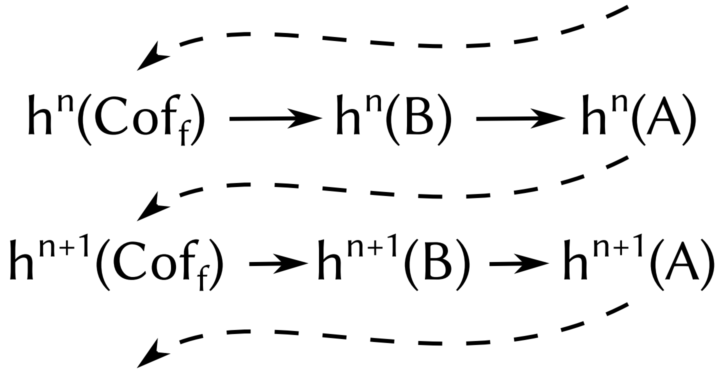

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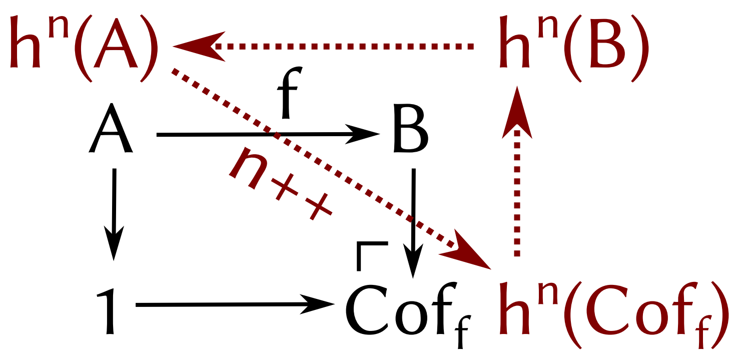


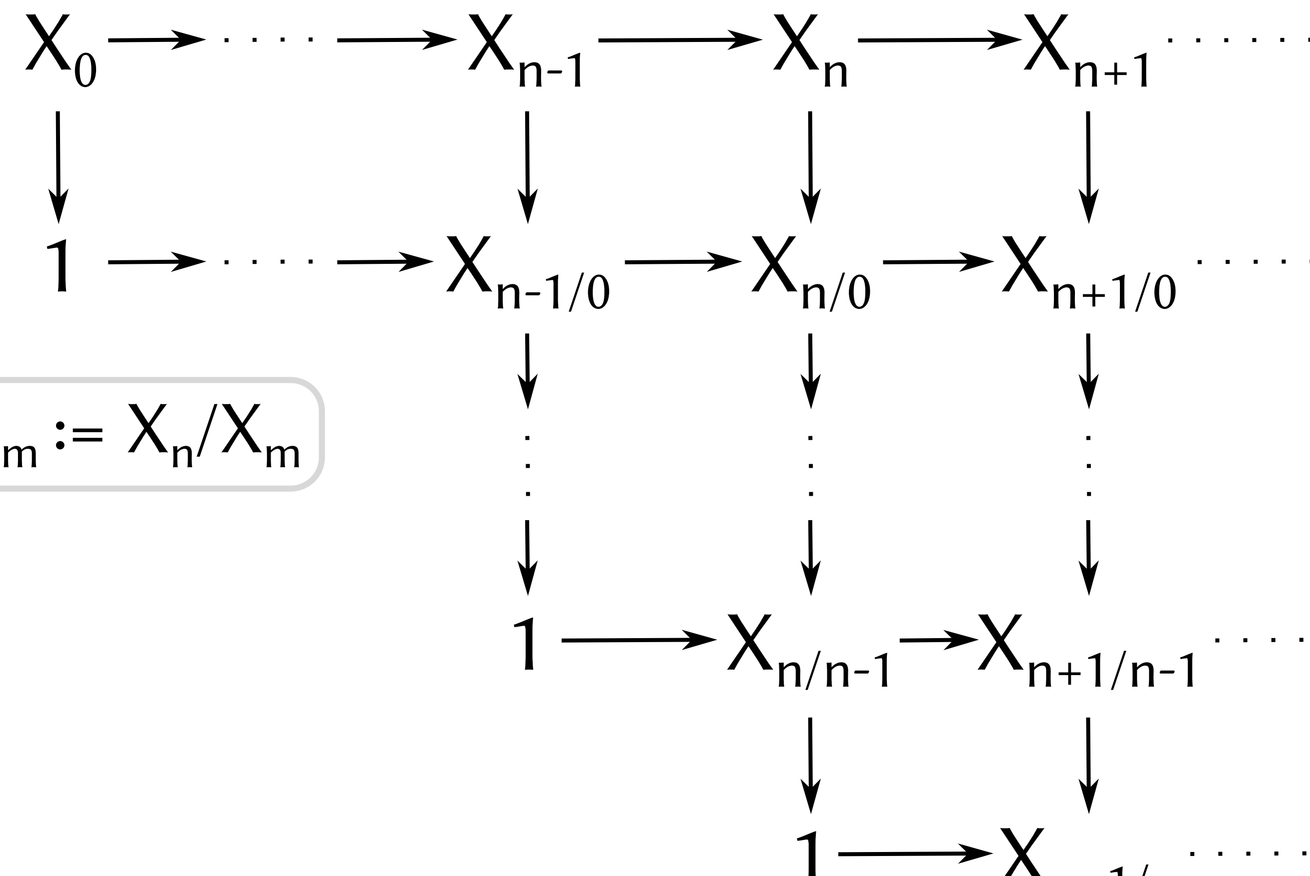
Wedges of cells

$h^m(X_n/X_{n-1}) \simeq \mathrm{hom}(Z[A_n], h^0(2))$
when m = n or trivial otherwise

$h^m(X_0) \simeq \mathrm{hom}(Z[A_0 \backslash \{pt\}], h^0(2))$
when m = 0 or trivial otherwise

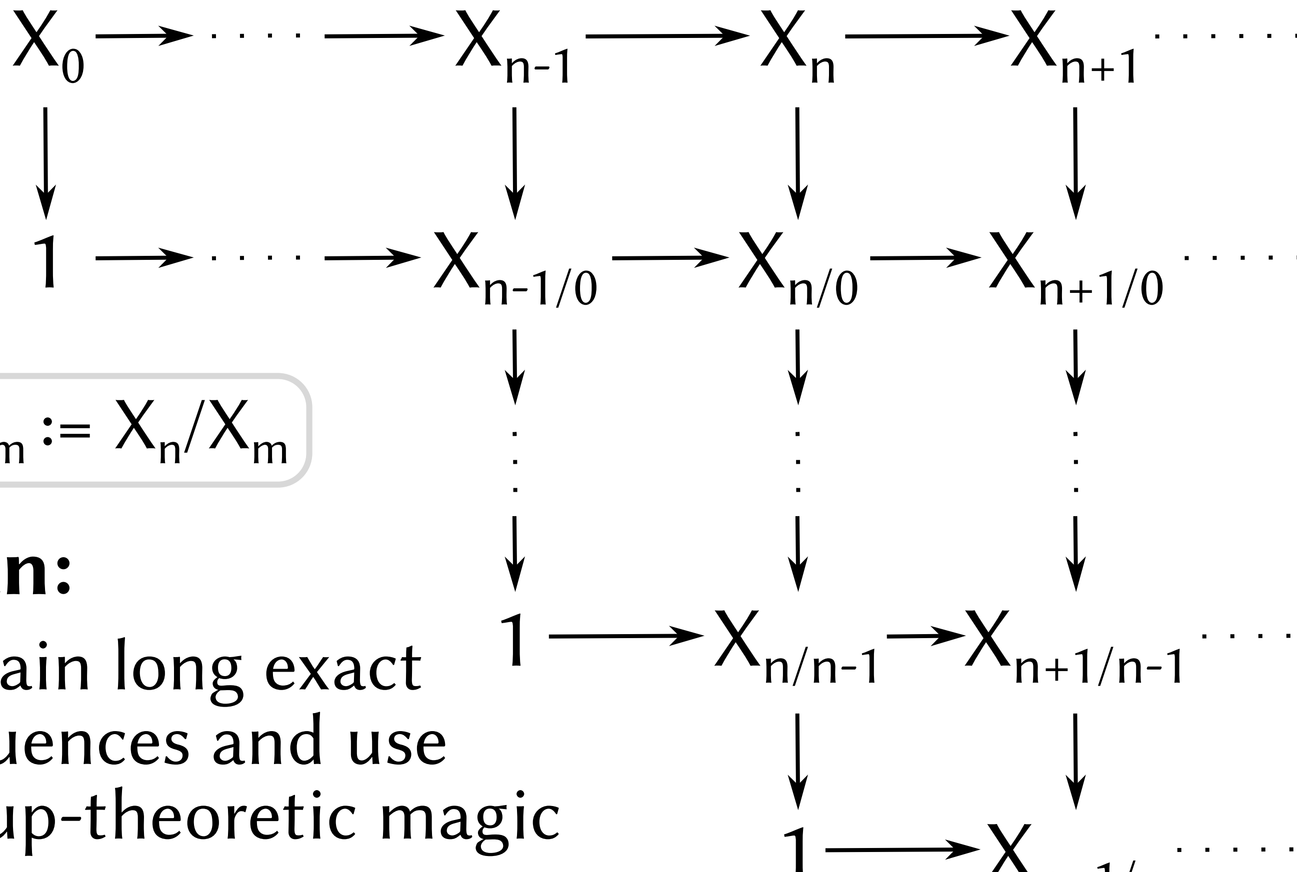
} trivial if $m \neq n$

Ultimate Cofiber Diagram



$X_{n/m} := X_n/X_m$

Ultimate Cofiber Diagram



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Plan:

Obtain long exact sequences and use group-theoretic magic

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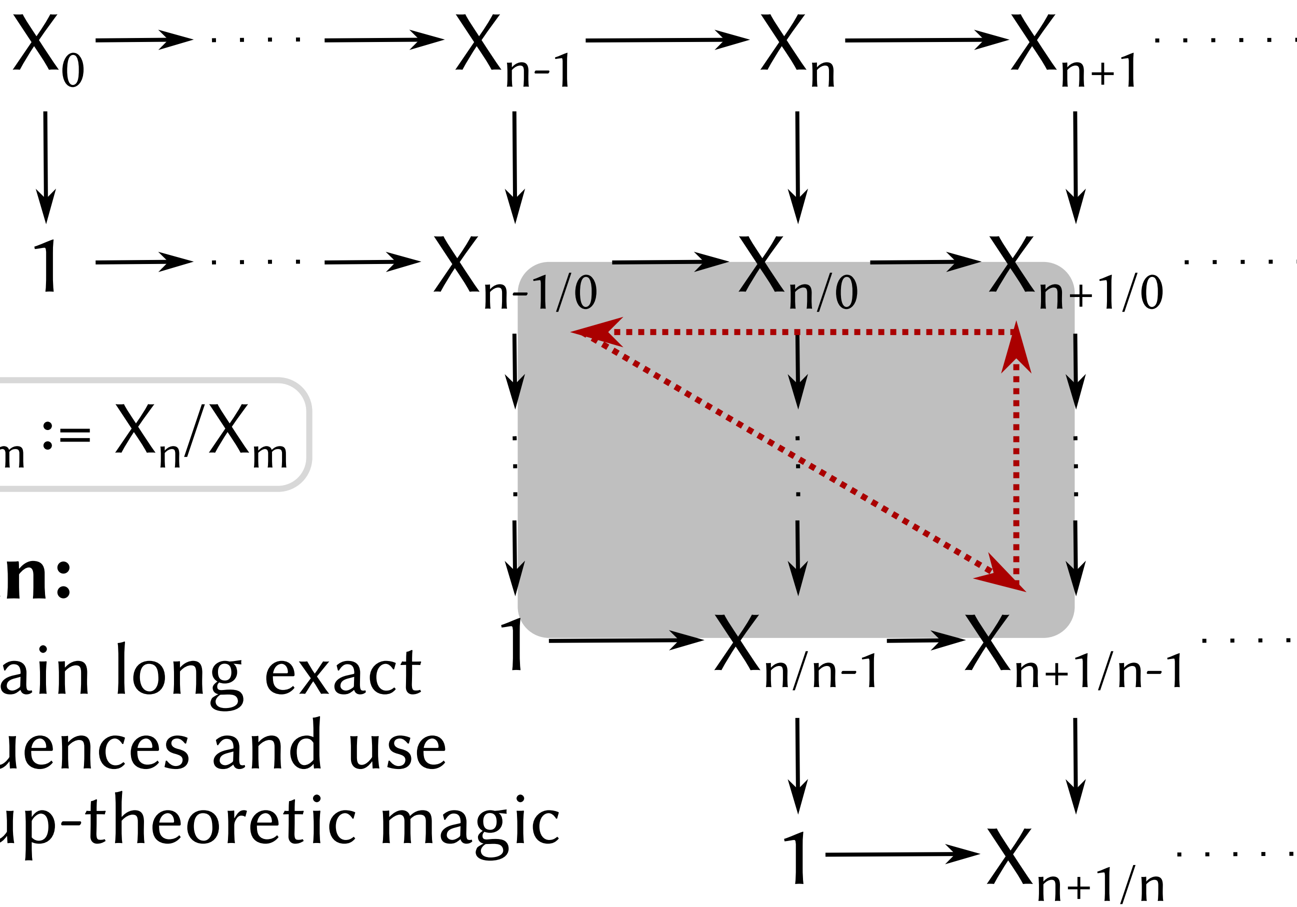

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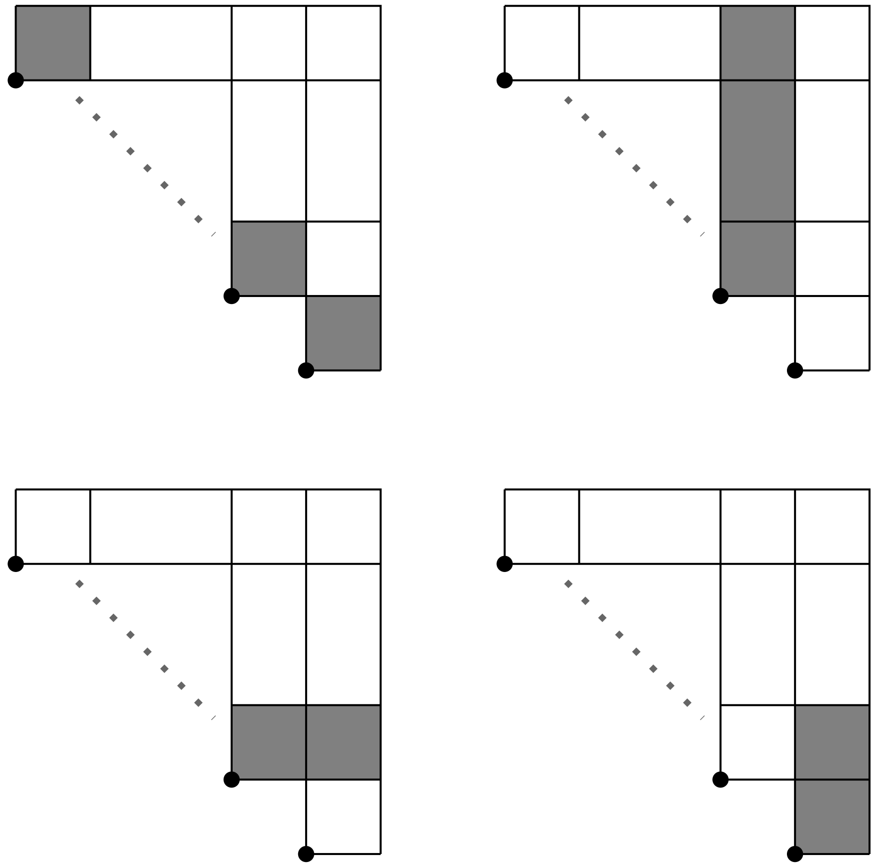

+ group theory

$$h^n(X) \simeq \ker(\delta'_{n+1})/\mathrm{im}(\delta'_n)$$

Proof Plan (updated)

$$H^n(X; h^0(2)) \simeq^{?} h^n(X)$$

$$\ker(\delta_{n+1})/\mathrm{im}(\delta_n) \qquad \ker(\delta'_{n+1})/\mathrm{im}(\delta'_n)$$

$$\delta \simeq \delta'$$

1. Find δ' such that $h^n(X) \simeq \ker(\delta'_{n+1})/\mathrm{im}(\delta'_n)$
2. Show δ and δ' are equivalent

Step 2: Calculation

$$
\begin{array}{ccccccccccccc}
\cdots & \longleftarrow & C^{n+2} & \overset{\delta_{n+2}}{\longleftarrow} & C^{n+1} & \overset{\delta_{n+1}}{\longleftarrow} & C^{n} & \overset{\delta_{n}}{\longleftarrow} & C^{n-1} & \overset{\delta_{n-1}}{\longleftarrow} & C^{n-2} & \longleftarrow & \cdots \\
 & & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \\
\cdots & \longleftarrow & D^{n+2} & \underset{\delta'_{n+2}}{\longleftarrow} & D^{n+1} & \underset{\delta'_{n+1}}{\longleftarrow} & D^{n} & \underset{\delta'_{n}}{\longleftarrow} & D^{n-1} & \underset{\delta'_{n-1}}{\longleftarrow} & D^{n-2} & \longleftarrow & \cdots
\end{array}
$$

The n=0 case ($C^0 \simeq D^0$) is interesting

Summary

Cellular cohomology groups	$\simeq$ (finite)	Ordinary reduced cohomology groups

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Cellular cohomology groups	$\simeq$ (finite)	Ordinary reduced cohomology groups

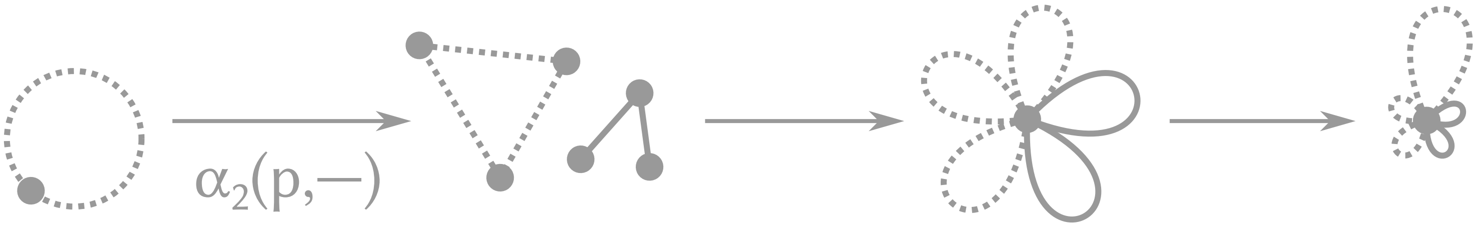
To-Do

Infinity: colimits

Homology ⇨ Poincaré duality, ...

Parametrization ⇨ non-orientability, ...

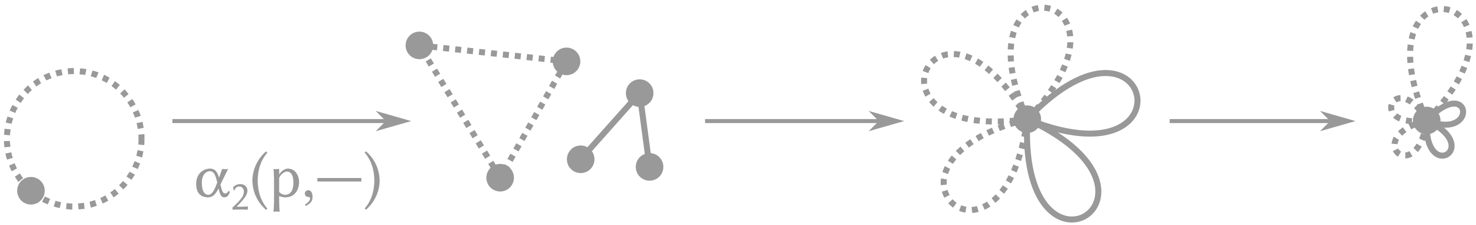
Higher-Dim. Boundary



$$S^n \xrightarrow[\alpha_{n+1}(p,-)]{} X_n \xrightarrow[\text{identify lower structs.}]{} X_n/X_{n-1} \simeq \bigvee S^n \xrightarrow[\text{squash}]{} S^n$$

coefficient = *degree* of this map

Higher-Dim. Boundary



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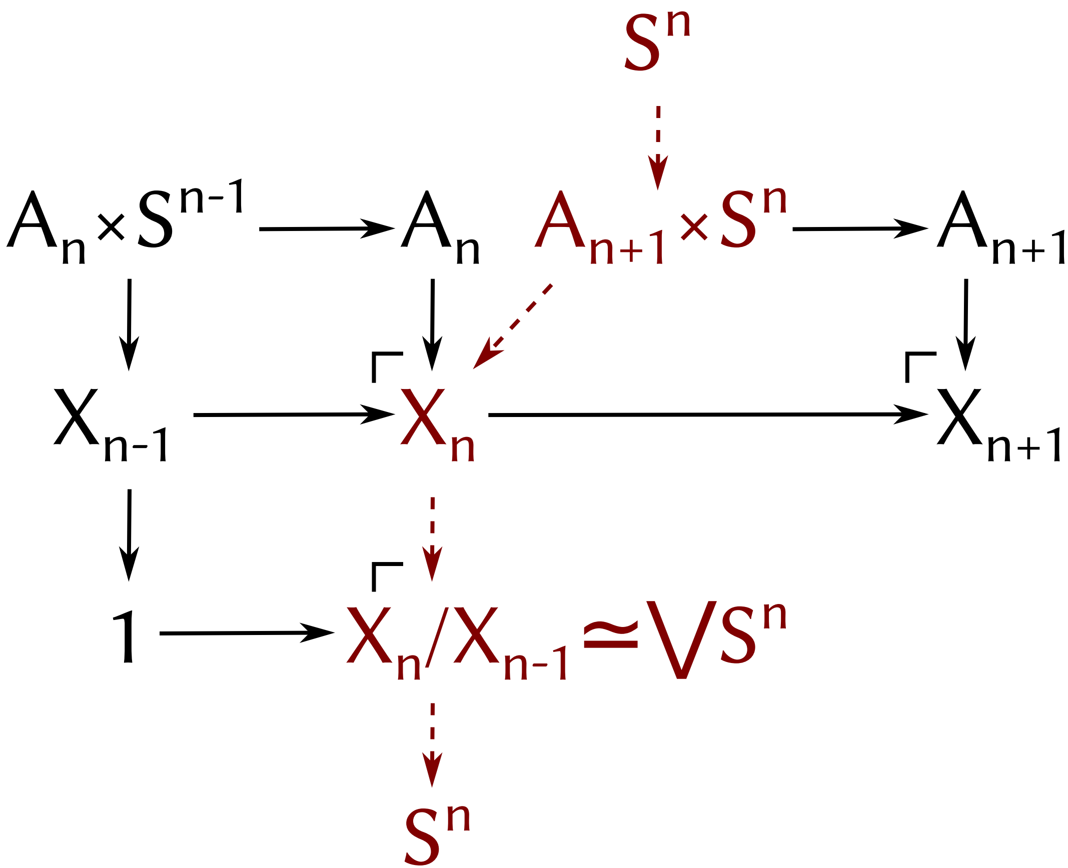
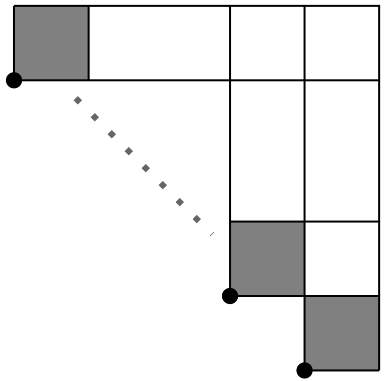
coefficient = *degree* of this map

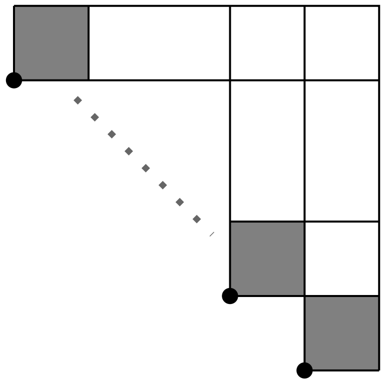
- squashing needs decidable equality
- linear sum needs closure-finiteness
  (free for finite cases)

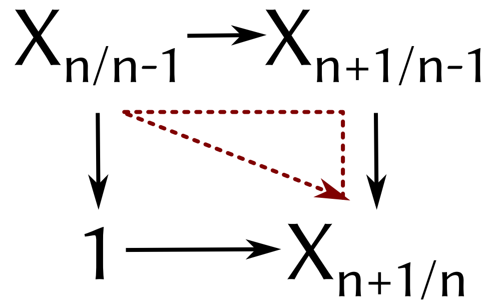
Higher-Dim. Boundary





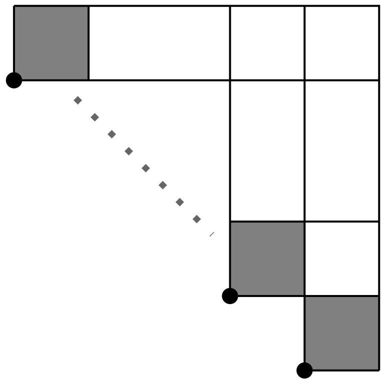
$$\begin{array}{ccc} X_{n/n-1} & \longrightarrow & X_{n+1/n-1} \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X_{n+1/n} \end{array}$$





$$h^n(X_{n+1/n}) \longrightarrow h^n(X_{n+1/n-1}) \longrightarrow h^n(X_{n/n-1})$$

$$h^{n+1}(X_{n+1/n}) \longrightarrow h^{n+1}(X_{n+1/n-1}) \longrightarrow h^{n+1}(X_{n/n-1})$$



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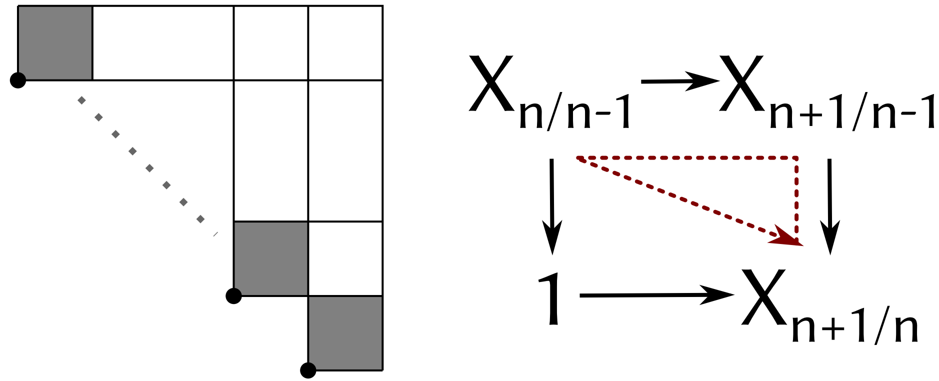
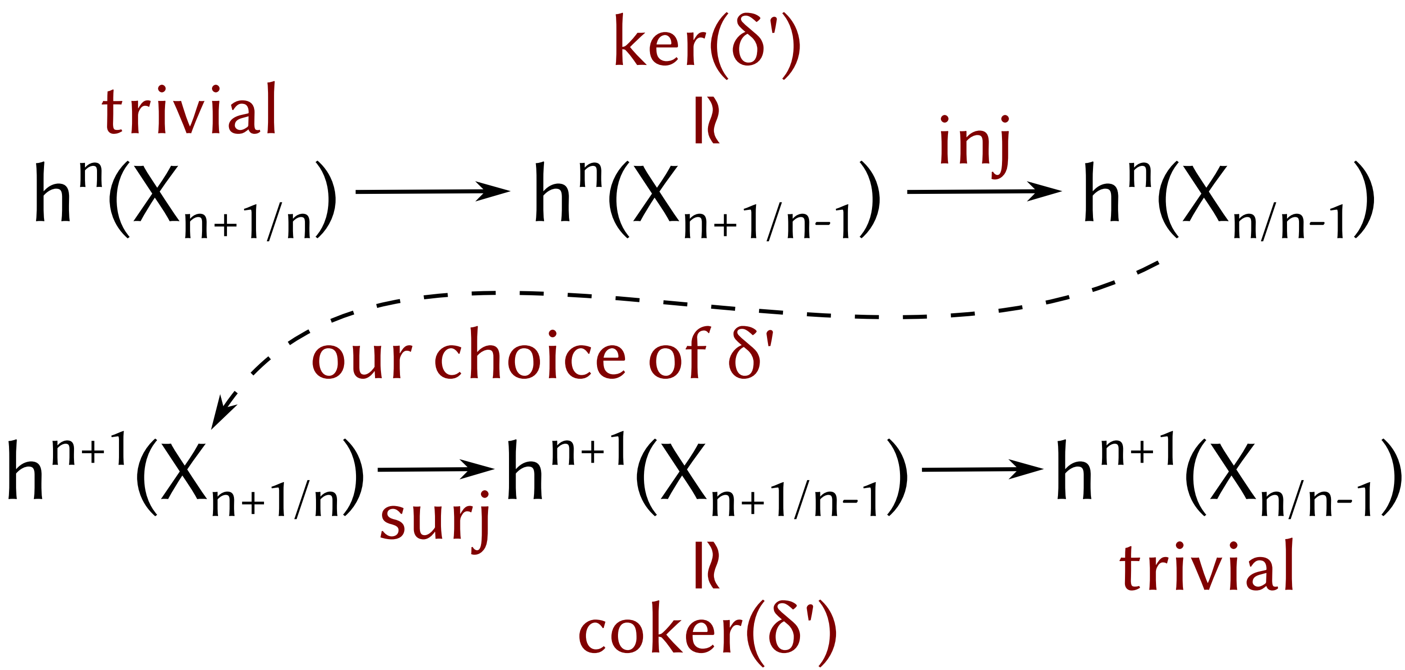
$$h^n(X_{n+1/n}) \longrightarrow h^n(X_{n+1/n-1}) \longrightarrow h^n(X_{n/n-1})$$

our choice of δ'

$$h^{n+1}(X_{n+1/n}) \longrightarrow h^{n+1}(X_{n+1/n-1}) \longrightarrow h^{n+1}(X_{n/n-1})$$


$X_{n/n-1}$
$X_{n+1/n-1}$
1
$X_{n+1/n}$


ker(δ')
≀
trivial
inj
$h^n(X_{n+1/n})$
$h^n(X_{n+1/n-1})$
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our choice of δ'
$h^{n+1}(X_{n+1/n})$
surj
$h^{n+1}(X_{n+1/n-1})$
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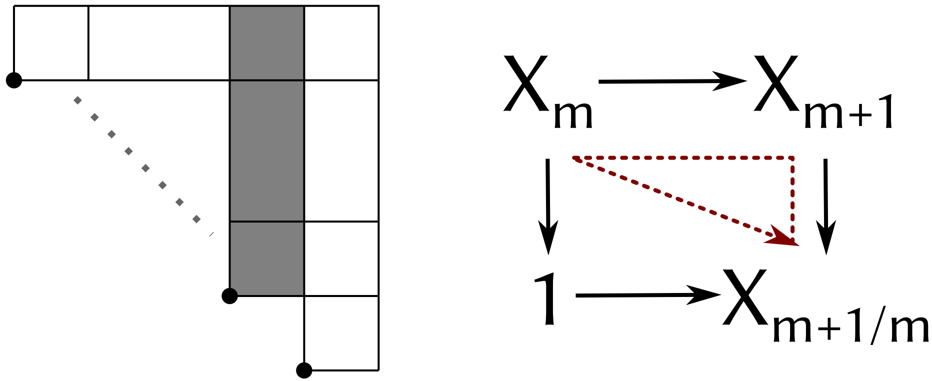
coker(δ')
trivial

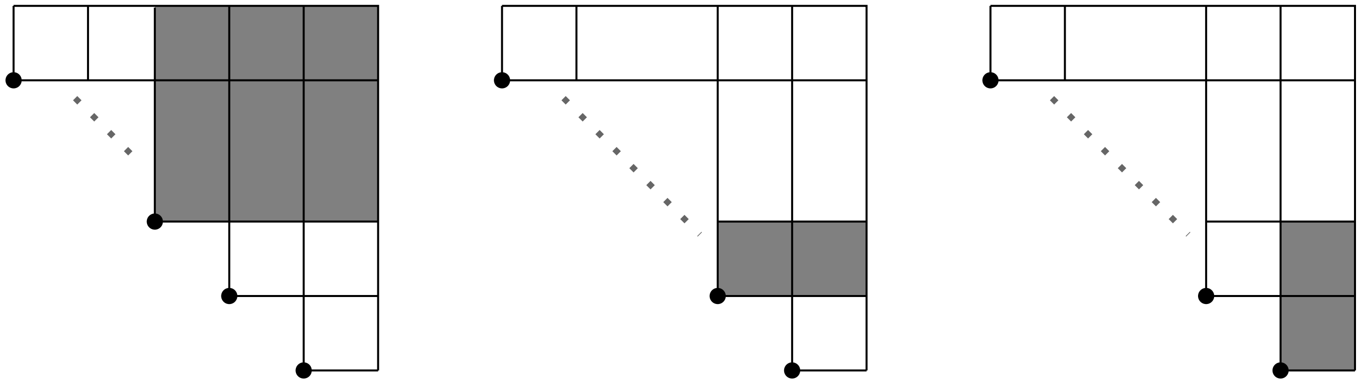


$$h^n(X_{m+1/m}) \longrightarrow h^n(X_{m+1}) \longrightarrow h^n(X_m) \longrightarrow h^{n+1}(X_{m+1/m})$$

If $n \neq m, m+1$, both ends trivial, $h^n(X_{m+1}) \simeq h^n(X_m)$

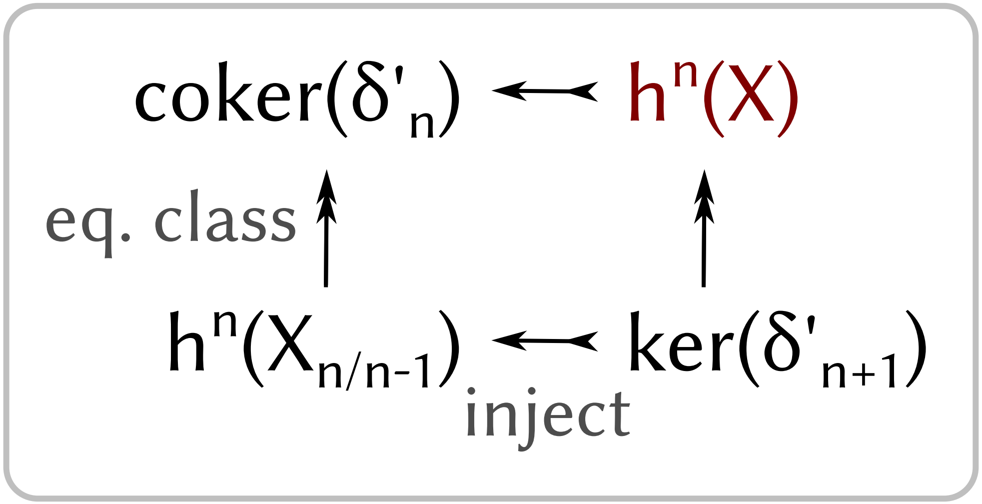
three possible values

$$\begin{cases} h^n(X_{n-1}) \simeq h^n(X_{n-2}) \simeq \cdots \simeq h^n(X_0), \text{ trivial} \\ h^n(X_n) \\ h^n(X_{n+1}) \simeq h^n(X_{n+2}) \simeq \cdots \simeq h^n(X) \end{cases}$$



$\mathrm{coker}(\delta'_n) \simeq$

$$
\begin{array}{ccc}
h^n(X_{n/n-2}) & \longleftarrow & h^n(X_{n+1/n-2}) \simeq h^n(X) \\
\text{eq. class} \uparrow & & \uparrow \\
h^n(X_{n/n-1}) & \underset{\text{inject}}{\longleftarrow} & h^n(X_{n+1/n-1}) \simeq \ker(\delta'_{n+1})
\end{array}
$$



Chasing the diagram,

$$h^n(X) \simeq \ker(\delta'_{n+1})/\mathrm{im}(\delta'_n)$$