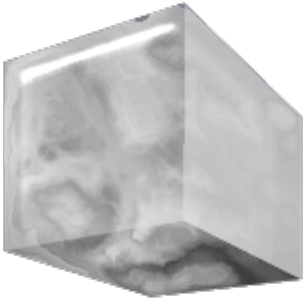
2018.07.24

Cartesian Cubical Computational Type Theory



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Jonathan Sterling
Todd Wilson

Cubical

features of homotopy type theory

univalence, higher inductive types

+

Computational

features of Nuprl and PVS

strict equality, strict quotients,
predicative subtypes...

Cartesian Cubical

features of homotopy type theory

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+

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Computational Types

programs/
realizers

computation

Computational Types

programs/ realizers	<------	computational type theory
computation		theory of computation

Computational Types

programs/ realizers <----- **computational type theory**

computation | theory of computation

meaning explanation <---- Martin-Löf type theory

pre-mathematical in M-L's work

A Minimum Example

```
M := a | bool | true | false | if(M,M,M)
```

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bool val       if(M,Mt,Mf) ↦ if(M',Mt,Mf)
true val       if(true,M,_) ↦ M
false val      if(false,_,M) ↦ M
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What are the types in canonical forms? **{bool}**

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What are the canonical forms of the types?

bool: {true, false}

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How they are equal? **syntactic equality**

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One Theory

A Minimum Example

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types: {bool} with syntactic equality $\approx$

bool: {true, false} with syntactic equality $\approx_{\text{bool}}$

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$A \doteq B$ type

$A \Downarrow A'$ $B \Downarrow B'$ and $A' \approx B'$

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$$A \doteq B \ \mathrm{type}$$

$$A \Downarrow A' \quad B \Downarrow B' \quad \text{and} \quad A' \approx B'$$

$$\mathrm{bool} \doteq \mathrm{bool}\ \mathrm{type}$$

$$\mathrm{if(true,bool,bool)} \doteq \mathrm{bool}\ \mathrm{type}$$

$$\Downarrow \mathrm{bool}$$

A Minimum Example

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---

bool $\doteq$ bool type

if(true,bool,bool) $\doteq$ bool type
$\Downarrow$ bool

if(true,bool,*any closed term*) $\doteq$ bool type

A Minimum Example

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bool: {true, false} with syntactic equality $\approx_{bool}$

$$M \doteq N \in A$$

$A \doteq A$ type, $M \Downarrow M'$, $N \Downarrow N'$, $A \Downarrow A'$ and $M' \approx_{A'} N'$

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$$\text{false} \doteq \text{false} \in \text{bool}$$

A Minimum Example

$M := a \mid bool \mid true \mid false \mid if(M,M,M)$

types: {bool} with syntactic equality $\approx$

bool: {true, false} with syntactic equality $\approx_{bool}$

$$M \doteq N \in A$$

$A \doteq A$ type, $M \Downarrow M'$, $N \Downarrow N'$, $A \Downarrow A'$ and $M' \approx_{A'} N'$

$$false \doteq false \in bool$$

$$\underset{\Downarrow true}{if(true,true,bool)} \doteq true \in \underset{\Downarrow bool}{if(true,bool,bool)}$$

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$$a{:}A \gg M \doteq N \in B$$

$P \doteq Q \in A$ implies $M[P/a] \doteq N[Q/a] \in B[P/a]$

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$P \doteq Q \in A$ implies $M[P/a] \doteq N[Q/a] \in B[P/a]$

$b{:}\mathrm{bool} \gg b \doteq \mathrm{if}(b,\mathrm{true},\mathrm{false}) \in \mathrm{bool}$?

A Functional Example

```
M := a | M1⟶M2 | \a.M | M1 M2 | ...

(M1⟶M2) val  \a.M val  (\a.M1)M2 ↦ M1[M2/a]
```

Another Language

A Functional Example

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M := a | M1⟶M2 | \a.M | M1 M2 | ...

(M1⟶M2) val   \a.M val   (\a.M1)M2 ↦ M1[M2/a]
```

Another Language

What are the types in canonical forms?

```
the least fixed point of
S ↦ {M⟶N | M⇓, N⇓ in S} union ...
```

What are the canonical forms of the types?

```
A⟶B: {\a.M}
```

How they are equal?

```
A1⟶B1 ≈ A2⟶B2 if A1 ≐ A2 and B1 ≐ B2
```

$\texttt{\textbackslash a.M1} \approx_{A\to B} \texttt{\textbackslash a.M2 if a:A >> M1} \doteq \texttt{M2} \in \texttt{B}$

Variables

Nuprl/...	Coq/Agda/...
Vars range over closed terms	Vars are indet.
Defined by transition b/w closed terms	Defined by conversion b/w open terms

Open-endedness

$$\begin{array}{c}\texttt{Proof theory/tactics/editors}\\ \downarrow \\ \texttt{Computational type theory}\\ \downarrow \\ \texttt{Programming language}\end{array}$$

Open-endedness

Proof theory/tactics/editors

↓

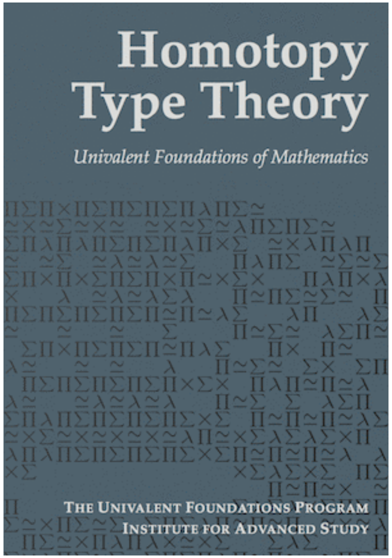
Computational type theory

↓

Programming language

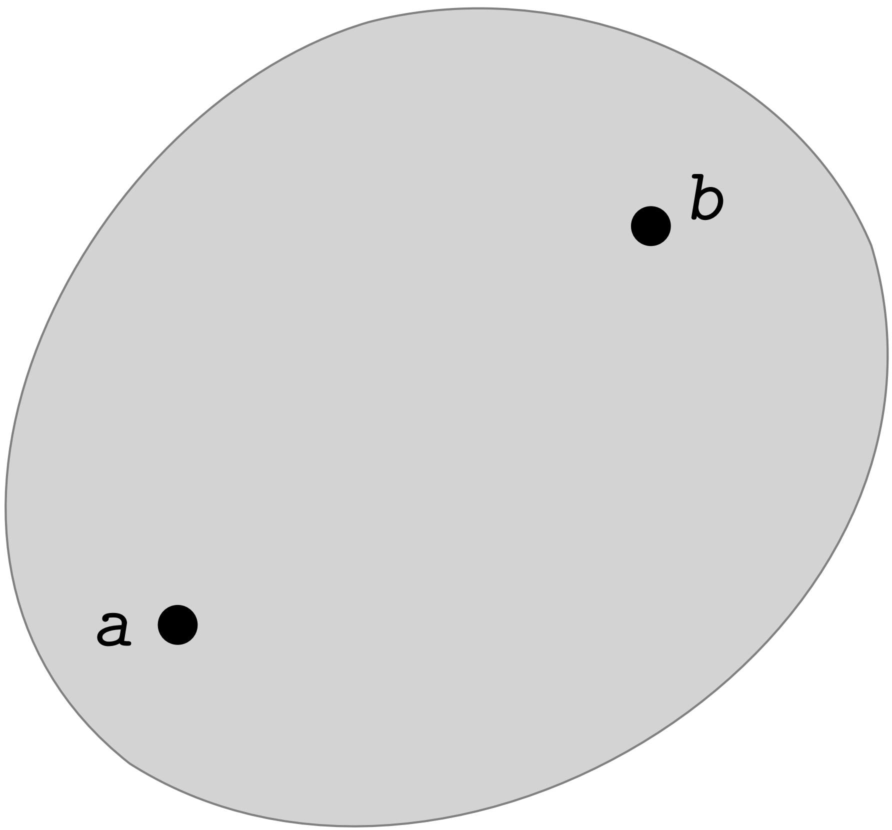
Canonicity always holds

Homotopy Type Theory

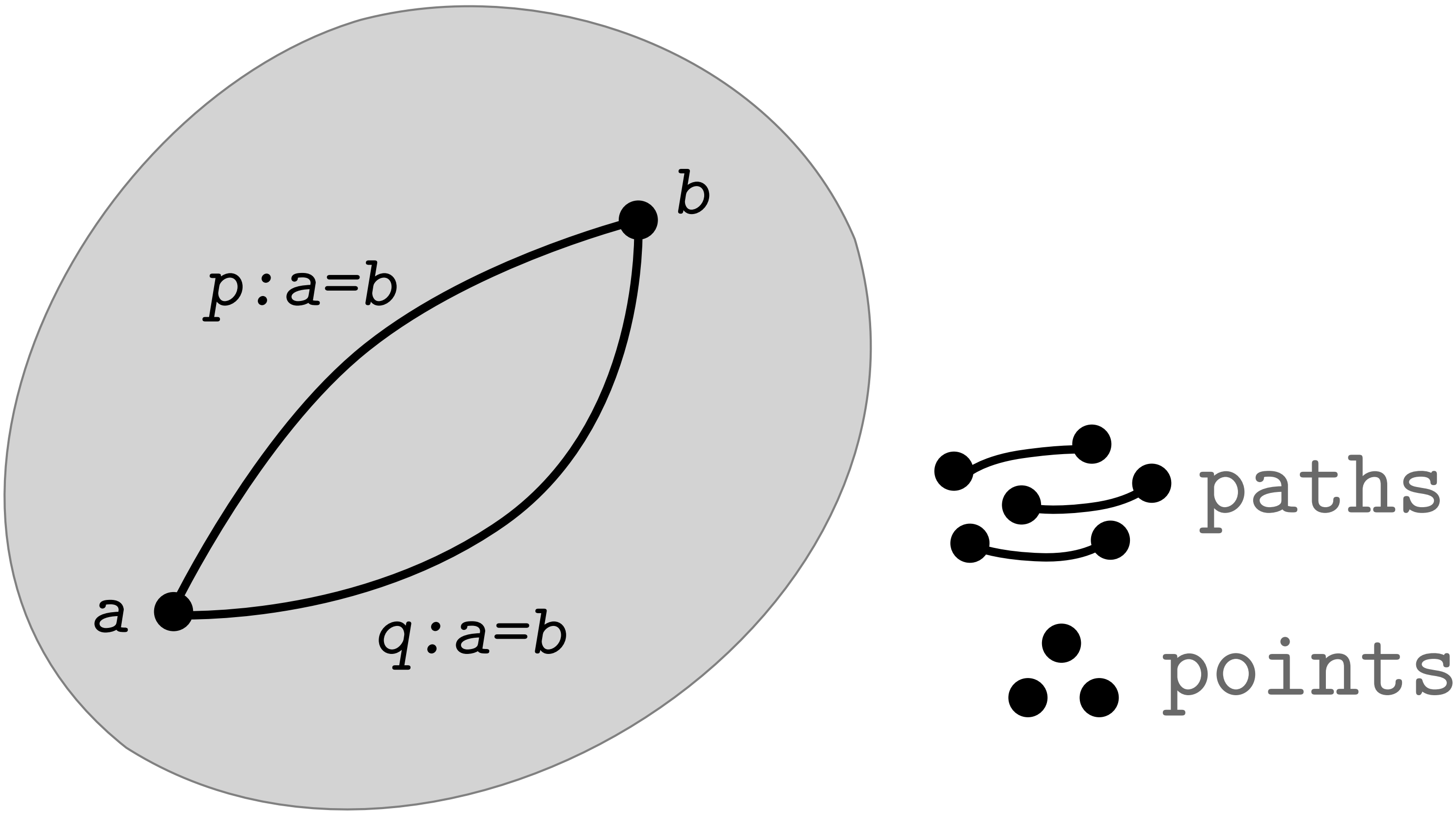


github.com/HoTT/book

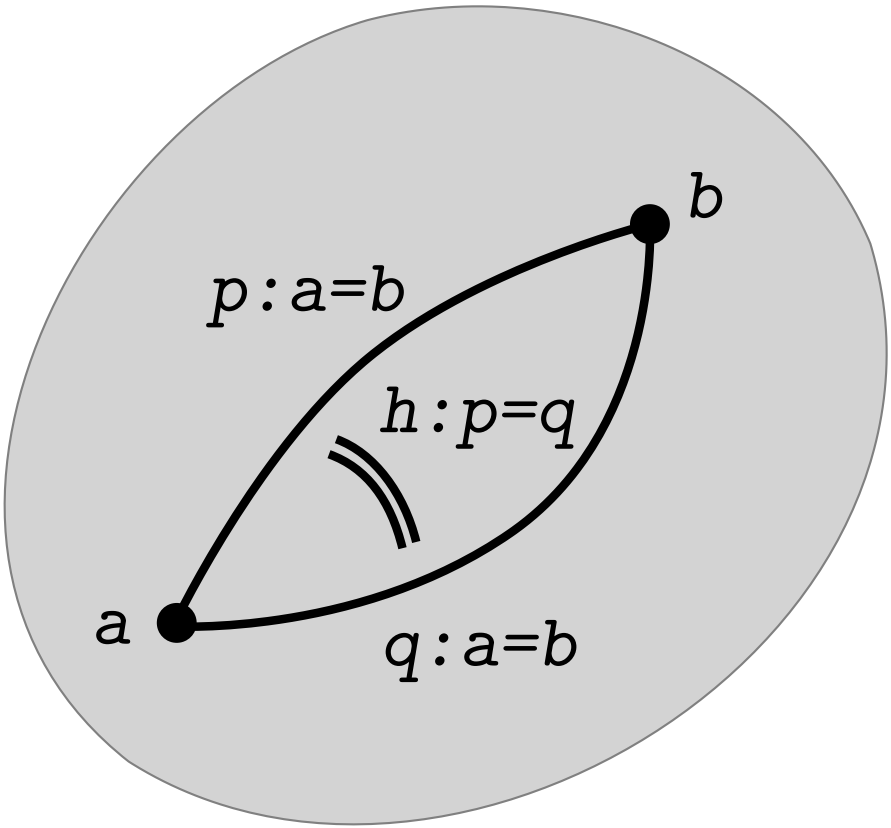
Homotopy Type Theory

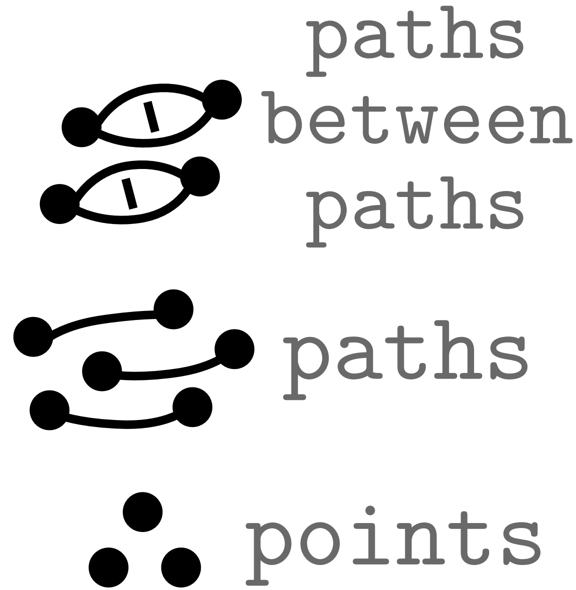


Homotopy Type Theory



Homotopy Type Theory





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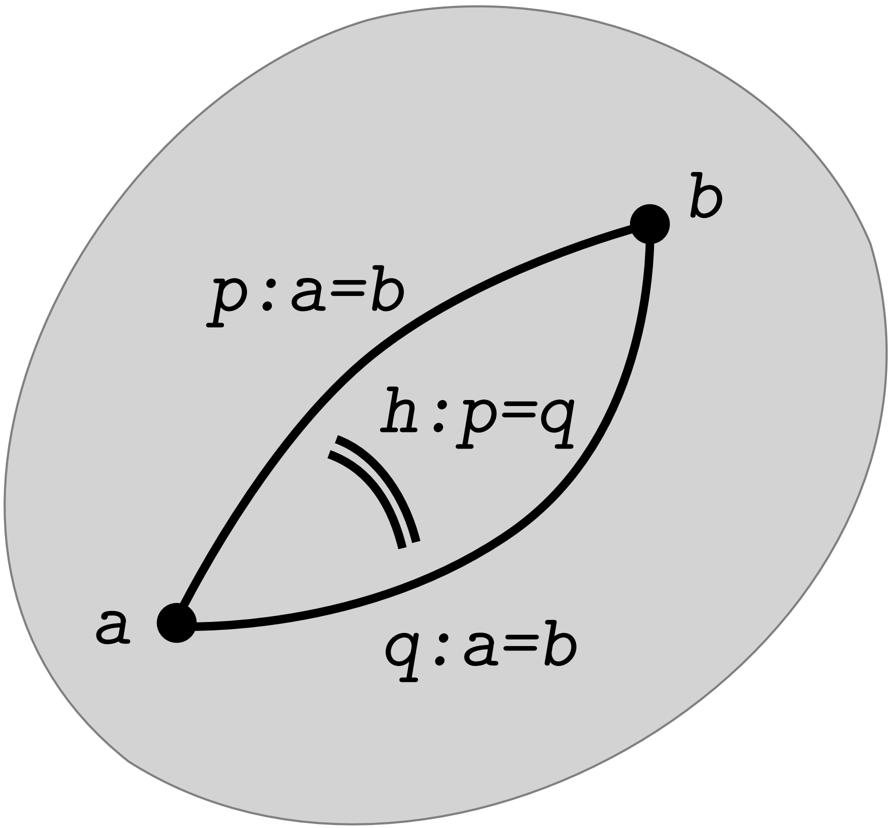

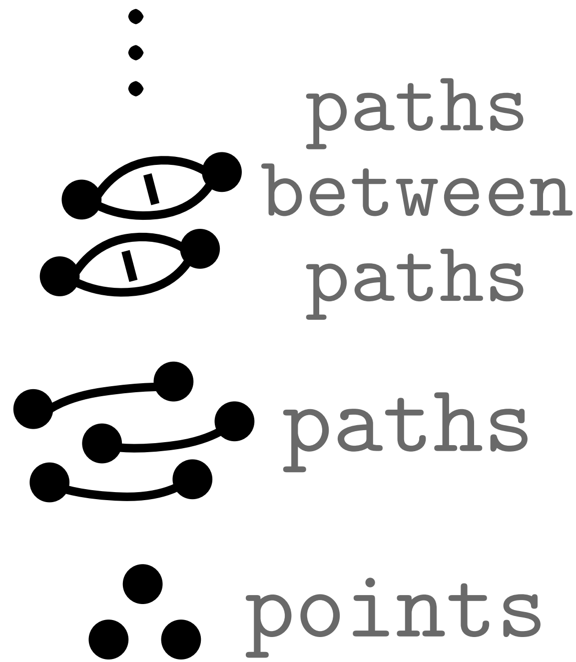

Equality and Paths

Equality ($\equiv$)

Silent in theory

$2 + 3 \equiv 5$

$fst\ \langle M, N \rangle \equiv M$

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If $A \equiv B$ and $M : A$ then $M : B$

Equality and Paths

Equality ($\equiv$)

Silent in theory

$2 + 3 \equiv 5$

$fst\ \langle M,N \rangle \equiv M$

If $A \equiv B$ and $M : A$ then $M : B$

Paths ($=$)

Visible in theory

If $P : A=B$ and $M : A$ then $transport(M,P) : B$

Homotopy Type Theory

[Awodey and Warren] [Voevodsky *et al*] [van den Berg and Garner]

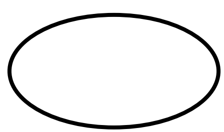
A	Type	Space
$a : A$	Element	Point
$f : A \to B$	Function	Continuous Mapping
$C : A \to Type$	Dependent Type	Fibration
$a =_A b$	Identification	Path

Features of HoTT

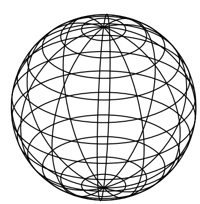
Univalence

If E is an equivalence between types A and B, then $ua(E) : A = B$

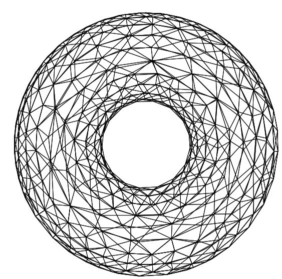
Higher Inductive Types



circle



sphere



torus

Canonicity?

Canonicity broken by
new features stated as axioms!

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Canonicity

For any $M : bool$, either
$M \equiv true : bool$ or $M \equiv false : bool$

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Canonicity

For any $M : bool$, either
$M \equiv true : bool$ or $M \equiv false : bool$

$ua(not) : bool = bool$

$transport(ua(not), true) \not\equiv false$

Canonicity for All

Canonicity for bool means
canonicity for *everyone*

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$$M : bool \times A$$

$$fst(M) \equiv ??? : bool$$

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Wants $M \equiv \langle P,Q \rangle$ and then
$fst(M) \equiv fst\langle P,Q \rangle \equiv P \equiv$ true or false

Canonicity for Paths?

$$\frac{M : A}{\mathit{refl}(M) : M =_A M}$$

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$$\frac{a{:}A \vdash R : C(a,a,\mathit{refl}(a)) \qquad P : M = N}{\mathit{path\text{-}ind}[C](a.R,P) : C(M,N,P)}$$

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$$\mathit{path\text{-}ind}[C](a.R,\mathit{ua}(E)) \equiv \;???$$

Restore Canonicity

Can we have a new TT with canonicity + univalence?

Yes with De Morgan cubes [CCHM 2016]
Yes with Cartesian cubes [AFH 2017]

... and higher inductive types?

Examples with De Morgan cubes [CHM 2018]
Yes with Cartesian cubes [CH 2018]

Restore Canonicity

Idea: each type manages its own paths

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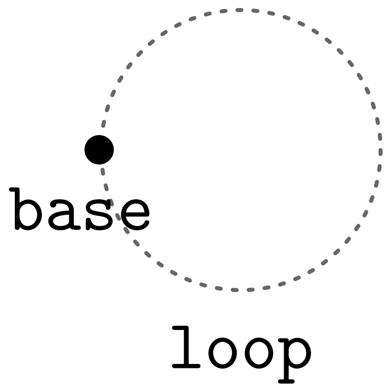
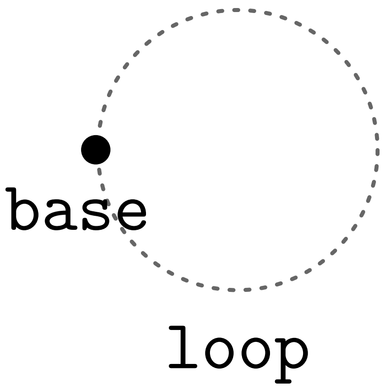


base : S1

Restore Canonicity

Idea: each type manages its own paths




```
base : S1
loop : base = base
```

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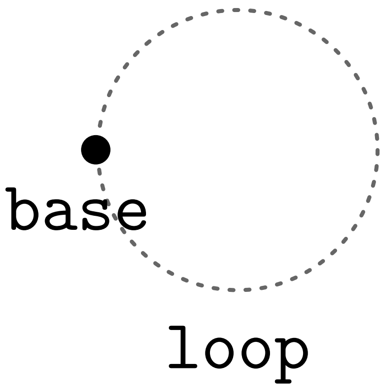

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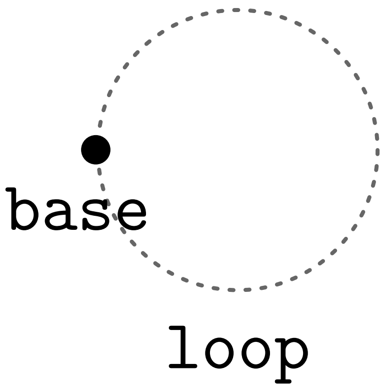


base : S1

~~loop : base = base~~

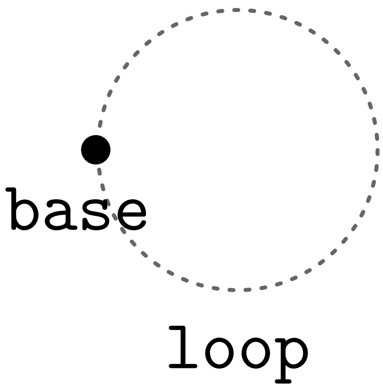
x:$\mathbb{I}$ ⊢ loop{x} : S1

loop{0} ≡ base : S1

loop{1} ≡ base : S1

Restore Canonicity

Idea: each type manages its own paths



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Kan structure:
sufficient to implement path-ind

Kan types: types with Kan structure

Cartesian Cubes

Introducing $\mathbb{I}$ the formal interval

Cartesian Cubes

Introducing $\mathbb{I}$ the formal interval

$$\Gamma \vdash 0 : \mathbb{I} \qquad \Gamma \vdash 1 : \mathbb{I}$$

$$\Gamma,\ x : \mathbb{I},\ \Gamma' \vdash x : \mathbb{I}$$

Cartesian Cubes

Introducing $\mathbb{I}$ the formal interval

$$\Gamma \vdash 0 : \mathbb{I} \qquad \Gamma \vdash 1 : \mathbb{I}$$

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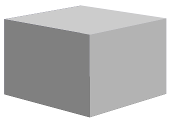
$$x_1 : \mathbb{I},\ x_2 : \mathbb{I},\ \ldots,\ x_n : \mathbb{I} \vdash M : A$$

$\Leftrightarrow$ M is an n-cube in A 

Cartesian Cubes

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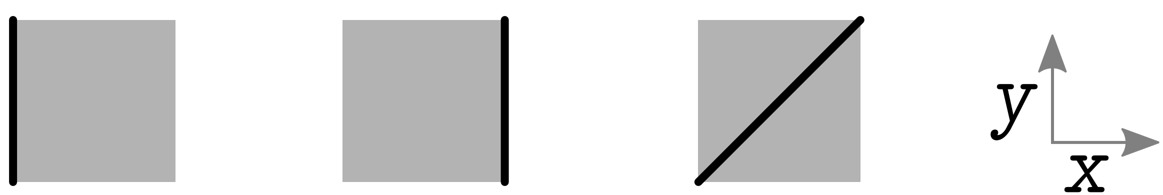
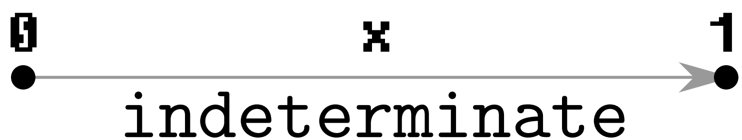
Cartesian: works as normal contexts

$M\langle 0/x \rangle \qquad M\langle 1/x \rangle \qquad M\langle y/x \rangle$

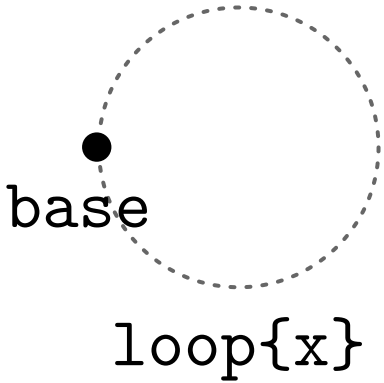


Cubical Programming

```
dim expr r := 0 | 1 | x
```



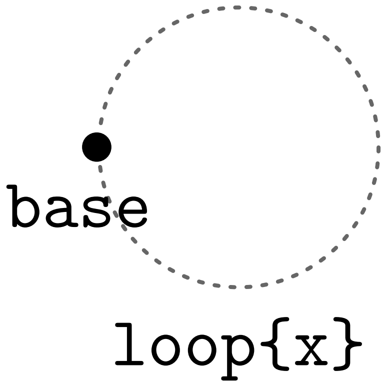
Circle



Circle

```
M := S1 | base | loop{r}
   | S1elim(a.M, M, M, x.M) | ...
```

dim expr



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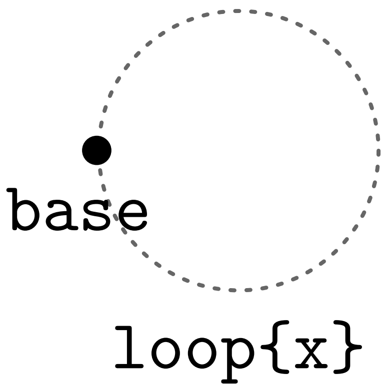

`S1 val`

Circle

dim expr

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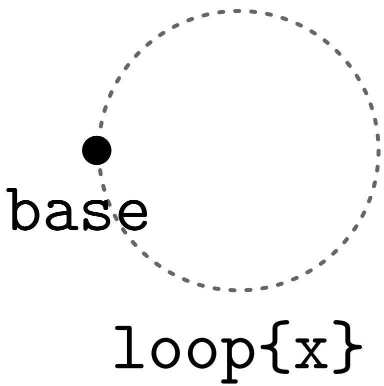


```
S1 val
```

```
base val
```

Circle

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M := S1 | base | loop{r}     dim expr
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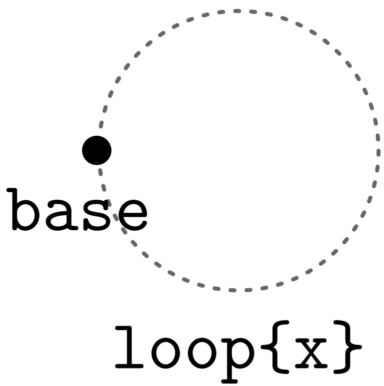


`S1 val`

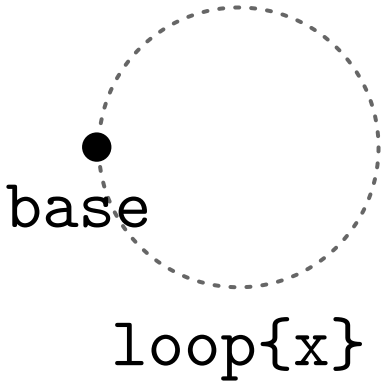
`base val`

`loop{x} val`

`loop{0} ↦ base`

`loop{1} ↦ base`

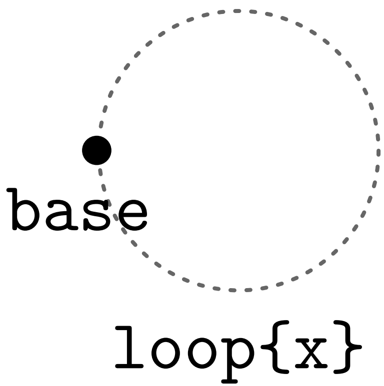
Circle




```
S1 val
```

```
M ↦ M'
————————————————————————————
S1elim(a.A, M, B, x.L)
  ↦ S1elim(a.A, M', B, x.L)
```

Circle

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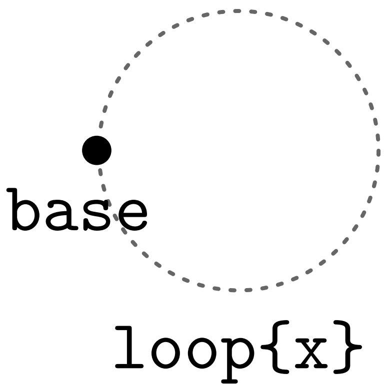
```
S1elim(a.A, base, B, x._)
  ↦ B
```

Circle

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```

```
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  ↦ B
```
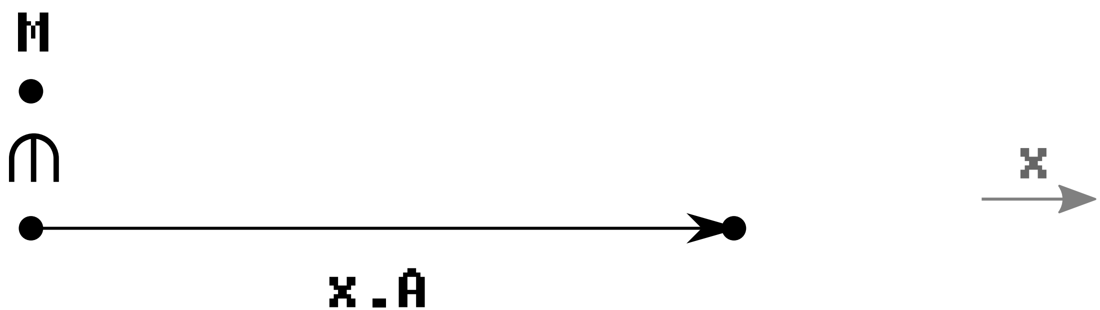
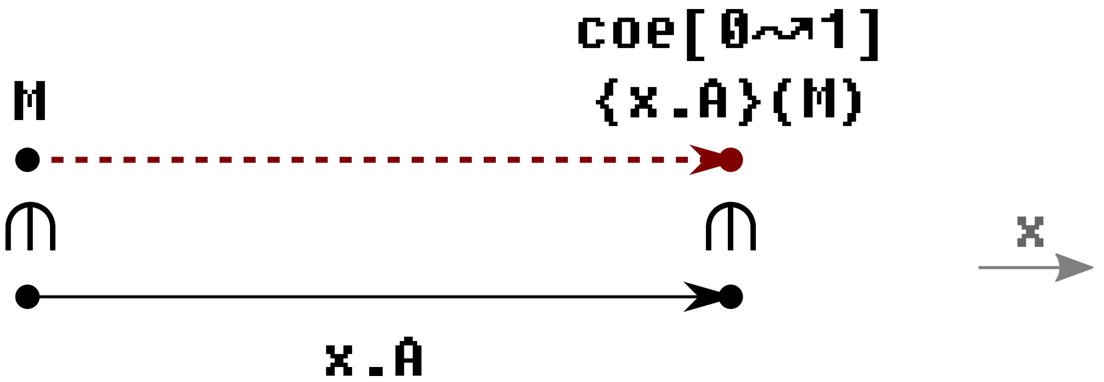

```
S1elim(a.A, loop{x}, _, y.L)
  ↦ L<x/y>
```

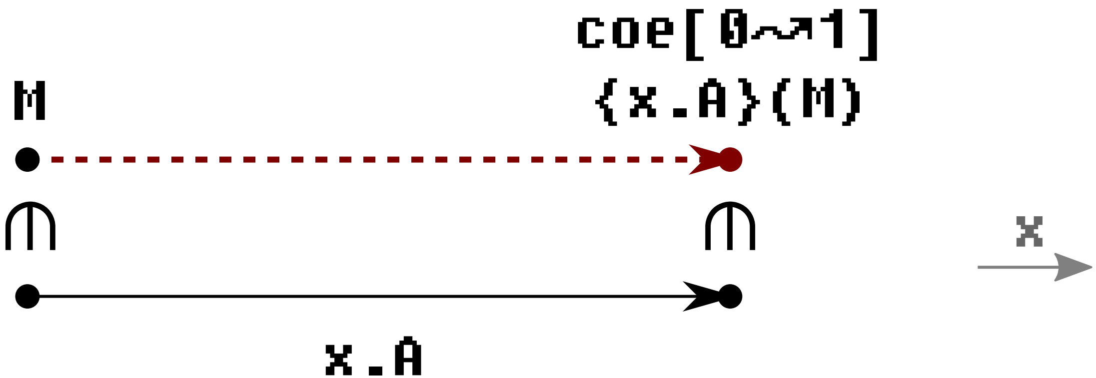
Kan 1/2: Coercion



Kan 1/2: Coercion



Kan 1/2: Coercion

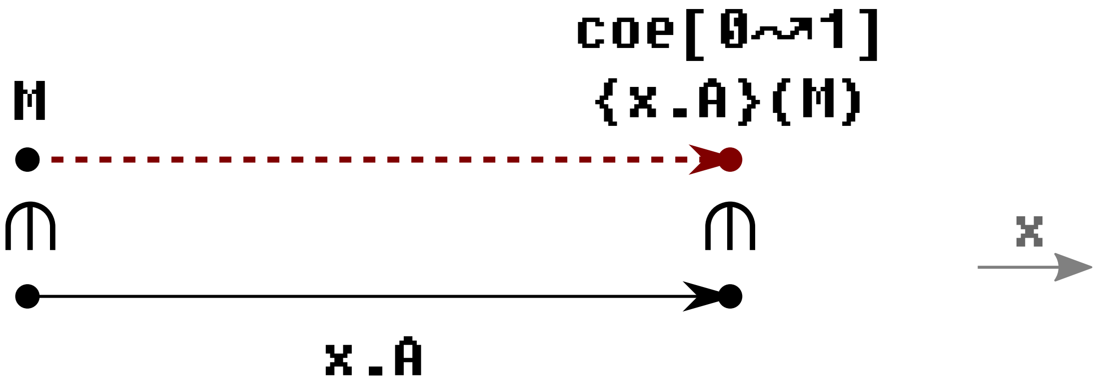


`coe[r⇝r']{x.A}(M)` ∈ `A<r'/x>`

⋔

`A<r/x>`

Kan 1/2: Coercion



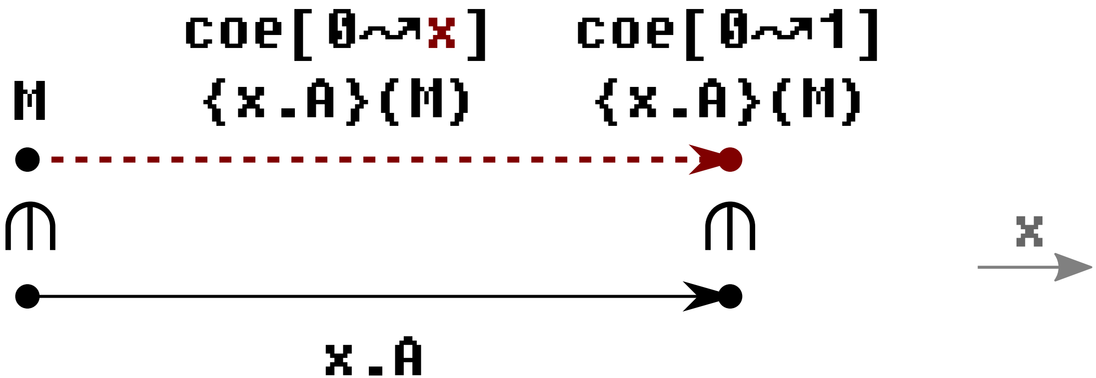
$$\mathtt{coe}[r \rightsquigarrow r']\{x.A\}(M) \in A\langle r'/x\rangle$$

$$\pitchfork$$

$$A\langle r/x\rangle$$

$$\mathtt{coe}[r \rightsquigarrow r]\{x.A\}(M) \doteq M \in A\langle r/x\rangle$$

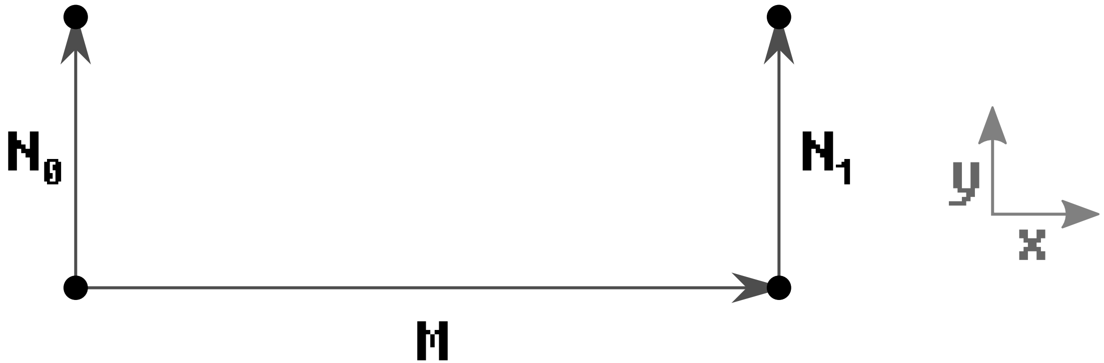
Kan 1/2: Coercion



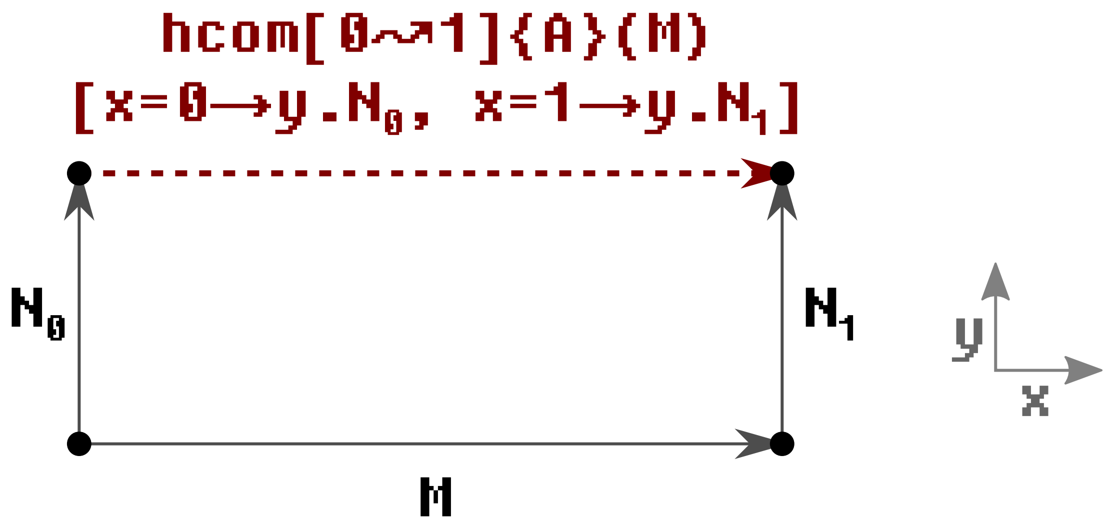
$$\texttt{coe}[r \rightsquigarrow r']\{x.A\}\underset{\displaystyle A\langle r/x\rangle}{\underset{\cap}{(M)}} \in A\langle r'/x\rangle$$

$$\texttt{coe}[r \rightsquigarrow r]\{x.A\}(M) \doteq M \in A\langle r/x\rangle$$

Kan 2/2: Homogeneous Comp.

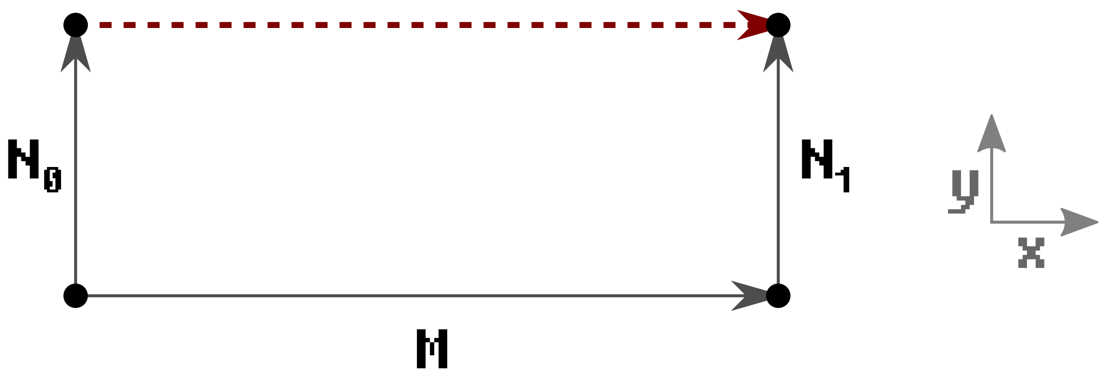


Kan 2/2: Homogeneous Comp.



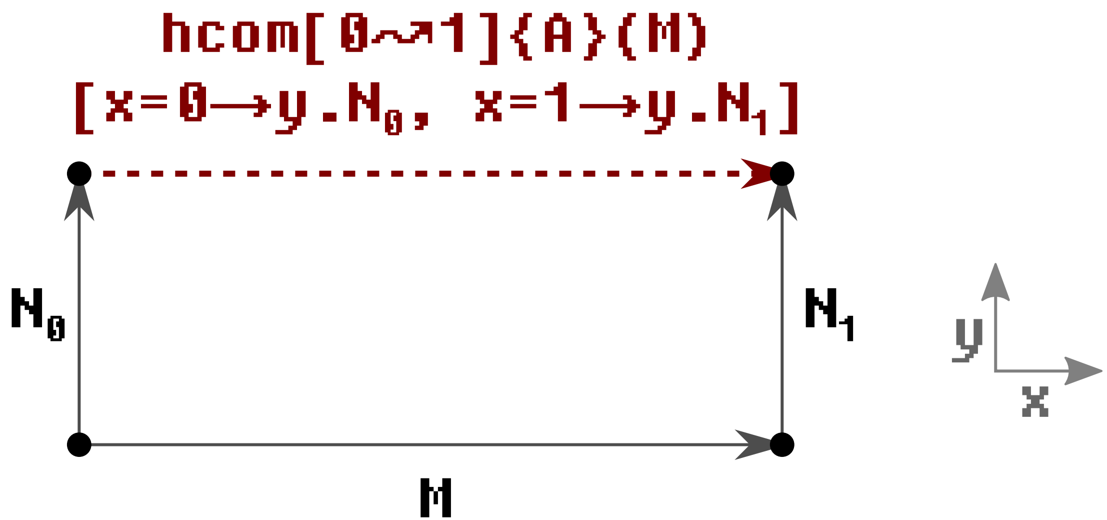
Kan 2/2: Homogeneous Comp.

hcom[0⤳1]{A}(M)
[x=0⟶y.N_0, x=1⟶y.N_1]



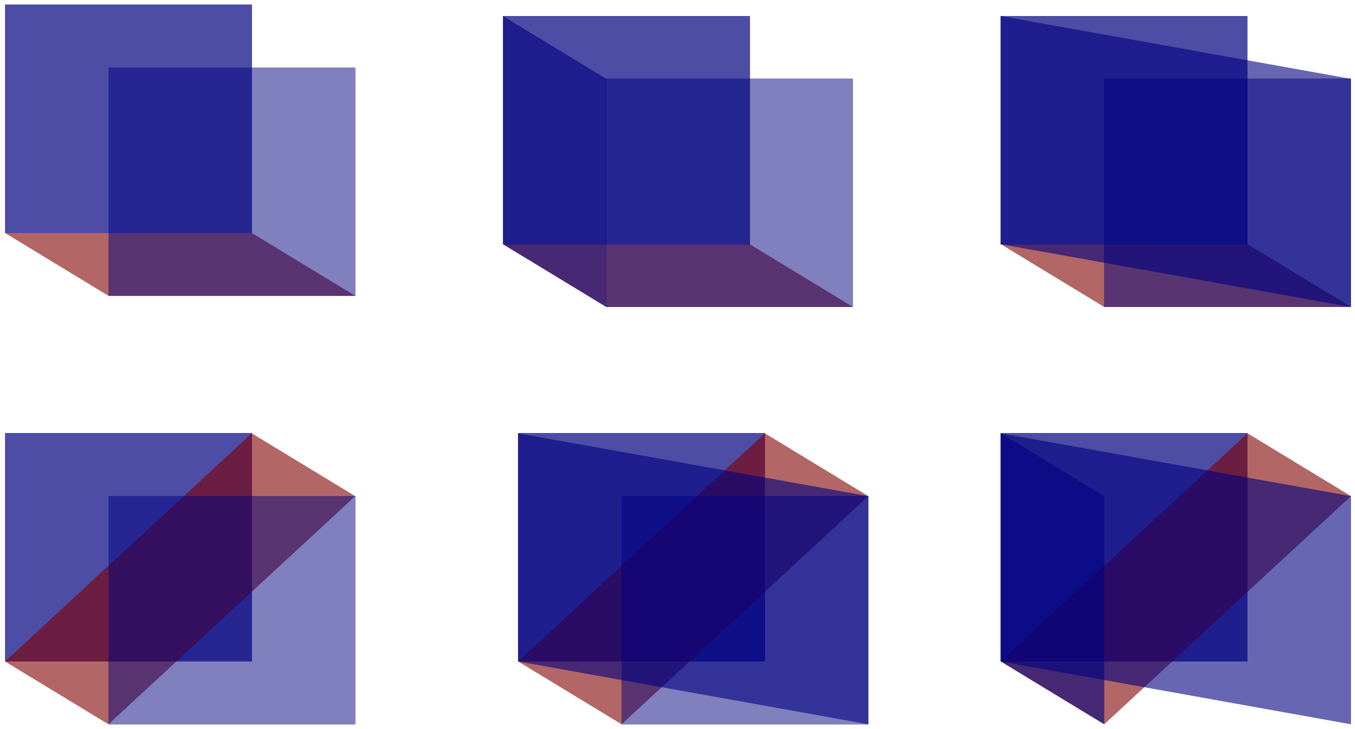
hcom[r⤳r']{A}(M) [..., r_i=r'_i⟶y.N_i, ...] ∈ A

Kan 2/2: Homogeneous Comp.




```
hcom[r⇝r']{A}(M) [..., r_i=r'_i⟶y.N_i, ...] ∈ A

hcom[r⇝r]{A}(M) ≐ M ∈ A

hcom[r⇝r']{A}(M)[..., r_i=r_i⟶y.N_i, ...]
  ≐ N_i<r'/y> ∈ A
```

Kan 2/2: Homogeneous Comp.



Kan Circle

```
coe[r~>r'](_.S1)(M) ↦ M
```

Kan Circle

```
coe[r~>r'](_.S1)(M) ↦ M
```

```
hcom[r~>r'](S1)(M)[…] ↦ fhcom[r~>r'](M)[…]
```

formal homo.
composition

Kan Circle

```
coe[r~>r'](_.S1)(M) ↦ M
```

```
hcom[r~>r'](S1)(M)[…] ↦ fhcom[r~>r'](M)[…]
```

formal homo.
composition

```
fhcom[r~>r](M)[…] ↦ M
```

Kan Circle

```
coe[r⇝r'](_.S1)(M) ↦ M
```

```
hcom[r⇝r'](S1)(M)[…] ↦ fhcom[r⇝r'](M)[…]
```

formal homo.
composition

```
fhcom[r⇝r](M)[…] ↦ M
```

```
r!=r'   r_i=r'_i (the first i)
────────────────────────────────────────────
fhcom[r⇝r'](M)[…, r_i=r'_i→y.N_i, …] ↦ N_i<r'/y>
```

Kan Circle

```
coe[r⇝r']{_.S1}(M) ↦ M
```

```
hcom[r⇝r']{S1}(M)[…] ↦ fhcom[r⇝r'](M)[…]
```

formal homo. composition

```
fhcom[r⇝r](M)[…] ↦ M
```

```
r≠r'  r_i=r'_i (the first i)
------------------------------------------------
fhcom[r⇝r'](M)[…, r_i=r'_i→y.N_i, …] ↦ N_i<r'/y>
```

```
r≠r'  r_i≠r'_i for all i
----------------------------
fhcom[r⇝r'](M)[…] val
```

Kan Circle

`S1elim` needs to handle `fcom`

Kan Circle

`S1elim` needs to handle `fcom`

```
r≠r'   rᵢ≠r'ᵢ
————————————————————————————————————
S1elim(a.A, fhcom[r⇝r'](M)[…], B, x.L)
  ↦ com[r⇝r']{y.A[fhcom[r⇝y](M)[…]/a}
        (S1elim(M, B, x.L))[…]
```

S1elim(composition) ↦ composition(S1elim)

Cubical Stability

Dimension substs. do not commute with evaluation!

Cubical Stability

Dimension substs. do not commute with evaluation!

```
S1elim(a.A,
 loop{x}, B, y.L)  |--------> L<x/y>
                                 |
                                 |  <0/x>
                                 v
                               L<0/y>
```

Cubical Stability

Dimension substs. do not commute with evaluation!

```
S1elim(a.A,
 loop{x}, B, y.L)   |-------->  L<x/y>

        |  <0/x>                  |  <0/x>
        v                         v

S1elim(a.A,
 base, B, y.L)  |--> B   <=??=>  L<0/y>
```

Cubical Stability

Dimension substs. do not commute with evaluation!

```
S1elim(a.A,
 loop{x}, B, y.L)   |-------->  L<x/y>
                                  |
       |  <0/x>                   |  <0/x>
       v                          v

S1elim(a.A,
 base, B, y.L)   |-->  B   <=??=>  L<0/y>
```

Restrict our theory to only cubically stable parts

Cubical Type Theory

stability: consider every substitution

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stability: consider every substitution

$$\mathtt{A} \doteq \mathtt{B}\ \mathtt{type}\ [\Psi]$$

dim context

A and B stably recognize the same stable values and have stably equal Kan structures

(see our arXiv papers)

Cubical Type Theory

stability: consider every substitution

$$A \doteq B \ \mathtt{type}\ [\Psi]$$

dim context

A and B stably recognize the same stable values and have stably equal Kan structures

$$M \doteq N \in A\ [\Psi]$$

$A \doteq A$ type $[\Psi]$,
M and N stably eval to M' and N',
A stably treats M' and N' as the same

(see our arXiv papers)

Variables

Nupri/...	Coq/Agda/...
Vars range over closed terms	Vars are indet.
Defined by transition b/w closed terms	Defined by conversion b/w open terms

exp vars dim vars

cubical computational TT

arXiv papers

CHTT Part I [AHW 2016]

Cartesian cubical + computational

CHTT Part II [AH 2017]

Dependent types

CHTT Part III [AFH 2017]

Univalent Kan universes

Strict equality

CHTT Part IV [AFH 2017]

Higher inductive types

Proof Assistants

RedPRL

In Nuprl style

redprl.org

redtt

(Work in progress)

github.com/RedPRL/redtt

---

yacctt

Proof of concept

modified from cubicaltt

github.com/mortberg/yacctt

Conclusion

We extended Nuprl semantics
by cubical structure which
justifies key features of HoTT

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Best of the two worlds!

We also built proof assistants

redprl.org
github.com/RedPRL/redtt
github.com/mortberg/yacctt