DELIGNE-LUSZTIG DUALITY ON THE MODULI STACK OF BUNDLES

LIN CHEN

Abstract. Let $\mathrm{Bun}_G(X)$ be the moduli stack of G-torsors on a smooth projective curve X for a reductive group G. We prove a conjecture made by Drinfeld-Wang and Gaitsgory on the Deligne-Lusztig duality for D-modules on $\mathrm{Bun}_G(X)$. This conjecture relates the pseudo-identity functors in [Gai17], [DG15] to the enhanced Eisenstein series and geometric constant term functors in [Gai15]. We also prove a "second adjointness" result for these enhanced functors.

Contents

Harvard Mathematics Department, 1 Oxford Street, Cambridge 02138, MA, USA
E-mail address: linchen@math.harvard.edu.

0. Introduction

0.1. **Motivation: Deligne-Lusztig duality.** The following pattern has been observed in several representation-theoretic contexts: The composition of two different duality functors on the category $\mathcal{C}_G$ attached to a reductive group G is isomorphic to a Deligne-Lusztig functor, given by a complex indexed by standard parabolic subgroups[1] P of G, whose terms are compositions

$$\mathcal{C}_G \xrightarrow{\mathrm{CT}_P} \mathcal{C}_M \xrightarrow{\mathrm{Eis}_P} \mathcal{C}_G,$$

where

- M is the Levi quotient group of P;
- $\mathcal{C}_M$ is the category attached to M;
- CT_P and Eis_P are adjoint functors connecting $\mathcal{C}_G$ and $\mathcal{C}_M$.

Notable examples include

- The work of Bernstein-Bezrukavnikov-Kazhdan ([BBK18]), where $\mathcal{C}_G$ is the category of representations of the group $G(K)$, where K is a non-Archimedian local field.
- The work of Yom Din ([YD19]), where $\mathcal{C}_G$ is the category of character D-modules on G.
- The work of Drinfeld-Wang ([DW16] and [Wan18]), where $\mathcal{C}_G$ is the space of automorphic functions for the group G. Note that this example is actually one categorical level down from the above partern (i.e., one needs to replace "categories" by "spaces" and "functors" by "operators").

In the present paper we establish yet another incarnation of this pattern. Namely, we take $\mathcal{C}_G$ to be the category of automorphic D-modules[2], i.e., $\mathcal{C}_G = \mathrm{Dmod}(\mathrm{Bun}_G(X))$, where $\mathrm{Bun}_G(X)$ is the moduli stack of G-torsors on a smooth complete curve X.

Our context can be viewed as directly categorifying that of Drinfeld-Wang. It is also closely connected to that of Yom Din because the category of character D-modules can be regarded as a genus 0 version of the automorphic category.

Below we will review the contexts mentioned above.

0.1.1. *Work of [BBK18].* Let G be defined over a number field and K be a non-archimedian local field. In [BBK18], the authors proved the following result about the derived category $G(K)$-mod of admissible representations of the p-adic group $G(K)$.

For any object $\mathcal{M} \in G(K)$-mod, consider the corresponding *Deligne-Lusztig complex*[3]

$$\mathrm{DL}(\mathcal{M}) := [\mathcal{M} \to \bigoplus_P i_P^G \circ r_P^G(\mathcal{M}) \to \cdots \to i_B^G \circ r_B^G(\mathcal{M})]$$

where (r_P^G, i_P^G) is the adjoint pair for the parabolic induction and Jacquet functors, and where the direct sum in the k-th term of the complex is taken over standard parabolic subgroups of corank k. The main theorem of [BBK18] says that

$$\mathrm{DL} \simeq \mathbb{D}^{\mathrm{coh}} \circ \mathbb{D}^{\mathrm{contr}}[\mathrm{rank}(G)],$$

where $\mathbb{D}^{\mathrm{contr}}$ and $\mathbb{D}^{\mathrm{coh}}$ are the contravariant endofunctors on $G(K)$-mod for the *contragredient* and *cohomological* dualities. In other words,

$$\mathbb{D}^{\mathrm{contr}}(\mathcal{M}) := \mathcal{M}^\vee,\ \mathbb{D}^{\mathrm{coh}}(\mathcal{M}) := \mathrm{RHom}_{G(K)}(\mathcal{M}, \mathcal{H}),$$

where $\mathcal{M}^\vee$ is the admissible dual, and where $\mathcal{H} = C_c^\infty(G(K))$ is the regular bimodule for the Hecke algebra. The proof in *loc.cit.* used an explicit resolution for $\mathcal{H}$ coming from the geometry of the wonderful compactification of G.

[1] We fix a Borel subgroup B of G. A parabolic subgroup is *standard* if it contains B.
[2] Our method can also be applied to the category of automoprhic sheaves with suitable modifications.
[3] Analogous complexes for finite Chevalley group were firstly studied by Deligne and Lusztig in [DL82].

0.1.2. *Work of [YD19].* Let G be defined over an algebraically closed field k of characteristic 0. In [YD19], the author proved the following result about the DG-category[4] $\mathrm{D}(G/\mathrm{Ad}(G))$ of character D-modules on G.

Let P be a standard parabolic subgroup and M be its Levi quotient group. Consider the diagram

$$G/\mathrm{Ad}(G) \xleftarrow{p} P/\mathrm{Ad}(P) \xrightarrow{q} M/\mathrm{Ad}(M).$$

The map p is projective and q is smooth. Hence we have the *parabolic restriction* functor

$$\mathrm{res}_P = q_* \circ p^! : \mathrm{D}(G/\mathrm{Ad}(G)) \to \mathrm{D}(M/\mathrm{Ad}(M))$$

and its left adjoint, a.k.a. the *parabolic induction* functor

$$\mathrm{ind}_P = p_* \circ q^! : \mathrm{D}(M/\mathrm{Ad}(M)) \to \mathrm{D}(G/\mathrm{Ad}(G)).$$

These functors are t-exact by [BYD18]. Let P^- be an opposite parabolic subgroup[5]. It is known (see [DG14, § 0.2.1]) that res_P is left adjoint to ind_{P^-}. This is analogous to the well-known Bernstein's second adjointness.

Consider the diagonal map $\Delta : G/\mathrm{Ad}(G) \to G/\mathrm{Ad}(G) \times G/\mathrm{Ad}(G)$ and the endofunctor on $\mathrm{D}(G/\mathrm{Ad}(G))$ given by the kernel $\Delta_!(k_{G/\mathrm{Ad}(G)})$, where $k_{G/\mathrm{Ad}(G)}$ is the constant D-module. Explicitly, this endofunctor is

$$\mathrm{pr}_{1,\blacktriangle}(\Delta_!(k_{G/\mathrm{Ad}(G)}) \overset{!}{\otimes} \mathrm{pr}_2^!(-)),$$

where $\mathrm{pr}_{1,\blacktriangle}$ is the *renormalized pushforward functor* in [DG13]. This endofunctor is the so-called *Drinfeld-Gaitsgory functor* for $\mathrm{D}(G/\mathrm{Ad}(G))$ in [YD19].

One of the main results of [YD19] says that the above Drinfeld-Gaitsgory functor can be "glued"[6] from the functors

$$\bigoplus_{\mathrm{rank}(P)=l} \mathrm{ind}_{P^-} \circ \mathrm{res}_P[l - \dim(T)].$$

The proof in *loc.cit.* used a filtration of $\Delta_!(k_{G/\mathrm{Ad}(G)})$ coming from the geometry of the wonderful compactification of G.

0.1.3. *The pseudo-identity functor(s).* As explained in [YD19], the above Drinfeld-Gaitsgory functor can be identified with

$$\text{Ps-Id}_{G/\mathrm{Ad}(G),!} \circ (\text{Ps-Id}_{G/\mathrm{Ad}(G),\mathrm{naive}})^{-1},$$

where Ps-Id are the pseudo-identity functors constructed in [DG13]. Here are more details.

Let Y be a QCAD algebraic stack[7] in the sense of *loc.cit.*. Consider the cocomplete DG-category $\mathrm{D}(Y)$ and its full subcategory $\mathrm{D}(Y)^c$ of compact objects. Verdier duality provides an equivalence

$$\mathbb{D}^{\mathrm{Ver}} : \mathrm{D}(Y)^c \to \mathrm{D}(Y)^{c,\mathrm{op}}.$$

Using ind-completion, we obtain an equivalence

$$\text{Ps-Id}_{Y,\mathrm{naive}} : \mathrm{D}(Y)^\vee \simeq \mathrm{D}(Y),$$

where $\mathrm{D}(Y)^\vee$ is the *Lurie dual*[8] of $\mathrm{D}(Y)$.

On the other hand, we have *the product formula*:

$$\mathrm{D}(Y \times Y) \simeq \mathrm{D}(Y) \underset{k}{\otimes} \mathrm{D}(Y),$$

[4] See Notation 0.5.4 for our conventions for DG-categories.

[5] In order to define P^-, we fix a Carton subgroup of G.

[6] Roughly speaking, this means that up to a cohomological shift, the Drinfeld-Gaitsgory functor sends an object $\mathcal{F} \in \mathrm{D}(G/\mathrm{Ad}(G))^\heartsuit$ in the heart of the t-structure to a *certain* complex

$$\mathcal{F} \to \bigoplus_P \mathrm{ind}_{P^-} \circ \mathrm{res}_P(\mathcal{F}) \to \cdots \to \mathrm{ind}_{B^-} \circ \mathrm{res}_B(\mathcal{F}).$$

However, [YD19] did *not* describe the connecting morphisms in the above complex. Nevertheless, we have confidence that one can use the method in the current paper to show that these connecting maps are given by the adjunction natural transformations of the second adjointness.

[7] This means Y is a quasi-compact algebraic stack whose automorphism groups of geometric points are affine.

[8] See Notation 0.5.4 for what this means.

where $\otimes_k$ is the *Lurie tensor product* of cocomplete DG-categories. The RHS can be identified with $\mathrm{LFun}_k(\mathrm{D}(Y)^\vee, \mathrm{D}(Y))$, i.e., the category of k-linear colimit-preserving functors $\mathrm{D}(Y)^\vee \to \mathrm{D}(Y)$. Hence the object $\Delta_!(k_Y) \in \mathrm{D}(Y \times Y)$ provides a functor

$$\text{Ps-Id}_{Y,!} : \mathrm{D}(Y)^\vee \to \mathrm{D}(Y).$$

In general, the functor $\text{Ps-Id}_{Y,!}$ is not an equivalence. We say Y is *miraculous* if it is an equivalence.

It is known that the equivalence $\text{Ps-Id}_{Y,\mathrm{naive}}$ can be obtained in the same way by replacing $\Delta_!(k_Y)$ by $\Delta_*(\omega_Y)$, where ω_Y is the dualizing D-module on Y. For this reason, the functor $\text{Ps-Id}_{Y,\mathrm{naive}}$ is also denoted by $\text{Ps-Id}_{Y,*}$ in the literature.

It follows from definitions that the composition $\text{Ps-Id}_{Y,!} \circ (\text{Ps-Id}_{Y,\mathrm{naive}})^{-1}$ is the functor given by the kernel $\Delta_!(k_Y)$. In other words, it is the Drinfeld-Gaitsgory functor for $\mathrm{D}(Y)$.

Remark 0.1.4. By [YD19, Proposition 5.5], the Drinfeld-Gaitsgory functor for $\mathrm{D}(G/\mathrm{Ad}(G))$ is invertible, hence so is $\text{Ps-Id}_{!,G/\mathrm{Ad}(G)}$. In other words, $G/\mathrm{Ad}(G)$ is miraculous.

0.1.5. *The current work.* Let G be as in § 0.1.2 and X be a connected smooth projective curve over k. Let Bun_G be the moduli stack of G-torsors on X. The purpose of this paper is to describe the Deligne-Lusztig duality on the DG-category $\mathrm{D}(\mathrm{Bun}_G)$ of D-modules on Bun_G.

Unlike $G/\mathrm{Ad}(G)$, the stack Bun_G is *not* quasi-compact. Nevertheless, the main theorem of [DG15] says that $\mathrm{D}(\mathrm{Bun}_G)$ is compactly generated and hence dualizable. Also, the product formula

$$\mathrm{D}(\mathrm{Bun}_G \times \mathrm{Bun}_G) \simeq \mathrm{D}(\mathrm{Bun}_G) \underset{k}{\otimes} \mathrm{D}(\mathrm{Bun}_G)$$

still holds (see [DG15, Remark 2.2.9]). Hence as before, we have equivalences

$$\begin{aligned} \mathrm{LFun}_k(\mathrm{D}(\mathrm{Bun}_G)^\vee, \mathrm{D}(\mathrm{Bun}_G)) &\simeq \mathrm{D}(\mathrm{Bun}_G) \underset{k}{\otimes} (\mathrm{D}(\mathrm{Bun}_G)^\vee)^\vee \simeq \\ &\simeq \mathrm{D}(\mathrm{Bun}_G) \underset{k}{\otimes} \mathrm{D}(\mathrm{Bun}_G) \simeq \mathrm{D}(\mathrm{Bun}_G \times \mathrm{Bun}_G), \end{aligned} \tag{0.1}$$

and we use the objects $\Delta_*(\omega_{\mathrm{Bun}_G})$, $\Delta_!(k_{\mathrm{Bun}_G})$ in the RHS to define functors

$$\text{Ps-Id}_{\mathrm{Bun}_G,\mathrm{naive}}, \text{Ps-Id}_{\mathrm{Bun}_G,!} : \mathrm{D}(\mathrm{Bun}_G)^\vee \to \mathrm{D}(\mathrm{Bun}_G).$$

From now on, we write them just as $\text{Ps-Id}_{\mathrm{naive}}$, $\text{Ps-Id}_!$.

Unlike the quasi-compact case, the functor $\text{Ps-Id}_{\mathrm{naive}}$ is *not* an equivalence. On the contrary, the main theorem of [Gai17] says:

- The functor $\text{Ps-Id}_! : \mathrm{D}(\mathrm{Bun}_G)^\vee \to \mathrm{D}(\mathrm{Bun}_G)$ is an equivalence, i.e., Bun_G is miraculous.

Accordingly, the *Deligne-Lusztig duality for* Bun_G in this paper is actually analogous to the "left adjoint version" of [BBK18] and [YD19]. Namely, we will show

- Up to cohomological shifts, the endofunctor $\text{Ps-Id}_{\mathrm{naive}} \circ \text{Ps-Id}_!^{-1}$ on $\mathrm{D}(\mathrm{Bun}_G)$ can be "glued" from the functors
$$\bigoplus_{\mathrm{corank}(P)=l} \mathrm{Eis}^{\mathrm{enh}}_{P\to G} \circ \mathrm{CT}^{\mathrm{enh}}_{G\to P},$$
where $\mathrm{Eis}^{\mathrm{enh}}_{P\to G}$ (resp. $\mathrm{CT}^{\mathrm{enh}}_{G\to P}$) is the enhanced Eisenstein series (resp. enhanced constant term) functor[9] on $\mathrm{D}(\mathrm{Bun}_G)$.

The precise statement of our main theorem will be given in § 0.3. See Theorem 0.3.1.

Remark 0.1.6. For $G = \mathrm{SL}_2$, our main theorem was conjectured by V. Drinfeld and J. Wang in [DW16, Conjecture C.2.1]. For general G, Wang made the following remark in [Wan18, Remark 6.6.5]:

> *...describing the functor inverse to* $\text{Ps-Id}_{\mathrm{Bun}_G,!}$ *(we expect that one can mimic the construction of the Deligne-Lusztig complex using the functors* $\mathrm{Eis}^{\mathrm{enh}}_P, \mathrm{CT}^{\mathrm{enh}}_P$*).*

However, as far as we know, the precise formulation[10] of the conjecture for general G was first made by D. Gaitsgory and recorded by D. Beraldo in [Ber19, § 1.5.5].

[9]We review the definitions of them in § 0.2. Let us emphasize that $\mathrm{Eis}^{\mathrm{enh}}_{P\to G}$ is the *left adjoint* of $\mathrm{CT}^{\mathrm{enh}}_{G\to P}$.

[10]Since the functors $\mathrm{Eis}^{\mathrm{enh}}_P, \mathrm{CT}^{\mathrm{enh}}_P$ are not t-exact, the naïve formation of the Deligne-Lusztig complex is not a well-defined object in the DG-category $\mathrm{D}(\mathrm{Bun}_G)$.

0.1.7. *Relation with the Drinfeld-Wang operator.* Drinfeld and Wang made their conjecture according to an analogous result on the space of automorphic forms proved by them in [DW16] and [Wan18]. Let us briefly explain their work.

Let F be a global function field over $\mathbb{F}_q$ and $\mathbb{A}$ be the adele ring of F. Let G be a split reductive group over $\mathbb{F}_q$ and $G(\mathbb{O})$ be the maximal compact subgroup of $G(\mathbb{A})$. Let $\mathcal{A}_c$ be the space of compactly supported smooth $G(\mathbb{O})$-finite functions on $G(\mathbb{A})/G(F)$. As explained in [DW16, Appendix A], the DG-category $\mathrm{D}(\mathrm{Bun}_G)^\vee$ (when G is in characteristic 0) can be viewed as a geometric analogue of $\mathcal{A}_c^{G(\mathbb{O})}$ (the subspace of $\mathcal{A}_c$ fixed by $G(\mathbb{O})$).

Drinfeld and Wang also defined a subspace $\mathcal{A}_{pc\text{-}c} \subset \mathcal{A}_c$ such that $\mathcal{A}_{pc\text{-}c}^{G(\mathbb{O})}$ is analogous to $\mathrm{D}(\mathrm{Bun}_G)$. They also constructed a $G(\mathbb{A})$-linear operator

$$L : \mathcal{A}_c \to \mathcal{A}_{pc\text{-}c}$$

such that $L^{G(\mathbb{O})} : \mathcal{A}_c^{G(\mathbb{O})} \to \mathcal{A}_{pc\text{-}c}^{G(\mathbb{O})}$ is analogous to the functor $\text{Ps-Id}_!$. Moreover, they proved L is invertible and gave the following explicit formula for its inverse:

$$L^{-1} f = \sum_P (-1)^{\dim Z_M} \mathrm{Eis}_P \circ \mathrm{CT}_P(f),$$

where $\mathrm{Eis}_P, \mathrm{CT}_P$ are the Eisenstein and constant term operators, and where Z_M is the center of M.

Our main theorem can be viewed as a categorification of the above formula (when restricted to $G(\mathbb{O})$-invariant subspaces). We refer the reader to [DW16, Appendix C] for more details on this analogy.

0.2. **Recollections: The parabolic category $\mathbf{I}(G,P)$.** From now on, we fix a connected reductive group[11] G defined over an algebraically closed field k of characteristic 0. For simplicity, we assume $[G,G]$ to be simply connected. We also fix a connected smooth projective curve X over k.

Let P be a parabolic subgroup of G and M be its Levi quotient group. Consider the diagram

$$\mathrm{Bun}_G \xleftarrow{\mathfrak{p}_P} \mathrm{Bun}_P \xrightarrow{\mathfrak{q}_P} \mathrm{Bun}_M .$$

In [BG02] and [DG16], the authors constructed the *geometric Eisenstein series functor* and the *geometric constant term functor*

$$\mathrm{Eis}_{P,!} = \mathfrak{p}_{P,!} \circ \mathfrak{q}_P^* : \quad \mathrm{D}(\mathrm{Bun}_M) \to \mathrm{D}(\mathrm{Bun}_G)$$
$$\mathrm{CT}_{P,*} = \mathfrak{q}_{P,*} \circ \mathfrak{p}_P^! : \quad \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_M).$$

However, they are not the functors that we will use (otherwise the main theorem would be false). Instead, we need to replace $\mathrm{D}(\mathrm{Bun}_M)$ by the category $\mathbf{I}(G,P)$ defined in [Gai15, § 6], and accordingly use the "enhanced" functors $(\mathrm{Eis}_P^{\mathrm{enh}}, \mathrm{CT}_P^{\mathrm{enh}})$ defined there. We review these functors in this subsection.

Remark 0.2.1. As explained in *loc.cit.*, one can think of $\mathbf{I}(G,P)$ as the DG-category of D-modules on a *non-existent* stack obtained by gluing all the connected components of Bun_M together. Since this imaginary stack has the same field-valued points as Bun_M, the difference between $\mathbf{I}(G,P)$ and $\mathrm{D}(\mathrm{Bun}_M)$ can not be seen in their "de-categorifications". In other words, both $\mathrm{Eis}_{P,!}$ and $\mathrm{Eis}_P^{\mathrm{enh}}$ are analogous to the Eisenstein series operator for automorphic functions.

0.2.2. *Prestack of generic reductions.* Let K be any algebraic group and H be a subgroup of K. In [Bar14, Example 2.2.5], J. Barlev constructed a lft prestack[12] $\mathrm{Bun}_K^{H\text{-gen}}$ classifying a K-torsor on X equipped with a generic reduction to H. In the notation of *loc.cit.*, it is defined by

$$\mathrm{Bun}_K^{H\text{-gen}} := \mathbf{Maps}(X, \mathbb{B}K) \underset{\mathbf{GMap}(X,\mathbb{B}K)}{\times} \mathbf{GMap}(X, \mathbb{B}H).$$

To simplify the notation, we write the RHS as $\mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}K \leftarrow \mathbb{B}H)$. More generally, for any map between lft prestacks $Y_1 \to Y_2$, we define

$$\mathbf{Maps}_{\mathrm{gen}}(X, Y_2 \leftarrow Y_1) := \mathbf{Maps}(X, Y_2) \underset{\mathbf{GMap}(X,Y_2)}{\times} \mathbf{GMap}(X, Y_1).$$

[11]See Notation 0.5.6 for our notations for concepts related to G.
[12]See Definition 0.5.2 for what this means.

For future use, let us mention that the functor $\mathbf{Maps}_{\mathrm{gen}}(X,-)$ commutes with finite limits (see [Bar14, Remark 2.2.6]).

Applying the above construction to the diagram

$$(G,P) \leftarrow (P,P) \rightarrow (M,M),$$

we obtain a diagram

$$\mathrm{Bun}_G^{P\text{-gen}} \xleftarrow{\iota_P} \mathrm{Bun}_P \xrightarrow{\mathfrak{q}_P} \mathrm{Bun}_M .$$

Remark 0.2.3. The prestack $\mathrm{Bun}_G^{P\text{-gen}}$ has the same field-valued points as Bun_P.

Definition 0.2.4. *The DG-category $\mathbf{I}(G,P)$ is defined as the fiber product of the following diagram:*

$$\begin{tikzcd} \mathbf{I}(G,P) \arrow[r, dotted] \arrow[d, dotted] & \mathrm{D}(\mathrm{Bun}_M) \arrow[d, "\mathfrak{q}_P^*"] \\ \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}}) \arrow[r, "\iota_P^!"] & \mathrm{D}(\mathrm{Bun}_P). \end{tikzcd}$$

Remark 0.2.5. The above definition is equivalent to that in [Gai15, § 6.1] by [Gai15, Lemma 6.3.3].

Remark 0.2.6. By [Gai15, Lemma 6.1.2], the functor $\iota_P^!$ is conservative. By [Gai15, § 6.2.1], the functor $\mathfrak{q}_P^*$ is fully faithful. Therefore the functor

$$\mathbf{I}(G,P) \rightarrow \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}})$$

is fully faithful and the functor $\mathbf{I}(G,P) \rightarrow \mathrm{D}(\mathrm{Bun}_M)$ is conservative. Following *loc.cit.*, we denote the last functor by

$$\iota_M^! : \mathbf{I}(G,P) \rightarrow \mathrm{D}(\mathrm{Bun}_M).$$

The following result was claimed in [Gai15, § 6.2.5]. We provide a proof in Appendix B.1.1.

Proposition 0.2.7. *(Gaitsgory)*
The partially defined left adjoint $\iota_{P,!}$ to $\iota_P^!$ is well-defined on the essential image of $\mathfrak{q}_P^*$, and its image is contained in $\mathbf{I}(G,P)$.

Corollary 0.2.8. *The functor $\iota_M^! : \mathbf{I}(G,P) \rightarrow \mathrm{D}(\mathrm{Bun}_M)$ has a left adjoint*

$$\iota_{M,!} : \mathrm{D}(\mathrm{Bun}_M) \rightarrow \mathbf{I}(G,P).$$

Proof. By Proposition 0.2.7, the functor $\iota_{P,!} \circ \mathfrak{q}_P^*$ uniquely factors through a functor $\mathrm{D}(\mathrm{Bun}_M) \rightarrow \mathbf{I}(G,P)$, which is the desired left adjoint.

□[Corollary 0.2.8]

Remark 0.2.9. Since $\iota_M^!$ is conservative, the image of its left adjoint $\iota_{M,!}$ generates $\mathbf{I}(G,P)$. Hence $\mathbf{I}(G,P)$ is compactly generated because so is $\mathrm{D}(\mathrm{Bun}_M)$. Note that $\mathbf{I}(G,P) \rightarrow \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}})$ preserves compact objects because so is $\iota_{P,!} \circ \mathfrak{q}_P^*$.

0.2.10. *Enhanced Eisenstein series functor.* Let Q be another parabolic subgroup of G that contains P. Consider the map

$$\mathfrak{p}_{P\rightarrow Q}^{\mathrm{enh}} : \mathrm{Bun}_G^{P\text{-gen}} \rightarrow \mathrm{Bun}_G^{Q\text{-gen}}$$

and the functor

$$\mathfrak{p}_{P\rightarrow Q}^{\mathrm{enh},!} : \mathrm{D}(\mathrm{Bun}_G^{Q\text{-gen}}) \rightarrow \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}})$$

The special case (when $Q = G$) of the following result was claimed in [Gai15, Lemma 6.3.3]. We provide a proof in Appendix B.1.2.

Proposition 0.2.11. *(Gaitsgory)*

(1) *The partially defined left adjoint $\mathfrak{p}_{P\rightarrow Q,!}^{\mathrm{enh}}$ to $\mathfrak{p}_{P\rightarrow Q}^{\mathrm{enh},!}$ is well-defined on $\mathbf{I}(G,P) \subset \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}})$, and sends $\mathbf{I}(G,P)$ into $\mathbf{I}(G,Q)$.*

(2) *Let*

$$\mathrm{Eis}^{\mathrm{enh}}_{P\to Q}:\mathbf{I}(G,P)\to\mathbf{I}(G,Q)$$

be the functor obtained from $\mathfrak{p}^{\mathrm{enh}}_{P\to Q,!}$. *Then* $\mathrm{Eis}^{\mathrm{enh}}_{P\to Q}$ *has a* continuous *right adjoint*

$$\mathrm{CT}^{\mathrm{enh}}_{Q\to P}:\mathbf{I}(G,Q)\to\mathbf{I}(G,P).$$

Remark 0.2.12. When $Q=G$, we also denote the adjoint pair $(\mathrm{Eis}^{\mathrm{enh}}_{P\to G},\mathrm{CT}^{\mathrm{enh}}_{G\to P})$ by $(\mathrm{Eis}^{\mathrm{enh}}_{P},\mathrm{CT}^{\mathrm{enh}}_{P})$.

Warning 0.2.13. *The functor* $\mathfrak{p}^{\mathrm{enh},!}_{P\to Q}$ *does not send* $\mathbf{I}(G,Q)$ *into* $\mathbf{I}(G,P)$. *Hence the functor* $\mathrm{CT}^{\mathrm{enh}}_{Q\to P}$ *is not the restriction of* $\mathfrak{p}^{\mathrm{enh},!}_{P\to Q}$. *Instead, it is given by* $\mathrm{Av}^{\mathbf{U}_P}\circ\mathfrak{p}^{\mathrm{enh},!}_{P\to Q}$, *where* $\mathrm{Av}^{\mathbf{U}_P}$ *is the right adjoint to the inclusion* $\mathbf{I}(G,P)\subset\mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}})$. *We refer the reader to [Gai15, § 6.1.3] for the meaning of the symbol* $\mathrm{Av}^{\mathbf{U}_P}$.

0.3. **The main theorem.** We fix a Borel subgroup B of G. Let Par be the poset of standard parabolic subgroups (i.e., parabolic subgroups containing B) of G. Let Par' be $\mathrm{Par}-\{G\}$. We view posets as categories in the standard way. It follows formally from Proposition 0.2.11 that we have a functor

$$\mathbf{DL}:\mathrm{Par}\to\mathrm{LFun}_k(\mathrm{D}(\mathrm{Bun}_G),\mathrm{D}(\mathrm{Bun}_G)),\ P\mapsto\mathrm{Eis}^{\mathrm{enh}}_{P\to G}\circ\mathrm{CT}^{\mathrm{enh}}_{G\to P}$$

such that a morphism $P\to Q$ in Par is sent to the composition

$$\mathrm{Eis}^{\mathrm{enh}}_{P\to G}\circ\mathrm{CT}^{\mathrm{enh}}_{G\to P}\simeq\mathrm{Eis}^{\mathrm{enh}}_{Q\to G}\circ\mathrm{Eis}^{\mathrm{enh}}_{P\to Q}\circ\mathrm{CT}^{\mathrm{enh}}_{Q\to P}\circ\mathrm{CT}^{\mathrm{enh}}_{G\to Q}\to\mathrm{Eis}^{\mathrm{enh}}_{Q\to G}\circ\mathrm{CT}^{\mathrm{enh}}_{G\to Q}.$$

Note that $\mathbf{DL}(G)$ is the identity functor $\mathbf{Id}$ on $\mathrm{D}(\mathrm{Bun}_G)$.

Our main theorem is

Theorem 0.3.1. *There is a canonical equivalence*

$$\mathrm{coFib}(\operatorname*{colim}_{P\in\mathrm{Par}'}\mathbf{DL}(P)\to\mathbf{DL}(G))\simeq\mathrm{Ps\text{-}Id}_{\mathrm{naive}}\circ\mathrm{Ps\text{-}Id}_!^{-1}[-2\dim(\mathrm{Bun}_G)-\dim(Z_G)]\tag{0.2}$$

in $\mathrm{LFun}_k(\mathrm{D}(\mathrm{Bun}_G),\mathrm{D}(\mathrm{Bun}_G))$.

Remark 0.3.2. Let $\mathcal{F}\in\mathrm{D}(\mathrm{Bun}_G)^{\heartsuit}$ be an object in the heart of the t-structure. If the functors $\mathbf{DL}(P)$ were t-exact (which is not true), then the value of the LHS of (0.2) on $\mathcal{F}$ can be calculated by a complex

$$\mathbf{DL}(B)(\mathcal{F})\to\cdots\to\bigoplus_{\mathrm{corank}(P)=1}\mathbf{DL}(P)(\mathcal{F})\to\mathbf{DL}(G)(\mathcal{F}).$$

Hence the LHS of (0.2) can be viewed as an ∞-categorical analogue for the Deligne-Lusztig complex.

0.3.3. *A stronger result.* As mentioned in [Ber19, Remark 1.5.6], Gaitsgory's strategy for the proof of the above theorem is to express both sides via the Drinfeld's compactification $\overline{\mathrm{Bun}}_G:=\mathrm{VinBun}_G/T$, where

$$\mathrm{VinBun}_G:=\mathbf{Maps}_{\mathrm{gen}}(X,G\backslash\mathrm{Vin}_G/G\supset G\backslash{}_0\mathrm{Vin}_G/G).$$

We refer the reader to [Sch16, § 2.2.4], [FKM20, § 2.3.3] and [Che21, § 1] for a detailed discussion about Vin_G and VinBun_G. For now, it is enough to know [13]:

- $\overline{\mathrm{Bun}}_G$ is an algebraic stack;
- There is a canonical map $\overline{\mathrm{Bun}}_G\to T^+_{\mathrm{ad}}/T$, where $T^+_{\mathrm{ad}}:=\mathbb{A}^{\mathcal{I}}\supset\mathbb{G}_m^{\mathcal{I}}\simeq T_{\mathrm{ad}}$ is the semi-group completion[14] of the adjoint torus T_{ad};
- The diagonal map $\Delta:\mathrm{Bun}_G\to\mathrm{Bun}_G\times\mathrm{Bun}_G$ canonically factors as

$$\mathrm{Bun}_G\xrightarrow{b}\overline{\mathrm{Bun}}_G\xrightarrow{\overline{\Delta}}\mathrm{Bun}_G\times\mathrm{Bun}_G$$

and the map $\overline{\Delta}$ is schematic and proper.

---

[13] It is helpful to have the case $G=\mathrm{SL}_2$ in mind. In this case, $\mathrm{VinBun}_G(S)$ classifies $\phi:E_1\to E_2$, where E_1 and E_2 are rank 2 vector bundles on $X\times S$ with trivialized determinant line bundles, and ϕ is a map between *coherent* $\mathcal{O}_{X\times S}$-modules such that for any geometric point s of S, the map $\phi|_{X\times s}$ is nonzero. The Cartan subgroup $\mathbb{G}_m$ acts on $\mathrm{VinBun}_{\mathrm{SL}_2}$ by scalar multiplication on ϕ.

We warn the reader that the projection $\mathrm{VinBun}_G\to\mathrm{Bun}_G\times\mathrm{Bun}_G$ below sends the above data to (E_2,E_1) rather than (E_1,E_2). See [Che21, Warning 1.1.3] for the reason of this convention.

[14] See Notation 0.5.7 for more information about T^+_{ad}.

- Let Z_G be the center of G. The map $b : \mathrm{Bun}_G \to \overline{\mathrm{Bun}}_G$ canonically factors as
$$\mathrm{Bun}_G \xrightarrow{r} \mathrm{Bun}_G \times \mathbb{B}Z_G \xrightarrow{j_G} \overline{\mathrm{Bun}}_G$$
and the map j_G is a schematic open embedding onto
$$\overline{\mathrm{Bun}}_{G,G} \coloneqq \overline{\mathrm{Bun}}_G \underset{T^+_{\mathrm{ad}}/T}{\times} T_{\mathrm{ad}}/T.$$

The coordinate stratification on $T^+_{\mathrm{ad}} \coloneqq \mathbb{A}^{\mathcal{I}}$ (see Notation 0.5.7) provides a stratification on $\overline{\mathrm{Bun}}_G$ labelled by Par, known as *the parabolic stratification*. For a standard parabolic $P \in \mathrm{Par}$, the P-stratum is defined as
$$\overline{\mathrm{Bun}}_{G,P} \coloneqq \overline{\mathrm{Bun}}_G \underset{T^+_{\mathrm{ad}}/T}{\times} T^+_{\mathrm{ad},P}/T.$$
We deonte the corresponding locally closed embedding by
$$i_P : \overline{\mathrm{Bun}}_{G,P} \to \overline{\mathrm{Bun}}_G.$$

Note that Par is isomorphic to the power poset $P(\mathcal{I})$ of $\mathcal{I}$. By a general construction (see Corollary C.1.7) for stacks stratified by power posets, we have a canonically defined functor[15]
$$\begin{aligned} \mathbf{K} \quad : \quad & \mathrm{Par} \to \mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_G), \\ & P \mapsto i_{P,!} \circ i_P^* \circ j_{G,*} \circ r_!(k_{\mathrm{Bun}_G})[\mathrm{rank}(M) - \mathrm{rank}(G)], \end{aligned} \tag{0.3}$$
and a canonical isomorphism (see Lemma C.1.9)
$$\mathrm{coFib}(\underset{P \in \mathrm{Par}'}{\mathrm{colim}}\, \mathbf{K}(P) \to \mathbf{K}(G)) \simeq j_{G,*} \circ r_!(k_{\mathrm{Bun}_G}). \tag{0.4}$$
Consider the composition
$$\mathbf{E} : \mathrm{LFun}_k(\mathrm{D}(\mathrm{Bun}_G), \mathrm{D}(\mathrm{Bun}_G)) \to \mathrm{LFun}_k(\mathrm{D}(\mathrm{Bun}_G)^\vee, \mathrm{D}(\mathrm{Bun}_G)) \simeq \mathrm{D}(\mathrm{Bun}_G \times \mathrm{Bun}_G), \tag{0.5}$$
where the first functor is given by precomposition with $\text{Ps-Id}_!$ and the last equivalence is (0.1). Equivalently, $\mathbf{E}$ sends an endo-functor F to
$$\mathbf{E}(F) \simeq (F \otimes \mathbf{Id}) \circ \Delta_!(k_{\mathrm{Bun}_G}),$$
where we view $F \otimes \mathbf{Id}$ as an endo-functor of $\mathrm{D}(\mathrm{Bun}_G \times \mathrm{Bun}_G) \simeq \mathrm{D}(\mathrm{Bun}_G) \otimes_k \mathrm{D}(\mathrm{Bun}_G)$.

Let us first deduce Theorem 0.3.1 from the following stronger result, which will be proved in § 1 (but with some details postponed to the latter sections).

Theorem 0.3.4. *There is a canonical commutative diagram*
$$\begin{array}{ccc} \mathrm{Par} & \xrightarrow{\mathbf{DL}} & \mathrm{LFun}_k(\mathrm{D}(\mathrm{Bun}_G), \mathrm{D}(\mathrm{Bun}_G)) \\ \downarrow{\mathbf{K}} & & \downarrow{\mathbf{E}} \\ \mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_G) & \xrightarrow{\overline{\Delta}_!} & \mathrm{D}(\mathrm{Bun}_G \times \mathrm{Bun}_G). \end{array}$$

0.3.5. *Deduction of Theorem 0.3.1 from Theorem 0.3.4.* We first deduce our main theorem from Theorem 0.3.4. This step is due to Gaitsgory.

First note that $\mathbf{E}$ is an equivalence because $\text{Ps-Id}_!$ is. By definition,
$$\mathbf{E}^{-1}(\Delta_*(\omega_{\mathrm{Bun}_G})) \simeq \text{Ps-Id}_{\mathrm{naive}} \circ \text{Ps-Id}_!^{-1}.$$

On the other hand, as in [Gai17, § 3.2.3], we have an isomorphism
$$\Delta_*(\omega_{\mathrm{Bun}_G})[-2\dim(\mathrm{Bun}_G) - \dim(Z_G)] \simeq \overline{\Delta}_! \circ j_{G,*} \circ r_!(k_{\mathrm{Bun}_G}),$$
where the cohomological shift by $-2\dim(\mathrm{Bun}_G)$ is due to the difference between ω_{Bun_G} and k_{Bun_G}, while that by $-\dim(Z_G)$ is due to the difference between r_* and $r_!$. Hence the isomorphism (0.4) implies
$$\mathrm{coFib}(\underset{P \in \mathrm{Par}'}{\mathrm{colim}}\, \mathbf{E}^{-1} \circ \overline{\Delta}_! \circ \mathbf{K}(P) \to \mathbf{E}^{-1} \circ \overline{\Delta}_! \circ \mathbf{K}(G)) \simeq \text{Ps-Id}_{\mathrm{naive}} \circ \text{Ps-Id}_!^{-1}[-2\dim(\mathrm{Bun}_G) - \dim(Z_G)].$$

[15] The functor $\mathbf{K}$ is given by $\mathbf{G}^*_{r_!(k_{\mathrm{Bun}_G}), \overline{\mathrm{Bun}}_G}$, which is defined in Corollary C.1.7.

Then we are done because $\mathbf{E}^{-1}\circ\overline{\Delta}_!\circ\mathbf{K}\simeq\mathbf{DL}$.

$\square$[Theorem 0.3.1]

Remark 0.3.6. As a first test for Thoerem 0.3.4, let us evaluate the above diagram at $G\in\mathrm{Par}$. By definition, $\mathbf{K}(G)\simeq j_{G,!}\circ r_!(k_{\mathrm{Bun}_G})$. Hence $\overline{\Delta}_!\circ\mathbf{K}(G)\simeq\Delta_!(k_{\mathrm{Bun}_G})$. On the other hand $\mathbf{DL}(G)\simeq\mathbf{Id}$, hence $\mathbf{E}\circ\mathbf{DL}(G)\simeq\Delta_!(k_{\mathrm{Bun}_G})$ by the definition of Ps-Id$_!$.

Remark 0.3.7. The statement of Theorem 0.3.1 depends on the miraculous duality on Bun_G (i.e., Ps-Id$_!$ is invertible) but that of Theorem 0.3.4 does not. Our proof of the latter will not depend on the miraculous duality either.

Remark 0.3.8. The following claim is neither proved nor used in this paper. The object $\mathbf{K}(P)$ can be obtained by the following nearby cycles construction. Choose a homomorphism $\gamma:\mathbb{A}^1\to T_{\mathrm{ad}}^+$ connecting the unit point C_G and the point C_P. In [Sch16], S. Schieder calculated the nearby cycles of the constant D-module for the $\mathbb{A}^1$-family

$$\mathrm{VinBun}_G^\gamma\coloneqq\mathrm{VinBun}_G\underset{T_{\mathrm{ad}}^+}{\times}\mathbb{A}^1.$$

Denote this nearby cycles sheaf by $\Psi^\gamma\in\mathrm{D}_{\mathrm{indhol}}(\mathrm{VinBun}_G|_{C_P})$. Then up to a cohomological shift, $\mathbf{K}(P)$ is isomorphic to the !-pushforward of Ψ^γ along $\mathrm{VinBun}_G|_{C_P}\to\overline{\mathrm{Bun}}_G$. Moreover, one can use Ψ^γ to construct a duality bewteen $\mathbf{I}(G,P)$ and $\mathbf{I}(G,P^-)$. See [Che21, Theorem E, Theorem H] for more details.

0.4. **Organization of this paper.**

The outline of the proof for Theorem 0.3.4 will be provided in § 1. Each other section corresponds to a step in that proof.

In Appendix A, we review the theory of D-modules.

In Appendix B, we provide proofs for the results mentioned in § 0.2 (which are due to Gaitsgory).

In Appendix C, we review the gluing functors for D-modules on stratified stacks.

In Appendix D, we prove some results about the group scheme $\widetilde{G}$, which is the stabilizer of the canonical section $\mathfrak{s}:T_{\mathrm{ad}}^+\to\mathrm{Vin}_G$ for the $(G\times G)$-action on Vin_G.

0.5. **Notations and conventions.**

Notation 0.5.1. *(∞-categories)*

We use the theory of $(\infty,1)$-categories developed in [Lur09]. We use same symbols to denote a $(1,1)$-category and its simplicial nerve. The reader can distinguish them according to the context.

For two objects $c_1,c_2\in C$ in an $(\infty,1)$-category C, we write $\mathrm{Maps}_C(c_1,c_2)$ *for the* mapping space *between them, which is an object in the homotopy category of spaces. We omit the subscript C if there is no ambiguity.*

We also need the theory of $(\infty,2)$-categories developed in [GR17]. For two objects $c_1,c_2\in\mathbb{S}$ in an $(\infty,2)$-category $\mathbb{S}$, we write $\mathbf{Maps}_{\mathbb{S}}(c_1,c_2)$ *for the* mapping $(\infty,1)$-category *between them.*

Definition 0.5.2. *(Algebraic geometry)*

Unless otherwise stated, all algebro-geometric objects are defined over a fixed algebraically closed ground field k of characteristic 0, and are classical (i.e. non-derived).

A locally finite type prestack *or* lft prestack *is a contravariant (accessible) functor*

$$(\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})^{\mathrm{op}}\to\mathrm{Groupoids}$$

from the category of affine k-schemes to the category of groupoids. The collection of them form a $(2,1)$-category $\mathrm{PreStk}_{\mathrm{lft}}$.

An algebraic stack *is a lft 1-Artin stack in the sense of [GR17, Chapter 2, § 4.1].*

Notation 0.5.3. *We fix a connected smooth projective curve X over k.*

Notation 0.5.4. *(DG-categories)*

We study DG-categories *over k. Unless otherwise stated, DG-categories are assumed to be* cocomplete *(i.e., containing small colimits), and functors between them are assumed to be k-linear and* continuous *(i.e. preserving small colimits). The $(\infty,1)$-category formed by them is denoted by* $\mathrm{DGCat}_{\mathrm{cont}}$*. The corresponding $(\infty,2)$-category is denoted by* $\mathbf{DGCat}_{\mathrm{cont}}$.

$\mathrm{DGCat}_{\mathrm{cont}}$ *carries a closed symmetric monoidal structrue, known as the* Lurie tensor product $\otimes_k$*. The unit object for it is the DG-category* Vect_k *of k-vector spaces. For $\mathcal{C},\mathcal{D}\in \mathrm{DGCat}_{\mathrm{cont}}$, we write* $\mathrm{LFun}_k(\mathcal{C},\mathcal{D})$ *for the object in* $\mathrm{DGCat}_{\mathrm{cont}}$ *characterized by the universal property*

$$\mathrm{Maps}(\mathcal{E},\mathrm{LFun}_k(\mathcal{C},\mathcal{D}))\simeq \mathrm{Maps}(\mathcal{E}\underset{k}{\otimes}\mathcal{C},\mathcal{D}).$$

A DG-category $\mathcal{M}$ is dualizable *if it is a dualizable object in* $\mathrm{DGCat}_{\mathrm{cont}}$*. We write $\mathcal{M}^\vee$ for its dual DG-category, which is canonically equivalent to* $\mathrm{LFun}_k(\mathcal{M},\mathrm{Vect}_k)$*. It is well-known that $\mathcal{M}$ is dualizable if it is compactly generated, and there is a canonical identification* $\mathcal{M}^\vee\simeq \mathrm{Ind}(\mathcal{M}^{c,\mathrm{op}})$.

Notation 0.5.5. *(D-modules)*

Let Y be a lft prestack. We write $\mathrm{D}(Y)$ for the DG-category of D-modules on Y. We write ω_Y for the dualizing D-module on Y. When Y is an algebraic stack, we write k_Y for the constant D-module. See Appendix A for more details on the theory of D-modules.

Notation 0.5.6. *(Reductive groups)*

We fix a connected reductive group G. For simplicity, we assume $[G,G]$ to be simply connected[16].

We fix a pair of opposite Borel subgroups (B,B^-) *of it, therefore a* Cartan subgroup T*. We write Z_G for the center of G and $T_{\mathrm{ad}}:=T/Z_G$ for the* adjoint torus.

We write $\mathcal{I}$ for the set of vertices in the Dynkin diagram *of G, Λ_G (resp. $\check{\Lambda}_G$) for the* coweight (resp. weight) lattice*, and $\Lambda_G^{\mathrm{pos}}\subset\Lambda_G$ for the sub-monoid spanned by all positive simple co-roots $(\alpha_i)_{i\in\mathcal{I}}$.*

We often use P to denote a standard parabolic subgroup *of G (i.e. a parabolic subgroup containing B). We write P^- for the corresponding* standard opposite parabolic subgroup *and $M:=P\cap P^-$ for the* Levi subgroup*. We write U (resp. U^-) for the* unipotent radical *of P (resp. P^-). When we need to use a different parabolic subgroup, we often denote it by Q and its Levi subgroup by L.*

We write Par *for the partially ordered set of standard parabolic subgroups of G. We write* $\mathrm{Par}'=\mathrm{Par}-\{G\}$*. We view them as categories in the standard way.*

Notation 0.5.7. *(Semi-group completion)*

The collection of simple positive roots of G provides an identification $T_{\mathrm{ad}}\simeq\mathbb{G}_m^{\mathcal{I}}:=\prod_{i\in\mathcal{I}}\mathbb{G}_m$. We define $T_{\mathrm{ad}}^+:=\mathbb{A}^{\mathcal{I}}\supset\mathbb{G}_m^{\mathcal{I}}\simeq T_{\mathrm{ad}}$, which is a semi-group completion of the adjoint torus T_{ad}.

Consider the coordinate stratification of the affine space T_{ad}^+. The set of strata can be identified with the underlying set of Par*. Moreover, the scheme T_{ad}^+ is stratified by the* poset Par*. See Appendix C for what this means and the relevant notations for it. In particular, we will use the notations $T^+_{\mathrm{ad},P}$, $T^+_{\mathrm{ad},\geq P}$ and $T^+_{\mathrm{ad},\leq P}$.*

Write C_P for the unique point in $T^+_{\mathrm{ad},P}$ whose coordinates are either 0 or 1. In particular C_B is the zero element in T^+_{ad} and C_G is the unit element. We use the same symbols to denote the images of these points in the quotient stack T^+_{ad}/T.

Consider the homomorphism $Z_M/Z_G\to T_{\mathrm{ad}}$. Let[17] *$T^+_{\mathrm{ad},\geq C_P}$ be its closure in T^+_{ad}. Note that it is a sub-semi-group of $T^+_{\mathrm{ad},\geq P}$ that contains C_P as an idempotent element.*

Acknowledgements: I want to thank my advisor Dennis Gaitsgory for teaching me all the important concepts in this paper, such as the pseudo-identity functor, the Vinberg-degeneration, Braden's theorem, etc.. I am also grateful for his sharing of notes on the category $\mathbf{I}(G,P)$ and his comments on the first draft of this paper.

[16]Such assumption was made in many references that we cite, but we do not know if our results and proofs really depend on it.

[17]It was denoted by $T^+_{\mathrm{ad},\geq P,\mathrm{strict}}$ in [Sch16].

1. Outline of the proof

In this subsection, we reduce Theorem 0.3.4 to a series of partial results, which will be proved in the latter sections.

1.1. **Step 1: constructing the natural transformation.** The first step is to construct a natural transformation from $\overline{\Delta}_! \circ \mathbf{K}$ to $\mathbf{E} \circ \mathbf{DL}$. Let us first explain how to construct the morphism

$$\overline{\Delta}_! \circ \mathbf{K}(P) \to \mathbf{E} \circ \mathbf{DL}(P). \tag{1.1}$$

For $P \in \mathrm{Par}$, let $\overline{\mathrm{Bun}}_{G,P}$ be the P-stratum of $\overline{\mathrm{Bun}}_G$. We will construct (see Proposition-Construction 1.1.2 and the remark below it) a canonical commutative diagram[18]

$$\begin{tikzcd} \overline{\mathrm{Bun}}_{G,P} \arrow[r, "\overline{\Delta}_P^{\mathrm{enh},l}"] \arrow[d, "i_P"] & \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G \arrow[d, "\mathfrak{p}^{\mathrm{enh}}_{P\times G\to G\times G}"] \\ \overline{\mathrm{Bun}}_G \arrow[r, "\overline{\Delta}"] & \mathrm{Bun}_G \times \mathrm{Bun}_G . \end{tikzcd} \tag{1.2}$$

Consider the object

$$\mathcal{F}_P := \overline{\Delta}_{P,!}^{\mathrm{enh},l} \circ i_P^! \circ \mathbf{K}(P) \in \mathrm{D}_{\mathrm{indhol}}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G).$$

Note that

$$\overline{\Delta}_! \circ \mathbf{K}(P) \simeq \mathfrak{p}^{\mathrm{enh}}_{P\times G\to G\times G,!}(\mathcal{F}_P) \tag{1.3}$$

because $\mathbf{K}(P) \simeq i_{P,!} \circ i_P^!(\mathbf{K}(P))$.

The following result will be proved in § 2.2:

Proposition 1.1.1. *The object $\mathcal{F}_P$ is contained in the full subcategory*

$$\mathbf{I}(G\times G, P\times G) \subset \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G).$$

Let $\mathcal{F}'_P$ be the corresponding object in $\mathbf{I}(G\times G, P\times G)$. By (1.3), we have

$$\overline{\Delta}_! \circ \mathbf{K}(P) \simeq \mathrm{Eis}^{\mathrm{enh}}_{P\times G\to G\times G}(\mathcal{F}'_P). \tag{1.4}$$

Hence by functoriality of the LHS, we obtain a morphism

$$\mathrm{Eis}^{\mathrm{enh}}_{P\times G\to G\times G}(\mathcal{F}'_P) \to \mathcal{F}'_G. \tag{1.5}$$

By adjunction, we have a morphism

$$\theta'_P : \mathcal{F}'_P \to \mathrm{CT}^{\mathrm{enh}}_{G\times G\to P\times G}(\mathcal{F}'_G). \tag{1.6}$$

Note that $\mathcal{F}'_G = \mathcal{F}_G \simeq \Delta_!(k_{\mathrm{Bun}_G})$.

On the other hand, it is easy to see[19]

$$\mathbf{E} \circ \mathbf{DL}(P) \simeq \mathrm{Eis}^{\mathrm{enh}}_{P\times G\to G\times G} \circ \mathrm{CT}^{\mathrm{enh}}_{G\times G\to P\times G}(\Delta_!(k_{\mathrm{Bun}_G})). \tag{1.7}$$

Now we declare the morphism (1.1) to be given by

$$\overline{\Delta}_! \circ \mathbf{K}(P) \simeq \mathrm{Eis}^{\mathrm{enh}}_{P\times G\to G\times G}(\mathcal{F}'_P) \xrightarrow{\mathrm{Eis}^{\mathrm{enh}}(\theta'_P)} \mathrm{Eis}^{\mathrm{enh}}_{P\times G\to G\times G} \circ \mathrm{CT}^{\mathrm{enh}}_{G\times G\to P\times G}(\mathcal{F}'_G) \simeq \mathbf{E} \circ \mathbf{DL}(P). \tag{1.8}$$

In order to obtain the desired natural transformation, we need the following stronger construction:

---

[18] In the case $G = \mathrm{SL}_2$, using the notations in Footnote 13, $\mathrm{VinBun}_{G,B}$ classifies (up to nil-isomorphisms) objects $(\phi : E_1 \to E_2)$ with $\det(\phi) = 0$. It follows from the definition that the subsheaf $\mathrm{im}(\phi)$ is a generic line bundle. Then the map $\mathrm{im}(\phi) \to E_2$ provides a generic B-reduction to the SL_2-torsor for E_2. This provides a map $\mathrm{VinBun}_{G,B} \to \mathrm{Bun}_G^{B\text{-gen}}$ that factors through the quotient $\mathrm{VinBun}_{G,B}/T = \overline{\mathrm{Bun}}_{G,B}$.

[19] One needs to use Lemma 2.3.2.

Proposition-Construction 1.1.2. *Let $\overline{\mathrm{Bun}}_{G,\le P}$ be the reduced closed substack of $\overline{\mathrm{Bun}}_G$ containing all the P'-strata with $P' \subset P$. Then there exist canonical maps*

$$\overline{\Delta}^{\mathrm{enh}}_{\le P} : \overline{\mathrm{Bun}}_{G,\le P} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{gen}}$$

that are functorial[20] *in P such that when $P = G$ we have $\overline{\Delta}^{\mathrm{enh}}_{\le G} = \overline{\Delta} : \overline{\mathrm{Bun}}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$.*

Remark 1.1.3. In particular, we have functorial maps

$$\overline{\Delta}^{\mathrm{enh},l}_{\le P} : \overline{\mathrm{Bun}}_{G,\le P} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G .$$

The map $\overline{\Delta}^{\mathrm{enh},l}_{P}$ in (1.2) is defined to be its restriction to the P-stratum.

Note that we also have

$$\mathcal{F}_P \simeq \overline{\Delta}^{\mathrm{enh},l}_{\le P,!} \circ i^!_{\le P} \circ \mathbf{K}(P).$$

Using this, it is pure formal to show that the morphisms (1.8) constructed above is functorial in P. Namely, in § 2.3, we will use the theory of (co)Cartesian fibrations to prove:

Proposition-Construction 1.1.4. *There exists a canonical natural transformation $\overline{\Delta}_! \circ \mathbf{K} \to \mathbf{E} \circ \mathbf{DL}$ whose value at $P \in \mathrm{Par}$ is equivalent to the morphism (1.8)*

1.2. **Step 2: translating by the second adjointness.** After obtaining the natural transformation, we only need to show its value at each $P \in \mathrm{Par}$ is invertible. From this step on, we fix such a standard parabolic P.

Unwinding the definitions, we need to show the map (1.6)

$$\theta'_P : \mathcal{F}'_P \to \mathrm{CT}^{\mathrm{enh}}_{G\times G \to P\times G}(\mathcal{F}'_G)$$

is invertible. Recall (see Remark 0.2.6) that the functor

$$\iota^!_{M\times G} : \mathbf{I}(G\times G, P\times G) \to \mathrm{D}(\mathrm{Bun}_M \times \mathrm{Bun}_G)$$

is conservative. Hence we only need to show the map $\iota^!_{M\times G}(\theta'_P)$ is invertible. By definition, $\iota^!_{M\times G}$ is isomorphic to

$$\mathbf{I}(G\times G, P\times G) \to \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G) \xrightarrow{\iota^!_{P\times G}} \mathrm{D}(\mathrm{Bun}_P \times \mathrm{Bun}_G) \xrightarrow{\mathfrak{q}_{P\times G,*}} \mathrm{D}(\mathrm{Bun}_M \times \mathrm{Bun}_G).$$

We denote the composition of the latter two functors by

$$\mathrm{CT}^{\mathrm{gen}}_{P\times G,*} : \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G) \xrightarrow{\iota^!_{P\times G}} \mathrm{D}(\mathrm{Bun}_P \times \mathrm{Bun}_G) \xrightarrow{\mathfrak{q}_{P\times G,*}} \mathrm{D}(\mathrm{Bun}_M \times \mathrm{Bun}_G).$$

Then the source of $\iota^!_{M\times G}(\theta'_P)$ is isomorphic to $\mathrm{CT}^{\mathrm{gen}}_{P\times G,*}(\mathcal{F}_P)$.

On the other hand, the functor $\iota^!_{M\times G} \circ \mathrm{CT}^{\mathrm{enh}}_{G\times G\to P\times G}$ is isomorphic to the usual geometric constant term functor

$$\mathrm{CT}_{P\times G,*} : \mathrm{D}(\mathrm{Bun}_G \times \mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_M \times \mathrm{Bun}_G)$$

(as can be seen by passing to left adjoints). Hence the target of $\iota^!_{M\times G}(\theta'_P)$ is isomorphic to $\mathrm{CT}_{P\times G,*}(\mathcal{F}_G)$. Let[21]

$$\gamma_P : \mathrm{CT}^{\mathrm{gen}}_{P\times G,*}(\mathcal{F}_P) \to \mathrm{CT}_{P\times G,*}(\mathcal{F}_G). \tag{1.9}$$

be the morphism obtained from $\iota^!_{M\times G}(\theta'_P)$ via the above isomorphisms. Then we have reduced the main theorem to:

- We only need to show γ_P is invertible.

[20] Note that for any $P \subset Q$, we have maps

$$\overline{\mathrm{Bun}}_{G,\le P} \to \overline{\mathrm{Bun}}_{G,\le Q} \text{ and } \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}} \to \mathrm{Bun}_G^{Q\text{-gen}} \times \mathrm{Bun}_G^{Q^-\text{-gen}} .$$

[21] We will give a more direct description of γ_P in § 4.2.2.

Recall that the main theorem of [DG16] says that when restricted to each connected component $\mathrm{Bun}_{M,\lambda}$ of Bun_M, the functor

$$\mathrm{CT}_{P,*,\lambda}: \mathrm{D}(\mathrm{Bun}_G) \xrightarrow{!\text{-pull}} \mathrm{D}(\mathrm{Bun}_{P,\lambda}) \xrightarrow{*\text{-push}} \mathrm{D}(\mathrm{Bun}_{M,\lambda})$$

is canonically left adjoint to

$$\mathrm{Eis}_{P^-,*,\lambda}: \mathrm{D}(\mathrm{Bun}_{M,\lambda}) \xrightarrow{!\text{-pull}} \mathrm{D}(\mathrm{Bun}_{P^-,\lambda}) \xrightarrow{*\text{-push}} \mathrm{D}(\mathrm{Bun}_G).$$

In particular, the functor $\mathrm{CT}_{P,*} \simeq \bigoplus \mathrm{CT}_{P,*,\lambda}$ preserves ind-holonomic objects and its restriction to $\mathrm{D}_{\mathrm{indhol}}(\mathrm{Bun}_G)$ is isomorphic to[22]

$$'\mathrm{CT}_{P^-,!}: \mathrm{D}_{\mathrm{indhol}}(\mathrm{Bun}_G) \xrightarrow{\mathfrak{p}^{-,*}} \mathrm{D}_{\mathrm{indhol}}(\mathrm{Bun}_{P^-}) \xrightarrow{\mathfrak{q}^-_!} \mathrm{D}_{\mathrm{indhol}}(\mathrm{Bun}_M).$$

Hence we obtain an isomorphism

$$\mathrm{CT}_{P\times G,*}(\mathcal{F}_G) \simeq {}'\mathrm{CT}_{P^-\times G,!}(\mathcal{F}_G).$$

Now there is a similar story when we replace Bun_G by $\mathrm{Bun}_G^{P\text{-gen}}$. To state it, we need to review the counit natural transformation for the adjoint pair $(\mathrm{CT}^\lambda_{P,*}, \mathrm{Eis}^\lambda_{P^-,*})$. In *loc.cit.*, the authors considered the correspondences

$$\alpha^+_{P,\lambda}: (\mathrm{Bun}_{M,\lambda} \leftarrow \mathrm{Bun}_{P,\lambda} \to \mathrm{Bun}_G),\ \alpha^-_{P,\lambda}: (\mathrm{Bun}_G \leftarrow \mathrm{Bun}_{P^-,\lambda} \to \mathrm{Bun}_{M,\lambda})$$

and a 2-morphism in $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}}$ (see Appendix A for what this means):

$$\alpha^+_{P,\lambda} \circ \alpha^-_{P,\lambda} \to \mathrm{Id}_{\mathrm{Bun}_{M,\lambda}}.$$

Explicitly, this 2-morphism is given by the schematic open embedding

$$\mathrm{Bun}_{M,\lambda} \to \mathrm{Bun}_{P,\lambda} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-,\lambda}.$$

Then the counit natural transformation is given by[23]

$$\mathrm{CT}^\lambda_{P,*} \circ \mathrm{Eis}^\lambda_{P^-,*} \simeq \mathrm{Dmod}^{\blacktriangle,!}(\alpha^+_{P,\lambda}) \circ \mathrm{Dmod}^{\blacktriangle,!}(\alpha^-_{P,\lambda}) \to \mathrm{Dmod}^{\blacktriangle,!}(\mathrm{Id}_{\mathrm{Bun}_{M,\lambda}}) \simeq \mathbf{Id}_{\mathrm{D}(\mathrm{Bun}_{M,\lambda})}.$$

Motivated by this construction, we prove the following two results in § 3.

Lemma 1.2.1. *We have:*

(1) *The correspondences*

$$\begin{aligned} \alpha^{+,\mathrm{gen}}_{P,\lambda} &: (\mathrm{Bun}_{M,\lambda} \leftarrow \mathrm{Bun}_{P,\lambda} \to \mathrm{Bun}_G^{P\text{-gen}}), \\ \alpha^{-,\mathrm{gen}}_{P,\lambda} &: (\mathrm{Bun}_G^{P\text{-gen}} \leftarrow \mathrm{Bun}^{M\text{-gen}}_{P^-,\lambda} \to \mathrm{Bun}_{M,\lambda}) \end{aligned}$$

are morphisms in $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}}$. *In fact, the first leftward map is* safe *(by [DG16, Footnote 2]) and the second leftward map is quasi-compact and schematic.*

(2) *There is a 2-morphism* $\alpha^{+,\mathrm{gen}}_{P,\lambda} \circ \alpha^{-,\mathrm{gen}}_{P,\lambda} \to \mathrm{Id}_{\mathrm{Bun}_{M,\lambda}}$ *given by the map*

$$\mathrm{Bun}_{M,\lambda} \to \mathrm{Bun}_{P,\lambda} \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}^{M\text{-gen}}_{P^-,\lambda}.$$

In other words, this map is a schematic open embedding[24].

Theorem 1.2.2. *The natural transformation*

$$\mathrm{Dmod}^{\blacktriangle,!}(\alpha^{+,\mathrm{gen}}_{P,\lambda}) \circ \mathrm{Dmod}^{\blacktriangle,!}(\alpha^{-,\mathrm{gen}}_{P,\lambda}) \to \mathrm{Dmod}^{\blacktriangle,!}(\mathrm{Id}_{\mathrm{Bun}_{M,\lambda}}) \simeq \mathbf{Id}_{\mathrm{D}(\mathrm{Bun}_{M,\lambda})}$$

is the counit natural transformation for an adjoint pair

$$(\mathrm{Dmod}^{\blacktriangle,!}(\alpha^{+,\mathrm{gen}}_{P,\lambda}), \mathrm{Dmod}^{\blacktriangle,!}(\alpha^{-,\mathrm{gen}}_{P,\lambda})).$$

[22] We use the notation $'\mathrm{CT}_{P^-,!}$ because $\mathrm{CT}_{P^-,!}$ was used in *loc.cit.* to denote the corresponding functor for all the D-modules.

[23] See (A.1) for the notation $\mathrm{Dmod}^{\blacktriangle,!}_{\mathrm{indhol}}$.

[24] In fact, $\mathrm{Bun}_P \times_{\mathrm{Bun}_G^{P\text{-gen}}} \mathrm{Bun}^{M\text{-gen}}_{P^-}$ is the open Zastava space in the literature.

As before, the above theorem implies that $\mathrm{CT}^{\mathrm{gen}}_{P\times G,*}$ preserves ind-holonomic objects and its restriction to $\mathrm{D}_{\mathrm{indhol}}(\mathrm{Bun}_G^{P\text{-gen}}\times \mathrm{Bun}_G)$ is isomorphic to

$$'\mathrm{CT}^{\mathrm{gen}}_{P^-\times G,!}: \mathrm{D}_{\mathrm{indhol}}(\mathrm{Bun}_G^{P\text{-gen}}\times \mathrm{Bun}_G) \xrightarrow{*\text{-pull}} \mathrm{D}(\mathrm{Bun}_{P^-}^{M\text{-gen}}\times \mathrm{Bun}_G) \xrightarrow{!\text{-push}} \mathrm{D}(\mathrm{Bun}_M\times \mathrm{Bun}_G).$$

Hence the morphism (1.9) is equivalent to a certain morphism

$$'\gamma_P : {}'\mathrm{CT}^{\mathrm{gen}}_{P^-\times G,!}(\mathcal{F}_P) \to {}'\mathrm{CT}_{P^-\times G,!}(\mathcal{F}_G).$$

Hence we have reduced the main theorem to the following problem:

- We only need to show $'\gamma_P$ is invertible.

Remark 1.2.3. It is easier to study $'\gamma_P$ than γ_P because we can use the base-change isomorphisms.

Remark 1.2.4. We believe Theorem 1.2.2 (and its proof) is of independent interest. For example, we can use them to give a description of the monad structure of $\widetilde{\Omega}_P := \iota_M^! \circ \iota_{M,!}$ via Verdier (co)specialization along Schieder's local models. This monad was the central concept in the paper [Gai11]. The details of it will be provided elsewhere.

1.3. **Step 3: diagram chasing.** Using the base-change isomorphisms, and using the facts that $\mathbf{K}(P)$ is a !-extension along $\overline{\mathrm{Bun}}_{G,P}\to\overline{\mathrm{Bun}}_G$, one can simplify the source and target of $'\gamma_P$. Let us state the result directly[25] Consider the correspondences

$$\begin{aligned}
\beta_P &: (\mathrm{Bun}_M\times\mathrm{Bun}_G \leftarrow \mathrm{Bun}_{P^-}^{M\text{-gen}}\times_{\mathrm{Bun}_G^{P\text{-gen}}}\overline{\mathrm{Bun}}_{G,P} \to \overline{\mathrm{Bun}}_G)\\
\beta_G &: (\mathrm{Bun}_M\times\mathrm{Bun}_G \leftarrow \mathrm{Bun}_{P^-}\times_{\mathrm{Bun}_G}\overline{\mathrm{Bun}}_{G,G} \to \overline{\mathrm{Bun}}_G),
\end{aligned}$$

where the left arm of β_P is given by

$$\mathrm{Bun}_{P^-}^{M\text{-gen}} \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \overline{\mathrm{Bun}}_{G,P} \to \mathrm{Bun}_M\times\mathrm{Bun}_G\times\mathrm{Bun}_G \xrightarrow{\mathrm{pr}_{13}} \mathrm{Bun}_M\times\mathrm{Bun}_G.$$

Then the base-change isomorphisms provide

$$\begin{aligned}
{}'\mathrm{CT}^{\mathrm{gen}}_{P^-\times G,!}(\mathcal{F}_P) &\simeq \mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta_P)\circ\mathbf{K}(P),\\
{}'\mathrm{CT}_{P^-\times G,!}(\mathcal{F}_G) &\simeq \mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta_G)\circ\mathbf{K}(G).
\end{aligned}$$

This motivates the following construction (see § 4.1):

Proposition-Construction 1.3.1. *There exists an open substack*[26]

$$(\mathrm{Bun}_{P^-}\underset{\mathrm{Bun}_G}{\times}\overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \subset \mathrm{Bun}_{P^-}\underset{\mathrm{Bun}_G}{\times}\overline{\mathrm{Bun}}_{G,\geq P}$$

such that the parameterized correspondence

$$\begin{tikzcd}
\beta: & \mathrm{Bun}_M\times\mathrm{Bun}_G & (\mathrm{Bun}_{P^-}\times_{\mathrm{Bun}_G}\overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \arrow[l] \arrow[r] \arrow[d] & \overline{\mathrm{Bun}}_G \\
& & T^+_{\mathrm{ad},\geq P}/T &
\end{tikzcd}$$

captures β_P (resp. β_G) as its restriction to the P-stratum (resp. G-stratum) of $T^+_{\mathrm{ad},\geq P}/T$.

Using the fact that $\mathbf{K}(P)$ is a !-extension along $\overline{\mathrm{Bun}}_{G,P}\to\overline{\mathrm{Bun}}_G$ again, we obtain isomorphisms[27]

$$\begin{aligned}
\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta_P)\circ\mathbf{K}(P) &\simeq \mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta)\circ\mathbf{K}(P),\\
\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta_G)\circ\mathbf{K}(G) &\simeq \mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta)\circ\mathbf{K}(G).
\end{aligned}$$

We will prove the following result in § 4.2 by a routine diagram chasing:

[25]The result below only serves as motivation and will be incorporated into Lemma 1.3.2.

[26] In the case $G = \mathrm{SL}_2$, recall that $\mathrm{Bun}_{B^-}\times_{\mathrm{Bun}_G}\overline{\mathrm{Bun}}_G$ classifies certain chains $E_1\to E_2\to L_2$. Then the desired open substack classifies those chains such that the restriction of $E_1\to L_2$ at any geometric point of S is nonzero.

[27]See (A.5) for the notation $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}$.

Lemma 1.3.2. *The morphisms γ_P and $'\gamma_P$ are both equivalent to the morphism*

$$\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta)\circ\mathbf{K}(P)\to\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta)\circ\mathbf{K}(G)$$

given by the functor $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta)\circ\mathbf{K}$.

Hence we have reduced the main theorem to the following problem:

- We only need to show the functor $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta)\circ\mathbf{K}$ sends the arrow $P\to G$ to an isomorphism.

1.4. **Step 4: restoring the symmetry.** In § 5.1, we will show

Proposition-Construction 1.4.1. *There exists a canonical factorization of the map*[28]

$$(\mathrm{Bun}_{P^-}\underset{\mathrm{Bun}_G}{\times}\overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}}\to\mathrm{Bun}_M\times\mathrm{Bun}_G$$

via $\mathrm{Bun}_M\times\mathrm{Bun}_G^{P^-\text{-gen}}$.

In particular we obtain a correspondence

$$\beta':(\mathrm{Bun}_M\times\mathrm{Bun}_G^{P^-\text{-gen}}\leftarrow(\mathrm{Bun}_{P^-}\underset{\mathrm{Bun}_G}{\times}\overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}}\to\overline{\mathrm{Bun}}_G)$$

and we only need to show $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta')\circ\mathbf{K}$ sends the arrow $P\to G$ to an isomorphism.

The following result will be proved in § 5.2:

Proposition 1.4.2. *The objects* $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta')\circ\mathbf{K}(P)$ *and* $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta')\circ\mathbf{K}(G)$ *are both contained in the full subcategory*

$$\mathbf{I}(M\times G,M\times P^-)\subset\mathrm{D}(\mathrm{Bun}_M\times\mathrm{Bun}_G^{P^-\text{-gen}}).$$

Consider the correspondence

$$\delta:(\mathrm{Bun}_M\times\mathrm{Bun}_M\leftarrow\mathrm{Bun}_M\times\mathrm{Bun}_P^{M\text{-gen}}\to\mathrm{Bun}_M\times\mathrm{Bun}_G^{P^-\text{-gen}})$$

and the functor

$$'\mathrm{CT}^{\mathrm{gen}}_{M\times P,!}:=\mathrm{D}^{!,*}_{\mathrm{indhol}}(\delta).$$

Similar to Step 2, we can use "the second adjointness" to reduce the main theorem to the following problem:

- We only need to show the functor $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\delta\circ\beta')\circ\mathbf{K}$ sends the arrow $P\to G$ to an isomorphism.

1.5. **Step 5: calculating via the local models.** Now comes the critical observation. Recall Schieder's (relative) local model[29] in [Sch16]:

$$Y^P_{\mathrm{rel}}:=\mathbf{Maps}_{\mathrm{gen}}(X,P^-\backslash\mathrm{Vin}_{G,\geq C_P}/P\supset P^-\backslash\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}/P),$$

where (see Notation 0.5.7)

$$\mathrm{Vin}_{G,\geq C_P}:=\mathrm{Vin}_G\underset{T^+_{\mathrm{ad}}}{\times}T^+_{\mathrm{ad},\geq C_P}.$$

The T-action on $G\backslash\mathrm{Vin}_{G,\geq P}/G$ induces a Z_M-action on $P^-\backslash\mathrm{Vin}_{G,\geq C_P}/P$. It is known that it induces a Z_M-action on Y^P_{rel}. Note that we have the following diagram of stacks equipped with group actions:

$$(\mathrm{pt}\curvearrowright\mathrm{Bun}_M\times\mathrm{Bun}_M)\leftarrow(Z_M\curvearrowright Y^P_{\mathrm{rel}})\to(T\curvearrowright\mathrm{VinBun}_{G,\geq P}).$$

The following result is proved in § 5.1.

Lemma 1.5.1. *The composition $\delta\circ\beta'$ is isomorphic to*

$$\mathrm{Bun}_M\times\mathrm{Bun}_M\leftarrow Y^P_{\mathrm{rel}}/Z_M\to\overline{\mathrm{Bun}}_G.$$

[28] In the case $G=\mathrm{SL}_2$, the map $(\mathrm{Bun}_{B^-}\times_{\mathrm{Bun}_G}\overline{\mathrm{Bun}}_G)^{\mathrm{gen}}\to\mathrm{Bun}_G^{B^-\text{-gen}}$ sends a chain $E_1\to E_2\to L_2$ in Footnote 26 to the generic B^--reduction provided by the map $E_1\to L_2$.

[29] In the case $G=\mathrm{SL}_2$, Y^B_{rel} classifies chains $L_1\to E_1\to E_2\to L_2$ where $L_1\to E_1$, $E_1\to E_2$ and $E_2\to L_2$ are respectively S-points of Bun_B, VinBun_G and Bun_{B^-} such that the restriction of $L_1\to L_2$ at any geometric point of S is nonzero.

It is known (see Construction 6.1.2) that $Y^P_{\mathrm{rel}}/Z_M \to \mathrm{Bun}_M \times \mathrm{Bun}_M$ factors via $H_{M,G\text{-pos}}/Z_M$, where $H_{M,G\text{-pos}}$ is the G-positive part of Hecke stack for M-torsors[30]:

$$H_{M,G\text{-pos}} := \mathbf{Maps}_{\mathrm{gen}}(X, M\backslash\overline{M}/M \supset M\backslash M/M).$$

Consider the correspondence

$$\psi_P : H_{M,G\text{-pos}}/Z_M \leftarrow Y^P_{\mathrm{rel}}/Z_M \to \overline{\mathrm{Bun}}_G.$$

We have reduced the main theorem to

Goal 1.5.2. *The functor* $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\psi_P) \circ \mathbf{K}$ *sends the arrow* $P \to G$ *to an isomorphism.*

We will prove a stronger result:

Goal 1.5.3. *For any* $Q \in \mathrm{Par}_{\geq P}$*, the functor* $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\psi_P) \circ \mathbf{K}$ *sends the arrow* $Q \to G$ *to an isomorphism.*

We prove this by induction on the relative rank between Q and G. When $Q = G$, there is nothing to prove. Hence we assume $Q \neq G$ and assume the above claim is correct for any Q' strictly greater than Q. Let L be the Levi subgroup of Q.

Consider the object

$$\mathcal{D}_Q := \mathrm{coFib}(\underset{Q' \in \mathrm{Par}' \cap \mathrm{Par}_{\geq Q}}{\mathrm{colim}} \mathbf{K}(Q') \to \mathbf{K}(G))$$

We claim

$$\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\psi_P)(\mathcal{D}_Q) \simeq 0. \tag{1.10}$$

Let us execute the induction step using this claim. Note that the category $\mathrm{Par}' \cap \mathrm{Par}_{\geq Q}$ is weakly contractible, hence

$$\mathcal{D}_Q \simeq \underset{Q' \in \mathrm{Par}' \cap \mathrm{Par}_{\geq Q}}{\mathrm{colim}} \mathrm{coFib}(\mathbf{K}(Q') \to \mathbf{K}(G)).$$

By induction hypothesis, the functor $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\psi_P)$ sends $\mathrm{coFib}(\mathbf{K}(Q') \to \mathbf{K}(G))$ to 0 unless $Q' = Q$. Hence $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\psi_P)(\mathcal{D}_Q)$ is isomorphic to[31]

$$\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\psi_P)(\mathrm{coFib}(\mathbf{K}(Q) \to \mathbf{K}(G))[\mathrm{rank}(G) - \mathrm{rank}(L) + 1].$$

Then the claim (1.10) implies $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\psi_P)$ sends $\mathbf{K}(Q) \to \mathbf{K}(G)$ to an isomorphism as desired.

It remains to prove (1.10). Consider the maps

$$\overline{\mathrm{Bun}}_{G,G} \xrightarrow{j_{G,\geq Q}} \overline{\mathrm{Bun}}_{G,\geq Q} \xrightarrow{j_{\geq Q}} \overline{\mathrm{Bun}}_G.$$

By Lemma C.1.9, we have $\mathcal{D}_Q \simeq j_{\geq Q,!}(\mathcal{F})$, where

$$\mathcal{F} := (j_{G,\geq Q})_* \circ r_!(k_{\mathrm{Bun}_G}).$$

Hence by the base-change isomorphism, $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\psi_P)(\mathcal{D}_Q)$ is isomorphic to $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\psi_{P,\geq Q})(\mathcal{F})$, where

$$\psi_{P,\geq Q} : (H_{M,G\text{-pos}}/Z_M \leftarrow (Y^P_{\mathrm{rel}}/Z_M)_{\geq Q} \to \overline{\mathrm{Bun}}_{G,\geq Q})$$

and $(Y^P_{\mathrm{rel}}/Z_M)_{\geq Q}$ is the open substack of Y^P_{rel}/Z_M containing those Q'-strata with $Q' \supset Q$. The following construction will be provided in § 6.1:

Proposition-Construction 1.5.4. *The correspondence* $\psi_{P,\geq Q}$ *is isomorphic to the composition of*

$$\psi_{Q,\geq Q} : (H_{L,G\text{-pos}}/Z_L \leftarrow Y^Q_{\mathrm{rel}}/Z_L \to \overline{\mathrm{Bun}}_{G,\geq Q})$$

by a certain correspondence from $H_{L,G\text{-pos}}/Z_L$ *to* $H_{M,G\text{-pos}}/Z_M$.

[30]In the case $G = \mathrm{SL}_2$, $H_{T,G\text{-pos}}$ classifies morphisms between line bundles $L_1 \to L_2$ whose restriction at any geometric point of S is nonzero. The map $Y^P_{\mathrm{rel}} \to H_{M,G\text{-pos}}$ sends the chain $L_1 \to E_1 \to E_2 \to L_2$ in Footnote 29 to $L_1 \to L_2$.

[31]We use the following formal fact. Let I be an index category obtained by removing the final object from $[1]^r$ ($r \geq 1$). Let C be any stable category. Suppose $F : I \to C$ is a functor such that $F(x) \simeq 0$ unless x is the initial object i_0. Then colim $F \simeq F(i_0)[r-1]$. This fact can be proven by induction on r.

Therefore we only need to show $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\psi_{Q,\geq Q})(\mathcal{F})\simeq 0$. We will prove the following stronger claim: for any $Q\in\mathrm{Par}$, we have

$$\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\psi_{Q,\geq Q})\circ(j_{G,\geq Q})_*\simeq 0.$$

To finish the proof, we need one more geometric input. In [Sch16], the author constructed a *defect stratification* on the P-stratum $\overline{\mathrm{Bun}}_{G,P}$. Let ${}_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P}$ be the disjiont union of all the defect strata. It is known (see § 2.2.2) that

$${}_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P}\simeq \mathrm{Bun}_P\underset{\mathrm{Bun}_M}{\times}(H_{M,G\text{-}\mathrm{pos}}/Z_M)\underset{\mathrm{Bun}_M}{\times}\mathrm{Bun}_{P^-}.$$

Consider the diagram

$$\begin{array}{ccccc} H_{M,G\text{-}\mathrm{pos}}/Z_M & \xleftarrow{\mathfrak{q}^+_{P,\mathrm{Vin}}} & {}_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P} & \xrightarrow{\mathfrak{p}^+_{P,\mathrm{Vin}}} & \overline{\mathrm{Bun}}_{G,\geq P}, \\ H_{M,G\text{-}\mathrm{pos}}/Z_M & \xleftarrow{\mathfrak{q}^-_{P,\mathrm{Vin}}} & Y^P_{\mathrm{rel}}/Z_M & \xrightarrow{\mathfrak{p}^-_{P,\mathrm{Vin}}} & \overline{\mathrm{Bun}}_{G,\geq P}. \end{array}$$

In § 6.2, we will prove the following "second-adjointness-style" result:

Theorem 1.5.5. *The functor*

$$\mathfrak{q}^{\mp}_{P,\mathrm{Vin},!}\circ\mathfrak{p}^{\mp,*}_{P,\mathrm{Vin}}:\mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,\geq P})\to\mathrm{D}_{\mathrm{indhol}}(H_{M,G\text{-}\mathrm{pos}}/Z_M)$$

is isomorphic to the restriction of the functor

$$\mathfrak{q}^{\pm}_{P,\mathrm{Vin},*}\circ\mathfrak{p}^{\pm,!}_{P,\mathrm{Vin}}:\mathrm{D}(\overline{\mathrm{Bun}}_{G,\geq P})\to\mathrm{D}(H_{M,G\text{-}\mathrm{pos}}/Z_M).$$

Now the Q-version of Theorem 1.5.5 says

$$\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\psi_{Q,\geq Q})\simeq\mathfrak{q}^+_{Q,\mathrm{Vin},*}\circ\mathfrak{p}^{+,!}_{Q,\mathrm{Vin}}.$$

Hence we have

$$\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\psi_{Q,\geq Q})\circ(j_{G,\geq Q})_*\simeq\mathfrak{q}^+_{Q,\mathrm{Vin},*}\circ\mathfrak{p}^{+,!}_{Q,\mathrm{Vin}}\circ(j_{G,\geq Q})_*.$$

Note that ${}_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,Q}$ and $\overline{\mathrm{Bun}}_{G,G}$ have empty intersection (because $Q\neq G$). Hence $\mathfrak{p}^{+,!}_{Q,\mathrm{Vin}}\circ(j_{G,\geq Q})_*\simeq 0$. This finishes the proof.

□[Theorem 0.3.4]

Remark 1.5.6. In the case $G=\mathrm{SL}_2$, one can use Theorem 1.5.5 to give a quicker proof of Goal 1.5.2. Namely, using the theorem, we only need to show $\mathfrak{q}^+_{B,\mathrm{Vin},*}\circ\mathfrak{p}^{+,!}_{B,\mathrm{Vin}}\circ\mathbf{K}$ sends the arrow $B\to G$ to an isomorphism. Recall that $\mathfrak{p}^+_{B,\mathrm{Vin}}$ factors through

$$i_B:\overline{\mathrm{Bun}}_{G,B}\to\overline{\mathrm{Bun}}_G.$$

Hence we only need to show $i^!_B\circ\mathbf{K}$ sends $B\to G$ to an isomorphism. However, this is obvious because the image of this arrow is the map (see Remark C.1.8):

$$i^*_B\circ j_{G,*}(r_!(k_{\mathrm{Bun}_G}))[-1]\to i^!_B\circ j_{G,!}(r_!(k_{\mathrm{Bun}_G})),$$

which is an isomorphism because i_B and j_G are complementary to each other.

2. Step 1

We have three results to prove in this step: Proposition-Construction 1.1.2, Proposition 1.1.1 and Proposition 1.1.4. Each subsection corresponds to a result. Note that we have to do them in this order because the statement of the second result depends on the construction of the first.

2.1. **Proof of Proposition-Construction 1.1.2.** Our goal is:

Goal 2.1.1. *Construct canonical maps*

$$\overline{\mathbf{\Delta}}^{\mathrm{enh}}_{\leq P} : \overline{\mathrm{Bun}}_{G,\leq P} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{gen}}$$

that are functorial in P such that when $P = G$ we have $\overline{\mathbf{\Delta}}^{\mathrm{enh}}_{\leq G} = \overline{\Delta}$.

By definition, we have $\overline{\mathrm{Bun}}_{G,\leq P} \simeq \mathrm{VinBun}_{G,\leq P}/T$ and

$$\mathrm{VinBun}_{G,\leq P} \simeq \mathbf{Maps}_{\mathrm{gen}}(X, G\backslash \mathrm{Vin}_{G,\leq P}/G \supset G\backslash {}_0\mathrm{Vin}_{G,\leq P}/G),$$

where the T-action on $\mathrm{VinBun}_{G,\leq P}$ is induced from the T-action on $\mathrm{Vin}_{G,\leq P}$. By Fact D.1.1(2), we have

$$G\backslash {}_0\mathrm{Vin}_{G,\leq P}/G \simeq \mathbb{B}\widetilde{G}_{\leq P}, \tag{2.1}$$

where $\widetilde{G}_{\leq P}$ is a subgroup scheme of $G \times G \times T^+_{\mathrm{ad},\leq P} \to T^+_{\mathrm{ad},\leq P}$.

By Lemma D.1.4, the map $\mathbb{B}\widetilde{G}_{\leq P} \to \mathbb{B}G \times \mathbb{B}G$ factors as $\mathbb{B}\widetilde{G}_{\leq P} \to \mathbb{B}P \times \mathbb{B}P^- \to \mathbb{B}G \times \mathbb{B}G$. Also, the maps

$$\mathbb{B}\widetilde{G}_{\leq P} \to \mathbb{B}P \times \mathbb{B}P^- \tag{2.2}$$

are functorial in P. Now we have the following commutative diagram of algebraic stacks:

$$\begin{tikzcd} G\backslash \mathrm{Vin}_{G,\leq P}/G \arrow[d] & G\backslash {}_0\mathrm{Vin}_{G,\leq P}/G \arrow[l] \arrow[d, "(2.2)\circ(2.1)"] \\ \mathbb{B}G \times \mathbb{B}G & \mathbb{B}P \times \mathbb{B}P^-. \arrow[l] \end{tikzcd} \tag{2.3}$$

Taking $\mathbf{Maps}_{\mathrm{gen}}(X, -)$, we obtain maps

$$\mathrm{VinBun}_{G,\leq P} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{gen}}$$

functorial in P. To finish the construction, we only need to show:

Lemma 2.1.2. *The map $G\backslash {}_0\mathrm{Vin}_{G,\leq P}/G \to \mathbb{B}P \times \mathbb{B}P^-$ constructed above can be uniquely lifted to a morphism*

$$(T \curvearrowright G\backslash {}_0\mathrm{Vin}_{G,\leq P}/G) \to (\mathrm{pt} \curvearrowright \mathbb{B}P \times \mathbb{B}P^-)$$

fitting into the following commutative diagram

$$\begin{tikzcd} (T \curvearrowright G\backslash \mathrm{Vin}_{G,\leq P}/G) \arrow[d] & (T \curvearrowright G\backslash {}_0\mathrm{Vin}_{G,\leq P}/G) \arrow[l] \arrow[d] \\ (\mathrm{pt} \curvearrowright \mathbb{B}G \times \mathbb{B}G) & (\mathrm{pt} \curvearrowright \mathbb{B}P \times \mathbb{B}P^-). \arrow[l] \end{tikzcd}$$

Proof. The uniqueness follows from the fact that $\mathbb{B}P \times \mathbb{B}P^- \to \mathbb{B}G \times \mathbb{B}G$ is schematic. It remains to prove the existence.

The map $G\backslash {}_0\mathrm{Vin}_{G,\leq P}/G \to \mathbb{B}P \times \mathbb{B}P^-$ induces a $(G \times G)$-equivariant map

$${}_0\mathrm{Vin}_{G,\leq P} \to G/P \times G/P^-. \tag{2.4}$$

We only need to show the T-action on ${}_0\mathrm{Vin}_{G,\leq P}$ preserves the fibers of this map.

Recall that any closed point in ${}_0\mathrm{Vin}_{G,\leq P}$ is of the form $g_1 \cdot \mathfrak{s}(s) \cdot g_2^{-1}$ where g_1 and g_2 are closed points of G, s is a closed point of $T^+_{\mathrm{ad},\leq P}$ and $\mathfrak{s}$ is the canonical section. Unwinding the definitions, the map (2.4) sends this point to (g_1, g_2). Now consider the T-action on Vin_G. It follows from definition that a closed point t of T sends the point $\mathfrak{s}(s)$ to the point $\iota(t) \cdot \mathfrak{s}(ts)$, where $\iota : T \hookrightarrow G$ is the embedding. Since the T-action commutes with the $(G \times G)$-action, the element t sends $g_1 \cdot \mathfrak{s}(s) \cdot g_2^{-1}$ to $g_1 \iota(t) \cdot \mathfrak{s}(s) \cdot g_2^{-1}$. This makes the desired claim manifest.

□[Lemma 2.1.2]

□[Proposition-Construction 1.1.2]

2.2. **Proof of Proposition 1.1.1.**

Goal 2.2.1. *The object $\mathcal{F}_P$ is contained in the full subcategory*
$$\mathbf{I}(G\times G, P\times G) \subset \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G).$$

2.2.2. *The $(\mathbf{U},\mathbf{U}^-)$-equivariant categories.* In order to prove Proposition 1.1.1, we will introduce a subcategory
$$\mathrm{D}(\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}\times\mathbf{U}^-} \subset \mathrm{D}(\overline{\mathrm{Bun}}_{G,P}),$$
whose definition is similar to
$$\mathbf{I}(G\times G, P\times P^-) \subset \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}}).$$

To define this subcategory, we use the *defect stratification* on $\overline{\mathrm{Bun}}_{G,P}$ defined[32] in [Sch16]. Recall the disjoint union of its strata is given by[33]
$$_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P} \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} (H_{M,G\text{-}\mathrm{pos}}/Z_M) \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-},$$
or more abstractly
$$_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P} \simeq \mathbf{Maps}_{\mathrm{gen}}(X, P\backslash\overline{M}/P^- \supset P\backslash M/P^-)/Z_M,$$
where $\overline{M}$ is the closure of the locally closed embedding
$$M \simeq (P\times P^-)/(P\underset{M}{\times}P^-) \hookrightarrow (G\times G)/(P\underset{M}{\times}P^-) \simeq {}_0\mathrm{Vin}_{G,C_P}.$$

It is well-known that the map $\mathrm{Bun}_P \to \mathrm{Bun}_M$ is *universally homological contractible*, or *UHC*. In other words, for any lft prestack $Y\to \mathrm{Bun}_M$, the !-pullback functor $\mathrm{D}(Y)\to \mathrm{D}(Y\times_{\mathrm{Bun}_M}\mathrm{Bun}_P)$ is fully faithful. In particular, the following !-pullback functor is fully faithful
$$_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P} \to (H_{M,G\text{-}\mathrm{pos}}/Z_M) \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}.$$
We denote its essential image by $\mathrm{D}({}_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}_P}$. Similarly we define $\mathrm{D}({}_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}_P^-}$ and $\mathrm{D}({}_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}_P\times\mathbf{U}_P^-}$.

Since $\mathrm{Bun}_P\to\mathrm{Bun}_M$ is smooth, in the previous definition, we can also use $*$-pullbacks instead of the !-pullbacks. The resulting subcategories are the same.

We define $\mathrm{D}(\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}_P}$ to fit into the following pullback diagram
$$\begin{tikzcd}
\mathrm{D}(\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}_P} \arrow[r, "\subset"] \arrow[d] & \mathrm{D}(\overline{\mathrm{Bun}}_{G,P}) \arrow[d, "!\text{-pull}"] \\
\mathrm{D}({}_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}_P} \arrow[r, "\subset"] & \mathrm{D}({}_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P}).
\end{tikzcd}$$
Similarly we define $\mathrm{D}(\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}_P^-}$ and $\mathrm{D}(\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}_P\times\mathbf{U}_P^-}$. We also define the version of these subcategories for ind-holonomic D-modules.

We will deduce Proposition 1.1.1 from the following three lemmas. The proof of the first one is completely similar to that in [Che21, Appendix § G.1]. We provide the proofs for the other two.

Lemma 2.2.3. *For any morphism $P\in\mathrm{Par}$, the object*
$$i_P^*\circ j_{G,*}\circ r_!(k_{\mathrm{Bun}_G}) \in \mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P})$$
is contained in $\mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}_P\times\mathbf{U}_P^-}$.

[32] More precisely, [Sch16] constructed the defect stratification on VinBun_{G,C_P}. It follows from the construction that the Z_M-action on VinBun_{G,C_P} preserves the defect strata. Hence we obtain a stratification on
$$\overline{\mathrm{Bun}}_{G,P} \simeq \mathrm{VinBun}_{G,P}/T \simeq \mathrm{VinBun}_{G,C_P}/Z_M.$$

[33] The corresponding Z_M-action on $\mathrm{Bun}_P\times_{\mathrm{Bun}_M} H_{M,G\text{-}\mathrm{pos}}\times_{\mathrm{Bun}_M}\mathrm{Bun}_{P^-}$ is the one induced by the Z_M-action on $H_{M,G\text{-}\mathrm{pos}}$. Note that the map $H_{M,G\text{-}\mathrm{pos}}\to\mathrm{Bun}_M\times\mathrm{Bun}_M$ is Z_M-equivariant for this action and the *trivial* action on $\mathrm{Bun}_M\times\mathrm{Bun}_M$.

Lemma 2.2.4. *The* !*-pushforward functor*

$$\mathrm{D}_{\mathrm{indhol}}({}_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P}) \to \mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P})$$

preserves $(\mathbf{U}_P \times \mathbf{U}_P^-)$*-equivariant objects.*

Proof. It suffices to prove the similar version after replacing $\overline{\mathrm{Bun}}_{G,P}$ by its smooth cover VinBun_{G,C_P}. By [Sch16, § 3.3.2], the map $f : {}_{\mathrm{dfstr}}\mathrm{VinBun}_{G,C_P} \to \mathrm{VinBun}_{G,C_P}$ factors as

$${}_{\mathrm{dfstr}}\mathrm{VinBun}_{G,C_P} \xrightarrow{j} \widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} \widetilde{\mathrm{Bun}}_{P^-} \xrightarrow{\overline{f}} \mathrm{VinBun}_{G,C_P}$$

such that j is a schematic open embedding and $\overline{f}$ is proper on each connected component. Recall that $\widetilde{\mathrm{Bun}}_P$ also has a defect stratification with

$${}_{\mathrm{dfstr}}\widetilde{\mathrm{Bun}}_P \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}}.$$

We define $\mathrm{D}({}_{\mathrm{dfstr}}\widetilde{\mathrm{Bun}}_P)^{\mathbf{U}_P}$ to be the full subcategory of $\mathrm{D}({}_{\mathrm{dfstr}}\widetilde{\mathrm{Bun}}_P)$ consisting of objects that are !-pullbacks from $\mathrm{D}(H_{M,G\text{-}\mathrm{pos}})$. We define $\mathrm{D}(\widetilde{\mathrm{Bun}}_P)^{\mathbf{U}_P}$ similarly as before. We also define

$$\mathrm{D}(\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} \widetilde{\mathrm{Bun}}_{P^-})^{\mathbf{U}_P \times \mathbf{U}_P^-}.$$

We claim the functor $\overline{f}_!$ preserves $(\mathbf{U}_P \times \mathbf{U}_P^-)$-equivariant objects. To prove the claim, we use the fact that $\overline{f}$ is compatible with the defect stratifications. In other words, we have

$$\begin{aligned} &{}_{\mathrm{dfstr}}\mathrm{VinBun}_{G,C_P} \underset{\mathrm{VinBun}_{G,C_P}}{\times} (\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} \widetilde{\mathrm{Bun}}_{P^-}) \simeq \\ \simeq\ &(\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}}) \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} (H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}), \end{aligned}$$

such that the porjection from the RHS to

$${}_{\mathrm{dfstr}}\mathrm{VinBun}_{G,C_P} \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}$$

is induced by the "composition" map

$$H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \to H_{M,G\text{-}\mathrm{pos}}.$$

Then the claim follows from the base-change isomorphisms (which exist because $\overline{f}$ is proper on each connected component).

It remains to show $j_!$ preserves $(\mathbf{U}_P \times \mathbf{U}_P^-)$-equivariant objects. Using the base-change isomorphism, it suffices to show that the !-pushforward functor

$$\mathrm{D}_{\mathrm{indhol}}(\mathrm{Bun}_P) \to \mathrm{D}_{\mathrm{indhol}}(\widetilde{\mathrm{Bun}}_P)$$

preserves $\mathbf{U}_P$-equivariant object. However, this is well-known and was proved in § B.1.1.

□[Lemma 2.2.4]

Lemma 2.2.5. *The functor*

$$\overline{\Delta}^{\mathrm{enh},l}_{P,!} : \mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P}) \to \mathrm{D}_{\mathrm{indhol}}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G)$$

sends objects in $\mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}_P}$ *to objects in* $\mathbf{I}(G \times G, P \times G)$.

Proof. Lemma 2.2.4 formally implies $\mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}_P}$ is generated under colimits and extensions by the image of the !-pushforward funnctor

$$\mathrm{D}_{\mathrm{indhol}}({}_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}_P} \to \mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}_P}.$$

Hence it suffices to show the !-pushforward along

$${}_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P} \to \overline{\mathrm{Bun}}_{G,P} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G$$

sends $\mathbf{U}_P$-equivariant objects into $\mathbf{I}(G\times G, P\times G)$. Unwinding the definitions, this map is isomorphic to

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} (H_{M,G\text{-pos}}/Z_M) \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-} \xrightarrow{a} \mathrm{Bun}_P \times \mathrm{Bun}_G \xrightarrow{b} \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G .$$

By the base-change isomorphism, $a_!$ preserves $\mathbf{U}_P$-equivariant objects. Then we are done because $b_!$ sends $\mathbf{U}_P$-equivariant objects into $\mathbf{I}(G\times G, P\times G)$ by Proposition 0.2.11(1).

□[Lemma 2.2.5]

2.2.6. *Finish the proof.* Recall

$$\mathcal{F}_P \simeq \overline{\mathbf{\Delta}}_{P,!}^{\mathrm{enh},l} \circ i_P^! \circ \mathbf{K}(P).$$

By definition, we have

$$i_P^! \circ \mathbf{K}(P) \simeq i_P^* \circ j_{G,*} \circ r_!(k_{\mathrm{Bun}_G})[\mathrm{rank}(M) - \mathrm{rank}(G)].$$

By Lemma 2.2.3, this is an $\mathbf{U}_P$-equivariant object. Then we are done by Lemma 2.2.5.

□[Proposition 1.1.1]

2.3. **Proof of Proposition-Construction 1.1.4.**

Goal 2.3.1. *Construct a canonical natural transformation $\overline{\Delta}_! \circ \mathbf{K} \to \mathbf{E}\circ \mathbf{DL}$ whose value at $P \in \mathrm{Par}$ is equivalent to the morphism (1.8)*

Proposition 0.2.11 provides a functor

$$\mathbf{I}(G,-) : \mathrm{Par} \to \mathrm{DGCat}_{\mathrm{cont}}$$

that sends an arrow $P \to Q$ to the functor $\mathrm{Eis}^{\mathrm{enh}}_{P\to Q}$. Hence we also have a functor

$$\mathbf{I}(G\times G, -\times G) : \mathrm{Par} \to \mathrm{DGCat}_{\mathrm{cont}} \tag{2.5}$$

that sends an arrow $P \to Q$ to the functor $\mathrm{Eis}^{\mathrm{enh}}_{P\times G\to Q\times G}$.

Lemma 2.3.2. *The functor (2.5) is canonically isomorphic to the functor*

$$\mathrm{Par} \to \mathrm{DGCat}_{\mathrm{cont}},\ P \mapsto \mathbf{I}(G,P) \underset{k}{\otimes} \mathrm{D}(\mathrm{Bun}_G).$$

Proof. By the proof of [DG13, Corollary 2.3.4], the functor $\mathrm{D}(Y)\otimes_k \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(Y\times \mathrm{Bun}_G)$ is an equivalence for any lft prestack Y. Then the lemma follows from definitions.

□[Lemma 2.3.2]

Let $\widetilde{\mathbf{I}} \to \mathrm{Par}$ be the presentable fibration[34] classifying the functor (2.5). Note that Par has a final object G, and the fiber of $\widetilde{\mathbf{I}}$ at this object is $\widetilde{\mathbf{I}}_G := \mathrm{D}(\mathrm{Bun}_G\times \mathrm{Bun}_G)$. Consider the trivial fibration $\widetilde{\mathbf{I}}_G \times \mathrm{Par} \to \mathrm{Par}$. It follows formally that we have an adjoint pair

$$\mathrm{Eis}^{\mathrm{enh}} : \widetilde{\mathbf{I}} \rightleftarrows \widetilde{\mathbf{I}}_G \times \mathrm{Par} : \mathrm{CT}^{\mathrm{enh}},$$

where $\mathrm{Eis}^{\mathrm{enh}}$ (resp. $\mathrm{CT}^{\mathrm{enh}}$) preserves co-Cartesian (resp. Cartesian) arrows and its fiber at $P\in \mathrm{Par}$ is $\mathrm{Eis}^{\mathrm{enh}}_{P\times G\to G\times G}$ (resp. $\mathrm{CT}^{\mathrm{enh}}_{G\times G\to P\times G}$). Using Lemma 2.3.2, the functor $\mathbf{E}\circ\mathbf{DL}$ is isomorphic to

$$\mathrm{Par} \xrightarrow{(\Delta_!(k_{\mathrm{Bun}_G}),-)} \widetilde{\mathbf{I}}_G \times \mathrm{Par} \xrightarrow{\mathrm{CT}^{\mathrm{enh}}} \widetilde{\mathbf{I}} \xrightarrow{\mathrm{Eis}^{\mathrm{enh}}} \widetilde{\mathbf{I}}_G \times \mathrm{Par} \xrightarrow{\mathrm{pr}} \widetilde{\mathbf{I}}_G.$$

Denote the composition of the first two functors by $\mathbf{S}_{\mathrm{CT}} : \mathrm{Par} \to \widetilde{\mathbf{I}}$. Note that it is the unique Cartesian section whose value at $G\in\mathrm{Par}$ is $\Delta_!(k_{\mathrm{Bun}_G}) \in \widetilde{\mathbf{I}}_G$.

We also have a functor

$$\mathrm{Par}^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},\ P \mapsto \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G)$$

that sends an arrow to the corresponding !-pullback functor. Let $\mathcal{D}_{\mathrm{gen}} \to \mathrm{Par}$ be the corresponding *Cartesian* fibration. By Proposition 0.2.11(1), we have a fully faithful functor $\widetilde{\mathbf{I}} \to \mathcal{D}_{\mathrm{gen}}$ that preserves *co-Cartesian* arrows (although $\mathcal{D}_{\mathrm{gen}}$ is not a co-Cartesian fibration).

[34] A presentable fibration is both a Cartesian fibration and a coCartesian fibration whose fibers are presentable $(\infty,1)$-categories. See [Lur09, Definition 5.5.3.2].

On the other hand, consider the functor

$$\mathrm{Par} \to \mathrm{DGCat}_{\mathrm{cont}},\ P \mapsto \mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,\leq P})$$

that sends an arrow to the corresponding !-extension functor. Let $\overline{\mathcal{D}} \to \mathrm{Par}$ be the presentable fibration classifying this functor. We have a fully faithful functor

$$\overline{\mathcal{D}} \to \mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_G) \times \mathrm{Par}$$

whose fiber at $P \in \mathrm{Par}$ is the corresponding !-extension functor. The graph of the functor $\mathbf{K}$:

$$\mathrm{Par} \to \mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_G) \times \mathrm{Par},\ P \mapsto (\mathbf{K}(P), P)$$

is contained in the above full subcategory $\overline{\mathcal{D}}$. Hence we obtain a section $\mathbf{S}_{\mathbf{K}} : \mathrm{Par} \to \overline{\mathcal{D}}$ to the projection $\overline{\mathcal{D}} \to \mathrm{Par}$.

By Proposition 1.1.2, we also have functorial maps

$$\overline{\mathbf{\Delta}}^{\mathrm{enh},l}_{\leq P} : \overline{\mathrm{Bun}}_{G,\leq P} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G\,.$$

Hence there is a functor

$$\overline{\mathcal{D}} \to \mathcal{D}_{\mathrm{gen}}$$

that preserves co-Cartesian arrows such that its fiber at $P \in \mathrm{Par}$ is the composition

$$\mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,\leq P}) \xrightarrow{\overline{\mathbf{\Delta}}^{\mathrm{enh},l}_{\leq P,!}} \mathrm{D}_{\mathrm{indhol}}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G).$$

By construction, the composition

$$\mathrm{Par} \xrightarrow{\mathbf{S}_{\mathbf{K}}} \overline{\mathcal{D}} \to \mathcal{D}_{\mathrm{gen}}$$

sends P to $\mathcal{F}_P$, viewed as an object in $\mathcal{D}_{\mathrm{gen}}$ over $P \in \mathrm{Par}$. Hence by Proposition 1.1.1, this functor factors through the full subcategory $\widetilde{\mathbf{I}} \subset \mathcal{D}_{\mathrm{gen}}$. Let $\mathbf{S}'_{\mathbf{K}} : \mathrm{Par} \to \widetilde{\mathbf{I}}$ be the corresponding functor. By constuction, $\overline{\Delta}_! \circ \mathbf{K}$ is isomorphic to the composition

$$\mathrm{Par} \xrightarrow{\mathbf{S}'_{\mathbf{K}}} \widetilde{\mathbf{I}} \xrightarrow{\mathrm{Eis}^{\mathrm{enh}}} \widetilde{\mathbf{I}}_G \times \mathrm{Par} \xrightarrow{\mathrm{pr}} \widetilde{\mathbf{I}}_G.$$

In summary, we have obtained two sections $\mathbf{S}_{\mathrm{CT}}$ and $\mathbf{S}'_{\mathbf{K}}$ to the Cartesian fibration $\widetilde{\mathbf{I}} \to \mathrm{Par}$ such that $\overline{\Delta}_! \circ \mathbf{K}$ and $\mathbf{E} \circ \mathbf{DL}$ are obtained respectively by composing them with

$$\widetilde{\mathbf{I}} \xrightarrow{\mathrm{Eis}^{\mathrm{enh}}} \widetilde{\mathbf{I}}_G \times \mathrm{Par} \xrightarrow{\mathrm{pr}} \widetilde{\mathbf{I}}_G.$$

Now the identification $\mathcal{F}'_G = \mathcal{F}_G \simeq \Delta_!(k_{\mathrm{Bun}_G})$ provides an isomorphism $\mathbf{S}'_{\mathbf{K}}(G) \simeq \mathbf{S}_{\mathrm{CT}}(G)$. Since $G \in \mathrm{Par}$ is the final object and since $\mathbf{S}_{\mathrm{CT}}$ is a Cartesian section, we obtain a natural transformation $\mathbf{S}'_{\mathbf{K}} \to \mathbf{S}_{\mathrm{CT}}$ whose value at $P \in \mathrm{Par}$ is the unique arrow $\mathbf{S}'_{\mathbf{K}}(P) \to \mathbf{S}_{\mathrm{CT}}(P)$ fitting into the following commutative diagram

$$\begin{tikzcd}
\mathbf{S}'_{\mathbf{K}}(P) \arrow[r] \arrow[d] & \mathbf{S}_{\mathrm{CT}}(P) \arrow[d] \\
\mathbf{S}'_{\mathbf{K}}(G) \arrow[r, "\simeq"] & \mathbf{S}_{\mathrm{CT}}(G).
\end{tikzcd}$$

By construction, when viewed as a morphism in $\widetilde{\mathbf{I}}_P \simeq \mathbf{I}(G \times G, P \times G)$, the arrow $\mathbf{S}'_{\mathbf{K}}(P) \to \mathbf{S}_{\mathrm{CT}}(P)$ is equivalent to (1.6). Now the desired natural transformation $\overline{\Delta}_! \circ \mathbf{K} \to \mathbf{E} \circ \mathbf{DL}$ is given by composing the above natural transformation $\mathbf{S}'_{\mathbf{K}} \to \mathbf{S}_{\mathrm{CT}}$ with the functor

$$\widetilde{\mathbf{I}} \xrightarrow{\mathrm{Eis}^{\mathrm{enh}}} \widetilde{\mathbf{I}}_G \times \mathrm{Par} \xrightarrow{\mathrm{pr}} \widetilde{\mathbf{I}}_G.$$

□[Proposition-Construction 1.1.4]

3. Step 2

We have two results to prove in this step: Lemma 1.2.1 and Theorem 1.2.2. We prove Lemma 1.2.1 in § 3.1, and prove Theorem 1.2.2 in § 3.3 after reviewing the work of [Dri13, Appendix C] in § 3.2.

3.1. **Proof of Lemma 1.2.1.**

Goal 3.1.1. *For any $\lambda \in \Lambda_{G,P}$, we have:*

(1) *The maps* $\mathrm{Bun}_{P^-,\lambda}^{M\text{-gen}} \to \mathrm{Bun}_G^{P\text{-gen}}$ *is quasi-compact and schematic;*

(2) *The map*

$$\mathrm{Bun}_{M,\lambda} \to \mathrm{Bun}_{P,\lambda} \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_{P^-,\lambda}^{M\text{-gen}}$$

is a schematic open embedding.

By definition, we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Bun}_{P^-}^{M\text{-gen}} \arrow[r] \arrow[d] & \mathrm{Bun}_G^{P\text{-gen}} \arrow[d] \\ \mathrm{Bun}_{P^-} \arrow[r] & \mathrm{Bun}_G . \end{tikzcd}$$

We claim it induces a schematic open embedding

$$\mathrm{Bun}_{P^-}^{M\text{-gen}} \to \mathrm{Bun}_G^{P\text{-gen}} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-} .$$

Indeed, the RHS is isomorphic to $\mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}P^- \leftarrow \mathbb{B}P \times_{\mathbb{B}G} \mathbb{B}P^-)$ and the above map is isomorphic to the map

$$\mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}P^- \leftarrow \mathbb{B}M) \to \mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}P^- \leftarrow \mathbb{B}P \underset{\mathbb{B}G}{\times} \mathbb{B}P^-)$$

induced by the map $\mathbb{B}M \to \mathbb{B}P \times_{\mathbb{B}G} \mathbb{B}P^-$. Then the claim follows from the fact that $\mathbb{B}M \to \mathbb{B}P \times_{\mathbb{B}G} \mathbb{B}P^-$ is a schematic open embedding.

Now (1) follows from the above claim and the well-known fact that $\mathrm{Bun}_{P^-,\lambda} \to \mathrm{Bun}_G$ is quasi-compact and schematic.

To prove (2), we only need to show

$$\mathrm{Bun}_M \to \mathrm{Bun}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_{P^-}^{M\text{-gen}}$$

is a schematic open embedding. As before, this follows from the fact that it is isomorphic to

$$\mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}M \leftarrow \mathbb{B}M) \to \mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}P \underset{\mathbb{B}G}{\times} \mathbb{B}P^- \leftarrow \mathbb{B}M)$$

and the fact that $\mathbb{B}M \to \mathbb{B}P \times_{\mathbb{B}G} \mathbb{B}P^-$ is a schematic open embedding.

□[Lemma 1.2.1]

3.2. **Recollections: Drinfeld's framework.** In [Dri13, Appendix C], Drinfeld set up a general framework to prove results like Theorem 1.2.2. We review this framework in this subsection. In fact, we slightly generalize it to the case of lft prestacks.

Definition 3.2.1. *We equip the category* $\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$ *with the Cartesian symmetric monoidal structure. Recall the notion of* enriched categories. *Following* loc.cit.*, we define a category* $\mathbf{P}_{\mathbb{A}^1}$ *enriched in* $\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$ *as follows:*

- *It has two objects: the "big" one* $\mathbf{b}$ *and the "small" one* $\mathbf{s}$.
- *The mapping scheme* $\mathbf{Hom}_{\mathbf{P}_{\mathbb{A}^1}}(\mathbf{b},\mathbf{b})$ *is defined to be* $\mathbb{A}^1$. *The other three mapping shemes are defined to be* pt*, viewed as the zero point in* $\mathbb{A}^1$. *The composition laws are all induced by the semi-group structure on* $\mathbb{A}^1$.

The unique morphism $\mathbf{s} \to \mathbf{b}$ *is denoted by* α^+ *and the unique morphism* $\mathbf{b} \to \mathbf{s}$ *is denoted by* α^-.

Definition 3.2.2. *Let* $\mathrm{AlgStk}_{\mathrm{QCAD}}$ *be the* $(2,1)$*-category of QCAD algebraic stacks equipped with the Cartesian symmetric monoidal structure. We define a category*[35] $\mathbf{Dri}$ *enriched in* $\mathrm{AlgStk}_{\mathrm{QCAD}}$ *by replacing* $\mathbb{A}^1$ *in Definition 3.2.1 by the quotient stack* $\mathbb{A}^1/\mathbb{G}_m$*, and replacing the zero map* $\mathrm{pt} \to \mathbb{A}^1$ *by the map* $\mathbb{B}\mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m$ *obtained by taking quotients.*

[35] It was denoted by $\mathbf{P}_{\mathbb{A}^1}/\mathbb{B}\mathbb{G}_m$ in [Dri13].

Note that there is an obvious functor $\mathbf{P}_{\mathbb{A}^1} \to \mathbf{Dri}$. *We use the same symbols* α^+ *and* α^- *to denote the corresponding morphisms in* $\mathbf{Dri}$.

Definition 3.2.3. *Let* $\mathcal{O}$ *be a monoidal* $(\infty,1)$*-category,* $\mathcal{A}$ *be a category enriched in* $\mathcal{O}$ *and* $\mathcal{C}$ *be a module* $(\infty,2)$*-category of* $\mathcal{O}$. *As explained in [Dri13, § C.13.1], there is a notion of* weakly $\mathcal{O}$-enriched (unital) right-lax functors[36] *from* $\mathcal{A}$ *to* $\mathcal{C}$. *We will review its explicit meaning later in our particular examples. For now, let us give the formal definition.*

We assume $\mathcal{O}$ *is small. Consider the* $(\infty,1)$*-category* $\mathrm{Funct}(\mathcal{O}^{\mathrm{op}},(\infty,1)\text{-Cat})$ *equipped with the Day convolution monoidal structure (see [Lur12, § 2.2.6]). Then* $\mathcal{C}$ *has a* $\mathrm{Funct}(\mathcal{O}^{\mathrm{op}},(\infty,1)\text{-Cat})$*-enriched structure such that for any* $x,y\in\mathcal{C}$*, the object*

$$\mathbf{Hom}_{\mathcal{C}}(x,y)\in \mathrm{Funct}(\mathcal{O}^{\mathrm{op}},(\infty,1)\text{-Cat})$$

is the functor $o\mapsto \mathbf{Maps}_{\mathcal{C}}(o\otimes x,y)$.

On the other hand, there is a right-lax *monoidal structure on the Yoneda functor*

$$\mathcal{O}\to \mathrm{Funct}(\mathcal{O}^{\mathrm{op}},(\infty,1)\text{-Cat}).$$

Then a weakly $\mathcal{O}$*-enriched functor (resp. right-lax functor)* $F:\mathcal{A}\dashrightarrow\mathcal{C}$ *is defined to be a functor (resp. right-lax functor)* F *that intertwines the enrichment via the above right-lax monoidal functor.*

Notation 3.2.4. *Consider the* $(3,2)$*-category* $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}}$. *We equip it with the obvious* $\mathrm{AlgStk}_{\mathrm{QCAD}}$*-action.*

A Drinfeld pre-input *is a weakly* $\mathrm{AlgStk}_{\mathrm{QCAD}}$*-enriched right-lax functor* $F:\mathbf{P}_{\mathbb{A}^1}\dashrightarrow\mathbf{Corr}$ *such that it is strict at the composition* $\alpha^+\circ\alpha^-$*, i.e., the* 2*-morphism* $F(\alpha^+)\circ F(\alpha^-)\to F(\alpha^+\circ\alpha^-)$ *is invertible.*

A Drinfeld input *is a weakly* $\mathrm{AlgStk}_{\mathrm{QCAD}}$*-enriched right-lax functor* $F^\sharp:\mathbf{Dri}\dashrightarrow\mathbf{Corr}$ *such that the composition* $\mathbf{P}_{\mathbb{A}^1}\to\mathbf{Dri}\dashrightarrow\mathbf{Corr}$ *is a Drinfeld pre-input.*

Remark 3.2.5. Unwinding the definitions, a Drinfeld pre-input provides

- Two lft prestacks $Z:=F(\mathbf{b})$ and $Z^0:=F(\mathbf{s})$;
- Two correspondences
$$F(\alpha^+):(Z\xleftarrow{p^+}Z^+\xrightarrow{q^+}Z^0)\text{ and }F(\alpha^-):(Z^0\xleftarrow{q^-}Z^-\xrightarrow{p^-}Z)$$
whose left arms are QCAD maps;
- An $\mathbb{A}^1$-family of correspondences:
$$\begin{tikzcd} Z & \widetilde{Z} \arrow[l] \arrow[r] \arrow[d] & Z; \\ & \mathbb{A}^1 & \end{tikzcd}$$
given by $\mathrm{Hom}(\mathbf{b},\mathbf{b})\times F(\mathbf{b})\to F(\mathbf{b})$;
- Isomorphisms
$$Z^+\underset{Z^0}{\times}Z^-\simeq\widetilde{Z}\underset{\mathbb{A}^1}{\times}0\text{ and }Z\simeq\widetilde{Z}\underset{\mathbb{A}^1}{\times}1$$
defined over $Z\times Z$, given respectively by the invertible 2-morphism $F(\alpha^+)\circ F(\alpha^-)\to F(\alpha^+\circ\alpha^-)$ and $\mathrm{Id}_{F(\mathbf{b})}\simeq F(\mathrm{Id}_{\mathbf{b}})$
- An open embedding
$$j:Z^0\to Z^-\underset{Z}{\times}Z^+$$
defined over $Z^0\times Z^0$, given by the lax composition law for $\mathbf{s}\leftarrow\mathbf{b}\leftarrow\mathbf{s}$;

---

[36] It was called just by *lax functors* in *loc.cit.*.

- Open embeddings
$$Z^+ \times \mathbb{A}^1 \to \widetilde{Z} \underset{Z}{\times} Z^+ \text{ and } Z^- \times \mathbb{A}^1 \to Z^- \underset{Z}{\times} \widetilde{Z},$$
defined respectively over $Z \times Z^0 \times \mathbb{A}^1$ and $Z^0 \times Z \times \mathbb{A}^1$, given respectively by the lax composition laws for $\mathbf{b} \leftarrow \mathbf{b} \leftarrow \mathbf{s}$ and $\mathbf{s} \leftarrow \mathbf{b} \leftarrow \mathbf{b}$;
- An open embedding[37]
$$\widetilde{Z} \underset{\mathbb{A}^1}{\times} \mathbb{A}^2 \to \widetilde{Z} \underset{Z}{\times} \widetilde{Z}$$
defined over $Z \times Z \times \mathbb{A}^2$, given by the lax composition law for $\mathbf{b} \leftarrow \mathbf{b} \leftarrow \mathbf{b}$.
- Some higher compatibilities.

Example 3.2.6. For any finite type scheme Z equipped with a $\mathbb{G}_m$-action, [Dri13] constructed a Drinfeld pre-input such that Z^+, Z^- and Z^0 are respectively the attractor, repeller and fixed loci of Z. Also, $\widetilde{Z}$ is the so-called Drinfeld-Gaitsgory interpolation, which is an $\mathbb{A}^1$-degeneration from Z to $Z^{\mathrm{att}} \times_{Z^{\mathrm{fix}}} Z^{\mathrm{rep}}$. Moreover, this construction is functorial in Z and compatible with Cartesian products.

When Z is affine, the corresponding right-lax functor $\mathbf{P}_{\mathbb{A}^1} \dashrightarrow \mathbf{Corr}$ is strict. In particular, we obtain a functor $\mathbf{P}_{\mathbb{A}^1} \to \mathrm{Corr}$.

It was also shown in *loc.cit.* that there is a Drinfeld input with $F^\sharp(\mathbf{b}) = Z/\mathbb{G}_m$ and $F^\sharp(\mathbf{s}) = Z^{\mathrm{fix}}/\mathbb{G}_m$

3.2.7. *Drinfeld's theorem on adjunctions.* Let $F^\sharp : \mathbf{Dri} \dashrightarrow \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}}$ be a Drinfeld input and F be the corresponding Drinfeld pre-input. We use the notations in Remark 3.2.5. Consider the composition
$$\mathbf{P}_{\mathbb{A}^1} \overset{F}{\dashrightarrow} \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}} \xrightarrow{\mathrm{Dmod}^{\blacktriangle,!}} \mathbf{DGCat}_{\mathrm{cont}}.$$
By construction, it sends α^+ and α^- respectively to the functors
$$\mathrm{Dmod}^{\blacktriangle,!} \circ F(\alpha^+) \simeq p^+_\blacktriangle \circ q^{+,!},\ \mathrm{Dmod}^{\blacktriangle,!} \circ F(\alpha^-) \simeq q^-_\blacktriangle \circ p^{-,!}.$$
The 2-morphism: $F(\alpha^-) \circ F(\alpha^+) \to F(\alpha^- \circ \alpha^+) = F(\mathrm{Id}_\mathbf{s})$ gives a natural transformation[38]
$$q^-_\blacktriangle \circ p^{-,!} \circ p^+_\blacktriangle \circ q^{+,!} \to \mathbf{Id}_{\mathrm{D}(Z^0)}. \tag{3.1}$$

The following result was proved in [Dri13, Appendix C].

Theorem 3.2.8. *(Drinfeld) In the above setting, there is an adjoint pair*
$$q^-_\blacktriangle \circ p^{-,!} : \mathrm{D}(Z) \rightleftarrows \mathrm{D}(Z^0) : p^+_\blacktriangle \circ q^{+,!}$$
with the counit adjunction natrual transformation given by (3.1).

Remark 3.2.9. The unit adjunction is given by a specialization construction along $\widetilde{Z} \to \mathbb{A}^1$. We do not need it in this paper.

Remark 3.2.10. More precisely, *loc.cit.* focused on the problem of reproving the Braden's theorem (see [Bra03]) using the Drinfeld input in Example 3.2.6. However, the proof there works for any Drinfeld input.

3.3. **Proof of Theorem 1.2.2.** Throughout this subsection, we fix a co-character $\gamma : \mathbb{G}_m \to Z_M$ dominant and regular with respect to P. Note that the homomorphism $\mathbb{G}_m \to Z_M \to Z_M/Z_G$ can be uniquely extended to a homomorphism between semi-groups $\overline{\gamma} : \mathbb{A}^1 \to T^+_{\mathrm{ad},\geq C_P}$. Via γ, the adjoint action $T_{\mathrm{ad}} \curvearrowright G$ induces an action $\mathbb{G}_m \curvearrowright G$.

We first deduce the theorem from the following result:

[37] The map $\mathbb{A}^2 \to \mathbb{A}^1$ in the formula is the multiplication map.

[38] Explicitly, the LHS is the !-pull-$\blacktriangle$-push along $Z^0 \leftarrow Z^- \times_Z Z^+ \to Z^0$, while the RHS is that along $Z^0 \leftarrow Z^0 \to Z^0$. The desired natural transformation is induced by the adjoint pair $(j^!, j_\blacktriangle)$ for the open embedding $j : Z^0 \to Z^- \times_Z Z^+$.

Proposition-Construction 3.3.1. *There exists a canonical Drinfeld input*

$$F^\sharp : \mathbf{Dri} \dashrightarrow \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}}$$

such that[39] *it sends α^+ and α^- respectively to*

$$\mathrm{Bun}_G^{P^-\text{-gen}}/\mathbb{G}_m \leftarrow \mathrm{Bun}_{P,\lambda}^{M\text{-gen}}/\mathbb{G}_m \to \mathrm{Bun}_{M,\lambda}/\mathbb{G}_m,$$
$$\mathrm{Bun}_{M,\lambda}/\mathbb{G}_m \leftarrow \mathrm{Bun}_{P^-,\lambda}/\mathbb{G}_m \to \mathrm{Bun}_G^{P^-\text{-gen}}/\mathbb{G}_m.$$

3.3.2. *Deduction Theorem 1.2.2.* We will use the mirror version of Proposition-Construction 3.3.1 by exchanging P and P^-. Using Theorem 3.2.8, we obtain the version of Theorem 1.2.2 after replacing the relevant stacks by their $\mathbb{G}_m$-quotients. The same proof of [DG14, Theorem 3.4.3] implies the following adjoint pair

$$\mathrm{Dmod}^{\blacktriangle,!}(\alpha^{+,\mathrm{gen}}_{P,\lambda}) : \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}})^{\mathbb{G}_m\text{-mon}} \rightleftarrows \mathrm{D}(\mathrm{Bun}_M) : \mathrm{Dmod}^{\blacktriangle,!}(\alpha^{-,\mathrm{gen}}_{P,\lambda}),$$

where

$$\mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}})^{\mathbb{G}_m\text{-mon}} \subset \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}})$$

is the full subcategory generated by the essential image of the !-pullback functor

$$\mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}}/\mathbb{G}_m) \to \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}}).$$

Then we are done because the $\mathbb{G}_m$-action on $\mathrm{Bun}_G^{P\text{-gen}}$ can be trivialized.

□[Theorem 1.2.2]

It remains to construct the Drinfeld input in Proposition-Construction 3.3.1.

Notation 3.3.3. *Let $\mathrm{Grp}^{\mathrm{aff}}_{\mathrm{ft}}$ be the category of group schemes $H \to S$ with H and S being finite type affine schemes. Consider its arrow category $\mathrm{Arr}(\mathrm{Grp}^{\mathrm{aff}}_{\mathrm{ft}})$. We equip the category*

$$\mathrm{Corr}(\mathrm{Arr}(\mathrm{Grp}^{\mathrm{aff}}_{\mathrm{ft}}))_{\mathrm{all},\mathrm{all}}$$

with the obvious $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$-action.

Construction 3.3.4. *Via the co-character γ, the adjoint actions $Z_M \curvearrowright G$ and $Z_M \curvearrowright P$ induces actions $\mathbb{G}_m \curvearrowright G$ and $\mathbb{G}_m \curvearrowright P$. The corresponding attractor, repeller and fixed loci are:*

$$G^{\mathrm{att},\gamma} = P,\ G^{\mathrm{rep},\gamma} = P^-,\ G^{\mathrm{fix},\gamma} = M,\ (P^-)^{\mathrm{att},\gamma} = M,\ (P^-)^{\mathrm{rep},\gamma} = P^-,\ (P^-)^{\mathrm{fix},\gamma} = M.$$

Using Example 3.2.6, we obtain a weakly $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$-enriched functor

$$\Theta_{P^-\to G} : \mathbf{P}_{\mathbb{A}^1} \to \mathrm{Corr}(\mathrm{Arr}(\mathrm{Grp}^{\mathrm{aff}}_{\mathrm{ft}}))_{\mathrm{all},\mathrm{all}}$$

sending α^+ and α^- respectively to

$$(P^- \to G) \leftarrow (M \to P) \to (M \to M),\ (M \to M) \leftarrow (P^- \to P^-) \to (P^- \to G).$$

Remark 3.3.5. By construction, $\mathrm{Hom}(\mathbf{b},\mathbf{b}) \times \Theta_{P^-\to G}(\mathbf{b}) \to \Theta_{P^-\to G}(\mathbf{b})$ corresponds to the following diagram

$$\begin{tikzcd} (P^- \to G) & (\widetilde{P^-}^\gamma \to \widetilde{G}^\gamma) \arrow[l] \arrow[r] \arrow[d] & (P^- \to G); \\ & \mathbb{A}^1, & \end{tikzcd}$$

where $\widetilde{G}^\gamma$ (resp. $\widetilde{P^-}^\gamma$) is the Drinfeld-Gaitsgory interpolation for the action $\mathbb{G}_m \curvearrowright G$ (resp. $\mathbb{G}_m \curvearrowright P$). Note that we have

$$\widetilde{G}^\gamma \simeq \widetilde{G} \underset{T^+_{\mathrm{ad}},\overline{\gamma}}{\times} \mathbb{A}^1,\ \widetilde{P^-}^\gamma \simeq P^- \underset{G}{\times} \widetilde{G}^\gamma.$$

[39]We also require that the 2-morphism $F^\sharp(\alpha^+) \circ F^\sharp(\alpha^-) \to F^\sharp(\mathrm{Id}_{\mathbf{s}})$ is given by the obvious open embedding.

Construction 3.3.6. *Consider the functor*

$$\mathbb{B} : \mathrm{Grp}_{\mathrm{ft}}^{\mathrm{aff}} \to \mathrm{AlgStk}_{\mathrm{lft}},\ (H \to S) \mapsto \mathbb{B}H,$$

where $\mathbb{B}H := S/H$ *is the quotient stack. Similarly we have a functor* $\mathrm{Arr}(\mathrm{Grp}_{\mathrm{ft}}^{\mathrm{aff}}) \to \mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}})$. *This functor does not commute with fiber products, hence we only have a* right-lax *functor*

$$\mathrm{Corr}(\mathrm{Arr}(\mathrm{Grp}_{\mathrm{ft}}^{\mathrm{aff}}))_{\mathrm{all,all}} \dashrightarrow \mathbf{Corr}(\mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}}))_{\mathrm{all,all}}^{\mathrm{all,2\text{-}op}}.$$

This right-lax functor has an $\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$*-linear structure. Hence by composing with* $\Theta_{P^- \to G}$, *we obtain a weakly* $\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$*-enriched right-lax functor*

$$\Theta_{\mathbb{B}P^- \to \mathbb{B}G} : \mathbf{P}_{\mathbb{A}^1} \dashrightarrow \mathbf{Corr}(\mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}}))_{\mathrm{all,all}}^{\mathrm{all,2\text{-}op}}.$$

Definition 3.3.7. *A morphism* $(Y_1 \to Y_2) \to (Y_1' \to Y_2')$ *in* $\mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}})$ *is called an* open embedding *if both* $Y_1 \to Y_1'$ *and* $Y_2 \to Y_2'$ *are schematic open embeddings.*

Lemma 3.3.8. *The right-lax functor* $\Theta_{\mathbb{B}P^- \to \mathbb{B}G}$ *factors through* $\mathbf{Corr}(\mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}}))_{\mathrm{all,all}}^{\mathrm{open,2\text{-}op}}$ *and is strict at the composition* $\alpha^+ \circ \alpha^-$.

Proof. Consider the two forgetful functors $\mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}}) \to \mathrm{AlgStk}_{\mathrm{lft}},\ (Y_1 \to Y_2) \mapsto Y_i$. We only need to prove the similar claims after applying these forgetful functors. Those claims for the first forgetful functor are obvious (because $\widetilde{P^-}^\gamma \simeq P^- \times \mathbb{A}^1$). It remains to prove those for the second forgetful functor.

To prove the claim on strictness, we only need to check $\mathbb{B}(P \times_M P^-) \to \mathbb{B}P \times_{\mathbb{B}M} \mathbb{B}P^-$ is an isomorphism. But this is obvious.

To prove the claim on openness, we only need to check that the following four maps are schematic open embeddings:

$$\mathbb{B}(P^- \underset{G}{\times} \widetilde{G}^\gamma) \to \mathbb{B}P^- \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}^\gamma, \quad \mathbb{B}(\widetilde{G}^\gamma \times_G P) \to \mathbb{B}\widetilde{G}^\gamma \times_{\mathbb{B}G} \mathbb{B}P,$$
$$\mathbb{B}(\widetilde{G}^\gamma \underset{G}{\times} \widetilde{G}^\gamma) \to \mathbb{B}\widetilde{G}^\gamma \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}^\gamma \quad \mathbb{B}(P^- \times_G P) \to \mathbb{B}P^- \times_{\mathbb{B}G} \mathbb{B}P$$

The claim for the last one is obvious. The claims for the first two maps follows from Corollary D.1.8. The proof for the third one is similar. Namely, consider the action

$$(G \times G \times G) \curvearrowright {}_0\mathrm{Vin}_G^\gamma \times {}_0\mathrm{Vin}_G^\gamma,\ (g_1, g_2, g_3) \cdot (x_1, x_2) \mapsto (g_1 x_1 g_2^{-1}, g_2 x_2 g_3^{-1}).$$

Its stablizer for the canonical section is the group scheme $\widetilde{G}^\gamma \times_G \widetilde{G}^\gamma$. We only need to prove the similar version of Lemma D.1.7, i.e., to show

$$(G \times G \times G \times \mathbb{A}^2)/(\widetilde{G}^\gamma \underset{G}{\times} \widetilde{G}^\gamma) \to {}_0\mathrm{Vin}_G^\gamma \times {}_0\mathrm{Vin}_G^\gamma$$

is an open embedding. As before, we only need to show the LHS is smooth. Now the functor $\Theta_{P^- \to G}$ provides an isomorphism $\widetilde{G}^\gamma \times_G \widetilde{G}^\gamma \simeq \widetilde{G}^\gamma \times_{\mathbb{A}^1} \mathbb{A}^2$ covering the map

$$\mathrm{pr}_{13} \times \mathrm{Id}_{\mathbb{A}^2} : (G \times G \times G) \times \mathbb{A}^2 \to (G \times G) \times \mathbb{A}^2.$$

Hence we have a map

$$(G \times G \times G \times \mathbb{A}^2)/(\widetilde{G}^\gamma \underset{G}{\times} \widetilde{G}^\gamma) \to (G \times G \times \mathbb{A}^2)/(\widetilde{G}^\gamma \underset{\mathbb{A}^1}{\times} \mathbb{A}^2) \simeq {}_0\mathrm{Vin}_G^\gamma \underset{\mathbb{A}^1}{\times} \mathbb{A}^2.$$

Then we are done because it is a smooth map to a smooth scheme.

□[Lemma 3.3.8]

Construction 3.3.9. *Consider the functor*

$$\mathbf{Maps}_{\mathrm{gen}}(X, -) : \mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}}) \to \mathrm{PreStk}_{\mathrm{lft}},\ (Y_1 \to Y_2) \mapsto \mathbf{Maps}_{\mathrm{gen}}(X, Y_1 \leftarrow Y_2).$$

It is easy to see that it sends open embeddings to schematic open embeddings. Hence we obtain a functor

$$\mathbf{Corr}(\mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}}))_{\mathrm{all,all}}^{\mathrm{open,2\text{-}op}} \to \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{all,all}}^{\mathrm{open,2\text{-}op}}.$$

This functor has an $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$*-linear structure*[40]*. Hence by composing with* $\Theta_{\mathbb{B}P^-\to\mathbb{B}G}$*, we obtain a weakly* $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$*-enriched right-lax functor*

$$\Theta : \mathbf{P}_{\mathbb{A}^1} \dashrightarrow \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{all},\mathrm{all}}$$

that is strict at the composition $\alpha^+ \circ \alpha^-$.

Remark 3.3.10. Explicitly, we have:

- The right-lax functor Θ sends α^+ and α^- respectively to

$$\mathrm{Bun}_G^{P^-\text{-gen}} \leftarrow \mathrm{Bun}_P^{M\text{-gen}} \to \mathrm{Bun}_M,\ \mathrm{Bun}_M \leftarrow \mathrm{Bun}_{P^-} \to \mathrm{Bun}_G^{P^-\text{-gen}}.$$

- The map $\mathrm{Hom}(\mathbf{b},\mathbf{b}) \times \Theta(\mathbf{b}) \to \Theta(\mathbf{b})$ is provided by the $\mathbb{A}^1$-family of correspondences:

$$\begin{tikzcd} \mathrm{Bun}_G^{P^-\text{-gen}} & \mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}\widetilde{G}^\gamma \leftarrow \mathbb{B}\widetilde{P^-}^\gamma) \arrow[l] \arrow[r] \arrow[d] & \mathrm{Bun}_G^{P^-\text{-gen}}. \\ & \mathbb{A}^1 & \end{tikzcd}$$

Construction 3.3.11. *We write*

$$\begin{aligned} \mathrm{VinBun}_G^{P^-\text{-gen},\gamma} &:= \mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}\widetilde{G}^\gamma \leftarrow \mathbb{B}\widetilde{P^-}^\gamma), \\ \mathrm{VinBun}_G^\gamma &:= \mathbf{Maps}(X, \mathbb{B}\widetilde{G}^\gamma). \end{aligned}$$

There is a map

$$\mathrm{VinBun}_G^{P^-\text{-gen},\gamma} \to \mathrm{Bun}_G^{P^-\text{-gen}} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_G^\gamma$$

induced by the map

$$\mathbb{B}\widetilde{P^-}^\gamma \simeq \mathbb{B}(P^- \underset{G}{\times} \widetilde{G}^\gamma) \to \mathbb{B}P^- \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}^\gamma.$$

By Corollary D.1.8, these maps are schematic open embeddings.

Construction 3.3.12. *Recall that* $\mathrm{VinBun}_{G,C_P} \simeq \mathrm{Bun}_P \times_{\mathrm{Bun}_M} \mathrm{Bun}_{P^-}$*. Hence there is a unique open substack* $\mathrm{VinBun}_{G,\lambda}^\gamma$ *of* VinBun_G^γ *obtained by removing all the connected components*

$$\mathrm{Bun}_{P,\mu} \underset{\mathrm{Bun}_{M,\mu}}{\times} \mathrm{Bun}_{P^-,\mu} \subset \mathrm{VinBun}_{G,C_P}$$

with $\mu \neq \lambda$ *from its 0-fiber. Let* $\mathrm{VinBun}_{G,\lambda}^{P^-\text{-gen},\gamma}$ *be the corresponding open sub-prestack. It is easy to see we can modify* Θ *to obtain*

$$\Theta_\lambda : \mathbf{P}_{\mathbb{A}^1} \dashrightarrow \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{all},\mathrm{all}}$$

such that

- *It sends* α^+ *and* α^- *respectively to*

$$\mathrm{Bun}_G^{P^-\text{-gen}} \leftarrow \mathrm{Bun}_{P,\lambda}^{M\text{-gen}} \to \mathrm{Bun}_{M,\lambda},\ \mathrm{Bun}_{M,\lambda} \leftarrow \mathrm{Bun}_{P^-,\lambda} \to \mathrm{Bun}_G^{P^-\text{-gen}}.$$

- *The map* $\mathrm{Hom}(\mathbf{b},\mathbf{b}) \times \Theta_\lambda(\mathbf{b}) \to \Theta_\lambda(\mathbf{b})$ *is provided by the* $\mathbb{A}^1$*-family of correspondences:*

$$\begin{tikzcd} \mathrm{Bun}_G^{P^-\text{-gen}} & \mathrm{VinBun}_{G,\lambda}^{P^-\text{-gen},\gamma} \arrow[l] \arrow[r] \arrow[d] & \mathrm{Bun}_G^{P^-\text{-gen}}. \\ & \mathbb{A}^1 & \end{tikzcd}$$

- *The other data are induced from* Θ.

Lemma 3.3.13. *The right-lax functor* Θ_λ *factors through* $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}}$.

---

[40] This is because for affine schemes Y, we have $\mathbf{Maps}(X,Y) \simeq Y$.

Proof. We only need to check all the three left arms in the above three correspondences are QCAD. The claims for the first two arms are just (the mirror version of) Lemma 1.2.1(1). To prove the claim for the third arm, using the open embedding in Construction 3.3.11, we only need to show $\mathrm{Bun}_G \leftarrow \mathrm{VinBun}^{\gamma}_{G,\lambda}$ is QCAD. It is well-known that VinBun_G is locally QCAD. Hence we only need to show $\mathrm{VinBun}^{\gamma}_{G,\lambda} \to \mathrm{Bun}_G$ is quasi-compact. Then we are done because both the $\mathbb{G}_m$-locus and the 0-fiber of $\mathrm{VinBun}^{\gamma}_{G,\lambda}$ is quasi-compact over Bun_G.

□[Lemma 3.3.13]

We are going to construct a Drinfeld input from Θ_λ by taking quotients for the torus actions. We first introduce some notations.

Notation 3.3.14. *Let* $\mathrm{ActSch}^{\mathrm{aff}}_{\mathrm{ft}}$ *be the category whose objects are* $(H \curvearrowright Y)$*, where* H *is an affine algberaic group and* $Y \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$*. We equip* $\mathrm{ActSch}^{\mathrm{aff}}_{\mathrm{ft}}$ *with the Cartesian symmetric monoidal structure. Note that the monoidal unit for it is* $(\mathrm{pt} \curvearrowright \mathrm{pt})$*. Also note that there is a symmetric monoidal forgetful functor* $\mathbf{oblv}_{\mathrm{Act}} : \mathrm{ActSch}^{\mathrm{aff}}_{\mathrm{ft}} \to \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

As in Definition 3.2.1, we define a category $\mathbf{P}_{\mathbb{G}_m \curvearrowright \mathbb{A}^1}$ *enriched in* $\mathrm{ActSch}^{\mathrm{aff}}_{\mathrm{ft}}$ *such that*

$$\mathbf{Hom}_{\mathbf{P}_{\mathbb{G}_m \curvearrowright \mathbb{A}^1}}(\mathbf{b}, \mathbf{b}) = (\mathbb{G}_m \curvearrowright \mathbb{A}^1)$$

and the other three mapping objects are $(\mathbb{G}_m \curvearrowright 0)$*. We use the same symbols* α^+ *and* α^- *to denote the canonical morphisms.*

Note that $\mathbf{P}_{\mathbb{A}^1}$ *can be obtained from* $\mathbf{P}_{\mathbb{G}_m \curvearrowright \mathbb{A}^1}$ *by the procudure of* changing of enrichment *along* $\mathbf{oblv}_{\mathrm{Act}}$*. In particular, there is a forgetful functor* $\mathbf{P}_{\mathbb{G}_m \curvearrowright \mathbb{A}^1} \to \mathbf{P}_{\mathbb{A}^1}$ *that intertwines the enrichment via* $\mathbf{oblv}_{\mathrm{Act}}$.

Let $\mathrm{ActPreStk}_{\mathrm{lft}}$ *be the similarly defined category. A morphism* $(H \curvearrowright Y_1) \to (H_2 \curvearrowright Y_2)$ *is said to be an open embedding if* $H_1 \simeq H_2$ *and* $Y_1 \to Y_2$ *is a schemtaic open embedding. It is said to be QCAD if* $Y_1 \to Y_2$ *is QCAD.*

Construction 3.3.15. *(c.f. [Dri13, § C.13.4])*

In the previous connstruction of Θ_λ*, we ignored the various* $\mathbb{G}_m$*-actions. If we keep tracking them, we can obtain a weakly* $\mathrm{ActSch}^{\mathrm{aff}}_{\mathrm{ft}}$*-enriched right-lax functor*

$$\Theta^{\mathrm{Act}}_\lambda : \mathbf{P}_{\mathbb{G}_m \curvearrowright \mathbb{A}^1} \dashrightarrow \mathbf{Corr}(\mathrm{ActPreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}}$$

such that

- *It sends* α^+ *and* α^- *respectively to*
$$(\mathbb{G}_m \curvearrowright \mathrm{Bun}_G^{P^-\text{-gen}}) \leftarrow (\mathbb{G}_m \curvearrowright \mathrm{Bun}_{P,\lambda}^{M\text{-gen}}) \to (\mathbb{G}_m \curvearrowright \mathrm{Bun}_{M,\lambda}),$$
$$(\mathbb{G}_m \curvearrowright \mathrm{Bun}_{M,\lambda}) \leftarrow (\mathbb{G}_m \curvearrowright \mathrm{Bun}_{P^-,\lambda}) \to (\mathbb{G}_m \curvearrowright \mathrm{Bun}_G^{P^-\text{-gen}}).$$
- *The map* $\mathrm{Hom}(\mathbf{b}, \mathbf{b}) \times \Theta^{\mathrm{Act}}_\lambda(\mathbf{b}) \to \Theta^{\mathrm{Act}}_\lambda(\mathbf{b})$ *is provided by the diagram:*
$$\begin{tikzcd} (\mathbb{G}_m \curvearrowright \mathrm{Bun}_G^{P^-\text{-gen}}) & (\mathbb{G}_m \times \mathbb{G}_m \curvearrowright \mathrm{VinBun}_{G,\lambda}^{P^-\text{-gen},\gamma}) \arrow[l] \arrow[r] \arrow[d] & (\mathbb{G}_m \curvearrowright \mathrm{Bun}_G^{P^-\text{-gen}}). , \\ & (\mathbb{G}_m \curvearrowright \mathbb{A}^1) & \end{tikzcd}$$
which is induced by the morphism
$$(\mathbb{G}_m \times \mathbb{G}_m \curvearrowright \widetilde{Z}) \to (\mathbb{G}_m \curvearrowright Z) \times (\mathbb{G}_m \curvearrowright Z) \times (\mathbb{G}_m \curvearrowright \mathbb{A}^1)$$
that exists for any Drinfeld-Gaitsgory interpolation $\widetilde{Z}$ *(see [DG14, § 2.2.3]).*
- *It is compatible with* Θ_λ *via the forgetful functors.*

Then as in [Dri13, Footnote 41], we obtain the desired Drinfeld input by passing to quotients and changing enrichment.

□[Proposition-Construction 3.3.1]

4. Step 3

We have two results to prove in this step: Proposition-Construction 1.3.1 and Lemma 1.3.2. Each subsection corresponds to a result.

4.1. **Proof of Proposition-Construction 1.3.1.**

Goal 4.1.1. *Construct a certain open embedding*

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \to \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P}$$

whose restriction to the G-stratum and P-stratum are canonically isomorphic to the maps

$$\begin{array}{rcl} \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,G} & \to & \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,G}, \\ \mathrm{Bun}_{P^-}^{M\text{-gen}} \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \overline{\mathrm{Bun}}_{G,P} & \to & \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,P}. \end{array}$$

By definition, we have

$$\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P} \simeq \mathbf{Maps}_{\mathrm{gen}}(X, P^-\backslash \mathrm{Vin}_{G,\geq C_P}/G \supset P^-\backslash {}_0\mathrm{Vin}_{G,\geq C_P}/G).$$

Note that

$$P^-\backslash {}_0\mathrm{Vin}_{G,\geq C_P}/G \simeq \mathbb{B}P^- \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}_{\geq C_P},$$

where $\widetilde{G}_{\geq C_P} \coloneqq \widetilde{G} \times_{T_{\mathrm{ad}}^+} T^+_{\mathrm{ad},\geq C_P}$. By Corollary D.1.8, the map

$$\mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P}) \to \mathbb{B}P^- \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}_{\geq C_P} \simeq P^-\backslash {}_0\mathrm{Vin}_{G,\geq C_P}/G$$

is a schematic open embedding. We define

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P})^{\mathrm{gen}} \coloneqq \mathbf{Maps}_{\mathrm{gen}}(X, P^-\backslash \mathrm{Vin}_{G,\geq C_P}/G \leftarrow \mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P})).$$

Then we have a schematic open embedding

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P})^{\mathrm{gen}} \to \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P}.$$

As in the proof of Lemma 2.1.2, a direct calculation shows that the Z_M-action on ${}_0\mathrm{Vin}_{G,\geq C_P}$ preserves the open substack

$$\mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P}) \underset{(P^-\backslash {}_0\mathrm{Vin}_{G,\geq C_P}/G)}{\times} {}_0\mathrm{Vin}_{G,\geq C_P}.$$

Hence it makes sense to define

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \coloneqq (\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P})^{\mathrm{gen}}/Z_M.$$

It is obvious that the restriction of

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \to \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P}$$

to the G-stratum is an isomorphism. It remains to identify its restriction to the P-stratum with the map

$$\mathrm{Bun}_{P^-}^{M\text{-gen}} \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \overline{\mathrm{Bun}}_{G,P} \to \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,P}.$$

Unwinding the definitions, we only need to identify the C_P-fiber of the open embedding

$$\mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P}) \to \mathbb{B}P^- \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}_{\geq C_P}$$

with the C_P-fiber of the map

$$\mathbb{B}M \underset{\mathbb{B}P}{\times} \mathbb{B}\widetilde{G}_{\leq P} \to \mathbb{B}P^- \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}_{\leq P}.$$

However, this follows from $\widetilde{G}_{C_P} \simeq P \times_M P^-$.

□[Proposition-Construction 1.3.1]

4.2. **Proof of Lemma 1.3.2.** We will introduce many temporary notations in this subsection. When we use an english letter, like c, to denote a correspondence, or when we use a letter of plain font, like K, to denote a D-module, it means such notations are only used in this subsection.

Goal 4.2.1. *The morphism γ_P and/or γ'_P are equivalent to the morphism*

$$\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta)\circ\mathbf{K}(P)\to\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta)\circ\mathbf{K}(G).$$

4.2.2. *The arrow γ_P.* We first give the following tautological description of

$$\gamma_P:\mathrm{CT}^{\mathrm{gen}}_{P\times G,*}(\mathcal{F}_P)\to\mathrm{CT}_{P\times G,*}(\mathcal{F}_G).$$

Recall the morphism (1.5):

$$\mathrm{Eis}^{\mathrm{enh}}_{P\times G\to G\times G}(\mathcal{F}'_P)\to\mathcal{F}'_G.$$

Its underlying morphism in $\mathrm{D}_{\mathrm{indhol}}(\mathrm{Bun}_G\times\mathrm{Bun}_G)$ is a map

$$\vartheta_P:\mathfrak{p}^{\mathrm{enh}}_{P\times G,!}(\mathcal{F}_P)\to\mathcal{F}_G,$$

which by adjunction induces a morphism

$$\theta_P:\mathcal{F}_P\to\mathfrak{p}^{\mathrm{enh},!}_{P\times G}(\mathcal{F}_G).$$

Then we have

$$\gamma_P\simeq\mathrm{CT}^{\mathrm{gen}}_{P\times G,*}(\theta_P).$$

Note that we indeed have $\mathrm{CT}_{P\times G,*}\simeq\mathrm{CT}^{\mathrm{gen}}_{P\times G,*}\circ\mathfrak{p}^{\mathrm{enh},!}_{P\times G}$.

4.2.3. *Second adjointness for left functors.* Next, we give a more convenient description for the second adjointness, when restricted to ind-holonomic objects.

Let $'\mathrm{CT}_{P,*}$ be the restriction of $\mathrm{CT}_{P,*}$ to the full subcategory of ind-holonomic objects. By construction, the natural transformation $'\mathrm{CT}_{P,*}\simeq{}'\mathrm{CT}_{P^-,!}$ is obtained as follows. We apply $\mathrm{Dmod}^{\blacktriangle,!}$ to the 2-morphism

$$\alpha^+_{P,\lambda}\circ\alpha^-_{P,\lambda}\to\mathrm{Id}_{\mathrm{Bun}_{M,\lambda}}$$

in $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD,all}}$ and obtain a natural transformation

$$'\mathrm{CT}_{P,*,\lambda}\circ({}'\mathrm{CT}_{P^-,!,\lambda})^R\to\mathbf{Id}_{\mathrm{D}_{\mathrm{indhol}}(\mathrm{Bun}_M)}.$$

Then we obtain the natural transfomation $'\mathrm{CT}_{P,*,\lambda}\to{}'\mathrm{CT}_{P^-,!,\lambda}$ by adjunction. Equivalently, we have the left adjoint version of the above picture. Namely, we start from the 2-morphism[41]

$$\mathrm{Id}_{\mathrm{Bun}_M}\to(\alpha^-_P)^{\mathrm{rev}}\circ(\alpha^+_P)^{\mathrm{rev}}$$

in $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open}}_{\mathrm{all,Stacky}}$, and use $\mathrm{D}^{!,*}_{\mathrm{indhol}}$ to obtain a natural transformation

$$\mathbf{Id}\to{}'\mathrm{CT}_{P^-,!}\circ({}'\mathrm{CT}_{P,*})^L.$$

Then we can obtain the same natural transfomation $'\mathrm{CT}_{P,*}\to{}'\mathrm{CT}_{P^-,!}$ by adjunction.

The advantage is: if we use left functors, we can work with all the connected components simultaneously.

Similarly, the natural transformation of $'\mathrm{CT}_{P\times G,*}\simeq{}'\mathrm{CT}_{P^-\times G,!}$ can be obtained by the same procedure from the correspondences

$$\begin{aligned} c^+ &: (\mathrm{Bun}_G\times\mathrm{Bun}_G\leftarrow\mathrm{Bun}_P\times\mathrm{Bun}_G\to\mathrm{Bun}_M\times\mathrm{Bun}_G),\\ c^- &: (\mathrm{Bun}_M\times\mathrm{Bun}_G\leftarrow\mathrm{Bun}_{P^-}\times\mathrm{Bun}_G\to\mathrm{Bun}_G\times\mathrm{Bun}_G),\end{aligned}$$

and the 2-morphism

$$\mathrm{Id}_{(\mathrm{Bun}_M\times\mathrm{Bun}_G)}\to c^-\circ c^+ \tag{4.1}$$

in $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open}}_{\mathrm{all,Stacky}}$.

[41] The superscript "rev" means exchanging the two arms of a correspondence.

Similarly, the natural transformation of $'\mathrm{CT}^{\mathrm{gen}}_{P\times G,*} \simeq {}'\mathrm{CT}^{\mathrm{gen}}_{P^-\times G,!}$ can be obtained by the same procedure from the correspondences

$$\begin{aligned} c^{+,\mathrm{gen}} &: (\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G \leftarrow \mathrm{Bun}_P \times \mathrm{Bun}_G \to \mathrm{Bun}_M \times \mathrm{Bun}_G), \\ c^{-,\mathrm{gen}} &: (\mathrm{Bun}_M \times \mathrm{Bun}_G \leftarrow \mathrm{Bun}_{P^-}^{M\text{-gen}} \times \mathrm{Bun}_G \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G). \end{aligned}$$

and the 2-morphism

$$\mathrm{Id}_{(\mathrm{Bun}_M \times \mathrm{Bun}_G)} \to c^{-,\mathrm{gen}} \circ c^{+,\mathrm{gen}} \tag{4.2}$$

in $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open}}_{\mathrm{all,Stacky}}$.

Notation 4.2.4. *To simplify the notations, for a correspondence c (in english letter), we use the symbol $\mathbf{c}$ to denote the corresponding functor $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(c)$. These shorthands are only used in this subsection.*

4.2.5. *Translation.* Using the above shorthands, the results in § 4.2.3 are translated as below. The 2-morphisms (4.1) and (4.2) induce natural transformations[42]

$$\mu : \mathbf{Id} \to \mathbf{c}^- \circ \mathbf{c}^+ \text{ and } \mu^{\mathrm{gen}} : \mathbf{Id} \to \mathbf{c}^{-,\mathrm{gen}} \circ \mathbf{c}^{+,\mathrm{gen}}$$

such that the following compositions are isomorphisms

$$(\mathbf{c}^{+,\mathrm{gen}})^R \xrightarrow{\mu^{\mathrm{gen}}} \mathbf{c}^{-,\mathrm{gen}} \circ \mathbf{c}^{+,\mathrm{gen}} \circ (\mathbf{c}^{+,\mathrm{gen}})^R \xrightarrow{\mathbf{counit}} \mathbf{c}^{-,\mathrm{gen}} \tag{4.3}$$

$$(\mathbf{c}^{+})^R \xrightarrow{\mu} \mathbf{c}^{-} \circ \mathbf{c}^{+} \circ (\mathbf{c}^{+})^R \xrightarrow{\mathbf{counit}} \mathbf{c}^{-}. \tag{4.4}$$

4.2.6. Consider the map $\mathfrak{p}^{\mathrm{enh}}_{P\times G} : \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$. Let

$$p : (\mathrm{Bun}_G \times \mathrm{Bun}_G \leftarrow \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G \xrightarrow{=} \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G)$$

be the corresponding correspondence. Note that we have $\mathbf{p} \simeq \mathfrak{p}^{\mathrm{enh}}_{P\times G,!}$.

By definition, we have $c^+ \simeq p \circ c^{+,\mathrm{gen}}$, which provides

$$\mathbf{c}^+ \simeq \mathbf{p} \circ \mathbf{c}^{+,\mathrm{gen}}.$$

We proved in § 3.1 that the map $\mathrm{Bun}_{P^-}^{M\text{-gen}} \to \mathrm{Bun}_{P^-} \times_{\mathrm{Bun}_G} \mathrm{Bun}_G^{P\text{-gen}}$ is a schematic open embedding. Hence we also have a 2-morphism $c^{-,\mathrm{gen}} \to c^- \circ p$, which provides

$$\nu : \mathbf{c}^{-,\mathrm{gen}} \to \mathbf{c}^- \circ \mathbf{p}.$$

By constuction, the 2-morphism (4.1) is equivalent to the composition

$$\mathrm{Id}_{(\mathrm{Bun}_M \times \mathrm{Bun}_G)} \xrightarrow{(4.2)} c^{-,\mathrm{gen}} \circ c^{+,\mathrm{gen}} \to c^- \circ p \circ c^{+,\mathrm{gen}} \simeq c^- \circ c^+.$$

Hence μ is isomorphic to

$$\mathbf{Id} \xrightarrow{\mu^{\mathrm{gen}}} \mathbf{c}^{-,\mathrm{gen}} \circ \mathbf{c}^{+,\mathrm{gen}} \xrightarrow{\nu(\mathbf{c}^{+,\mathrm{gen}})} \mathbf{c}^- \circ \mathbf{p} \circ \mathbf{c}^{+,\mathrm{gen}} \simeq \mathbf{c}^- \circ \mathbf{c}^+. \tag{4.5}$$

Lemma 4.2.7. *The arrow γ_P is equivalent to the composition*

$$(\mathbf{c}^{-,\mathrm{gen}})(\mathcal{F}_P) \xrightarrow{\nu} \mathbf{c}^- \circ \mathbf{p}(\mathcal{F}_P) \xrightarrow{\mathbf{c}^-(\vartheta_P)} \mathbf{c}^-(\mathcal{F}_G).$$

Proof. By definition, the arrow $\theta_P : \mathcal{F}_P \to \mathbf{p}^R(\mathcal{F}_G)$ is isomorphic to

$$\mathcal{F}_P \xrightarrow{\mathbf{unit}} \mathbf{p}^R \circ \mathbf{p}(\mathcal{F}_P) \xrightarrow{\mathbf{p}^R(\vartheta_P)} \mathbf{p}^R(\mathcal{F}_G).$$

Hence by definition, γ_P is isomorphic to

$$(\mathbf{c}^{+,\mathrm{gen}})^R(\mathcal{F}_P) \xrightarrow{\mathbf{unit}} (\mathbf{c}^{+,\mathrm{gen}})^R \circ \mathbf{p}^R \circ \mathbf{p}(\mathcal{F}_P) \simeq (\mathbf{c}^+)^R \circ \mathbf{p}(\mathcal{F}_P) \xrightarrow{(\mathbf{c}^+)^R(\vartheta_P)} (\mathbf{c}^+)^R(\mathcal{F}_G).$$

[42]The functor $\mathbf{Id}$ below is the identity functor for $\mathrm{D}_{\mathrm{indhol}}(\mathrm{Bun}_M \times \mathrm{Bun}_G)$.

Hence we only need to show the following diagram of functors commute

$$\begin{tikzcd}
(\mathbf{c}^{+,\mathrm{gen}})^R \arrow[r, "\mathbf{unit}"] \arrow[d, "(4.3)"', "\simeq"] & (\mathbf{c}^{+,\mathrm{gen}})^R \circ \mathbf{p}^R \circ \mathbf{p} \arrow[r, "\simeq"] & (\mathbf{c}^+)^R \circ \mathbf{p} \arrow[d, "(4.4)"', "\simeq"] \\
\mathbf{c}^{-,\mathrm{gen}} \arrow[rr, "\nu"] & & \mathbf{c}^- \circ \mathbf{p}.
\end{tikzcd} \tag{4.6}$$

Note that we have

$$\mathrm{Maps}((\mathbf{c}^{+,\mathrm{gen}})^R, \mathbf{c}^- \circ \mathbf{p}) \simeq \mathrm{Maps}(\mathbf{Id}, \mathbf{c}^- \circ \mathbf{p} \circ \mathbf{c}^{+,\mathrm{gen}}) \simeq \mathrm{Maps}(\mathbf{Id}, \mathbf{c}^- \circ \mathbf{c}^+).$$

Via this isomorphism, the top arc in (4.6), which is a point of the LHS, is given by the following point of the RHS:

$$\mathbf{Id} \xrightarrow{\mathbf{unit}} (\mathbf{c}^{+,\mathrm{gen}})^R \circ \mathbf{c}^{+,\mathrm{gen}} \xrightarrow{\mathbf{unit}} (\mathbf{c}^{+,\mathrm{gen}})^R \circ \mathbf{p}^R \circ \mathbf{p} \circ \mathbf{c}^{+,\mathrm{gen}} \simeq (\mathbf{c}^+)^R \circ \mathbf{c}^+ \to$$
$$\xrightarrow{\mu} \mathbf{c}^- \circ \mathbf{c}^+ \circ (\mathbf{c}^+)^R \circ \mathbf{c}^+ \xrightarrow{\mathbf{counit}} \mathbf{c}^- \circ \mathbf{c}^+.$$

The first row in the above composition is just $\mathbf{unit} : \mathbf{Id} \to (\mathbf{c}^+)^R \circ \mathbf{c}^+$. Hence this composition is isomorphic to

$$\mathbf{Id} \xrightarrow{\mu} \mathbf{c}^- \circ \mathbf{c}^+ \xrightarrow{\mathbf{unit}} \mathbf{c}^- \circ \mathbf{c}^+ \circ (\mathbf{c}^+)^R \circ \mathbf{c}^+ \xrightarrow{\mathbf{counit}} \mathbf{c}^- \circ \mathbf{c}^+,$$

which is just $\mathbf{Id} \to \mathbf{c}^- \circ \mathbf{c}^+$ by the axioms for **unit** and **counit**.

Similarly, one shows that the bottom arc corresponds to natural transformation (4.5). Then we are done by the discussion above the lemma.

□[Lemma 4.2.7]

4.2.8. *Finish of the proof.* We give temporary labels to the following correspondences

$$\begin{aligned}
i &: (\overline{\mathrm{Bun}}_{G,\geq P} \leftarrow \overline{\mathrm{Bun}}_{G,P} \xrightarrow{=} \overline{\mathrm{Bun}}_{G,P}),\\
d^{\mathrm{gen}} &: (\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G \leftarrow \overline{\mathrm{Bun}}_{G,P} \xrightarrow{=} \overline{\mathrm{Bun}}_{G,P}),\\
d &: (\mathrm{Bun}_G \times \mathrm{Bun}_G \leftarrow \overline{\mathrm{Bun}}_{G,\geq P} \xrightarrow{=} \overline{\mathrm{Bun}}_{G,\geq P}),\\
j &: (\overline{\mathrm{Bun}}_{G,\geq P} \xleftarrow{=} \overline{\mathrm{Bun}}_{G,\geq P} \to \overline{\mathrm{Bun}}_G),\\
b &: (\mathrm{Bun}_M \times \mathrm{Bun}_G \leftarrow (\mathrm{Bun}_{P^-} \times_{\mathrm{Bun}_G} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \to \overline{\mathrm{Bun}}_{G,\geq P}).
\end{aligned}$$

Note that we have an obvious isomorphism $\beta \simeq b \circ j$, hence

$$\mathrm{D}^{!,*}_{\mathrm{indhol}}(\beta) \simeq \mathbf{b} \circ \mathbf{j}.$$

We have an isomorphism $p \circ d^{\mathrm{gen}} \simeq d \circ i$ because both sides are just

$$\mathrm{Bun}_G \times \mathrm{Bun}_G \leftarrow \overline{\mathrm{Bun}}_{G,P} \xrightarrow{=} \overline{\mathrm{Bun}}_{G,P}.$$

Hence $\mathbf{p} \circ \mathbf{d}^{\mathrm{gen}} \simeq \mathbf{d} \circ \mathbf{i}$. We have an isomorphism $b \circ i \simeq c^{-,\mathrm{gen}} \circ d^{\mathrm{gen}}$ because both sides are just

$$\mathrm{Bun}_M \times \mathrm{Bun}_G \leftarrow \mathrm{Bun}_{P^-}^{M\text{-gen}} \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \overline{\mathrm{Bun}}_{G,P} \to \overline{\mathrm{Bun}}_{G,P}.$$

Hence $\mathbf{b} \circ \mathbf{i} \simeq \mathbf{c}^{-,\mathrm{gen}} \circ \mathbf{d}^{\mathrm{gen}}$. We have a 2-morphism $b \to c^- \circ d$ induced by the open embedding

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \subset \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P}.$$

Hence we have a natural transformation

$$\xi : \mathbf{b} \to \mathbf{c}^- \circ \mathbf{d}.$$

Moreover, the 2-morphism

$$b \circ i \simeq c^{-,\mathrm{gen}} \circ d^{\mathrm{gen}} \to c^- \circ p \circ d^{\mathrm{gen}} \simeq c^- \circ d \circ i$$

is isomorphic to the 2-morphism induced from $b \to c^- \circ d$. Hence we have the following commutative diagram of functors

$$\begin{tikzcd} \mathbf{b}\circ\mathbf{i} \arrow[r, "\xi(\mathbf{i})"] \arrow[d, "\simeq"] & \mathbf{c}^- \circ \mathbf{d}\circ\mathbf{i} \arrow[d, "\simeq"] \\ \mathbf{c}^{-,\mathrm{gen}} \circ \mathbf{d}^{\mathrm{gen}} \arrow[r, "\nu(\mathbf{d}^{\mathrm{gen}})"] & \mathbf{c}^- \circ \mathbf{p} \circ \mathbf{d}^{\mathrm{gen}}. \end{tikzcd} \tag{4.7}$$

After these preparations, we are ready to finish the proof. Recall that $\mathbf{K}(P)$ is a !-extension along $\overline{\mathrm{Bun}}_{G,P} \to \overline{\mathrm{Bun}}_G$. Let K_1 be the corresponding object in $\mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P})$. We also write $K_2 := j^*_{\geq P}(\mathbf{K}(G))$, where $j^*_{\geq P} : \overline{\mathrm{Bun}}_{G,\geq P} \to \overline{\mathrm{Bun}}_G$ is the open embedding. The morphism $\mathbf{K}(P) \to \mathbf{K}(G)$ is sent by $\mathbf{j} = j^*_{\geq P}$ to a morphism

$$\eta : \mathbf{i}(K_1) \to K_2.$$

It follows from definition that the arrow $\vartheta_P : \mathbf{p}(\mathcal{F}_P) \to \mathcal{F}_G$ is equivalent to

$$\mathbf{p}\circ\mathbf{d}^{\mathrm{gen}}(K_1) \simeq \mathbf{d}\circ\mathbf{i}(K_1) \xrightarrow{\mathbf{d}(\eta)} \mathbf{d}(K_2),$$

where $\mathcal{F}_P \simeq \mathbf{d}^{\mathrm{gen}}(K_1)$ and $\mathcal{F}_G \simeq \mathbf{d}(K_2)$. Hence by Lemma 4.2.7, the arrow γ_P is equivalent to

$$\mathbf{c}^{-,\mathrm{gen}} \circ \mathbf{d}^{\mathrm{gen}}(K_1) \xrightarrow{\nu(\mathbf{d}^{\mathrm{gen}})} \mathbf{c}^- \circ \mathbf{p}\circ \mathbf{d}^{\mathrm{gen}}(K_1) \simeq \mathbf{c}^- \circ \mathbf{d}\circ\mathbf{i}(K_1) \xrightarrow{\mathbf{c}^-\circ\mathbf{d}(\eta)} \mathbf{c}^- \circ \mathbf{d}(K_2).$$

By (4.7), this arrow is equivalent to

$$\mathbf{b}\circ\mathbf{i}(K_1) \xrightarrow{\xi(\mathbf{i}(K_1))} \mathbf{c}^- \circ \mathbf{d}\circ\mathbf{i}(K_1) \xrightarrow{\mathbf{c}^-\circ\mathbf{d}(\eta)} \mathbf{c}^- \circ \mathbf{d}(K_2),$$

or equivalently

$$\mathbf{b}\circ\mathbf{i}(K_1) \xrightarrow{\mathbf{b}(\eta)} \mathbf{b}(K_2) \xrightarrow{\xi(K_2)} \mathbf{c}^- \circ \mathbf{d}(K_2).$$

We claim $\xi(K_2)$ is invertible. Indeed, this is because K_2 is a !-extension from the G-stratum, and the open embedding

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \subset \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P}$$

is an isomorphism when restricted to the G-stratum. Hence γ_P is equivalent to $\mathbf{b}(\eta)$, which by definition is the image of $\mathbf{K}(P) \to \mathbf{K}(G)$ under $\mathbf{b}\circ\mathbf{j} \simeq \mathrm{D}^{!,*}_{\mathrm{indhol}}(\beta)$.

□[Lemma 1.3.2]

5. Step 4

We have two results to prove in this step: Proposition-Construction 1.4.1 and Proposition 1.4.2. Each subsection corresponds to a result.

To avoid jumping between topics, we also prove Lemma 1.5.1 (from Step 5) in § 5.1.

5.1. Proof of Proposition-Construction 1.4.1 and Lemma 1.5.1.

Goal 5.1.1. *Construct a canonical factorization of the map*

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \to \mathrm{Bun}_M \times \mathrm{Bun}_G$$

via $\mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}}$ *such that we have an isomorphism*

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \underset{\mathrm{Bun}_G^{P^-\text{-gen}}}{\times} \mathrm{Bun}_P^{M\text{-gen}} \simeq Y^P_{\mathrm{rel}}/Z_M \tag{5.1}$$

defined over $\mathrm{Bun}_M \times \overline{\mathrm{Bun}}_G \times \mathrm{Bun}_M$.

The proof below is similar to that in § 2.1. Hence we omit some details.

Recall in § 4.1, we defined

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P})^{\mathrm{gen}} := \mathbf{Maps}_{\mathrm{gen}}(X, P^-\backslash \mathrm{Vin}_{G,\geq C_P}/G \leftarrow \mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P})).$$

By Lemma D.1.5, the *right* projection map $P^- \times_G \widetilde{G}_{\geq C_P} \to G$ factors through P^-. Hence we obtain the following commutative diagram of algebraic stacks

$$\begin{tikzcd} P^-\backslash \mathrm{Vin}_{G,\geq C_P}/G \arrow[d] & \mathbb{B}(P^- \times_G \widetilde{G}_{\geq C_P}) \arrow[l] \arrow[d] \\ \mathbb{B}M \times \mathbb{B}G & \mathbb{B}M \times \mathbb{B}P^- \arrow[l] \end{tikzcd}$$

Taking $\mathbf{Maps}_{\mathrm{gen}}(X,-)$, we obtain a map

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P})^{\mathrm{gen}} \to \mathrm{Bun}_M \times \mathrm{Bun}_G^{P\text{-gen}}.$$

To obtain the map $(\mathrm{Bun}_{P^-} \times_{\mathrm{Bun}_G} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \to \mathrm{Bun}_M \times \mathrm{Bun}_G^{P\text{-gen}}$, as before, we show that the map $\mathbb{B}(P^- \times_G \widetilde{G}_{\geq C_P}) \to \mathbb{B}M \times \mathbb{B}P^-$ can be uniquely lifted to a morphism

$$(Z_M \curvearrowright \mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P})) \to (\mathrm{pt} \curvearrowright \mathbb{B}M \times \mathbb{B}P^-)$$

fitting into the diagram

$$\begin{tikzcd} (Z_M \curvearrowright P^-\backslash \mathrm{Vin}_{G,\geq C_P}/G) \arrow[d] & (Z_M \curvearrowright \mathbb{B}(P^- \times_G \widetilde{G}_{\geq C_P})) \arrow[l] \arrow[d] \\ (\mathrm{pt} \curvearrowright \mathbb{B}M \times \mathbb{B}G) & (\mathrm{pt} \curvearrowright \mathbb{B}M \times \mathbb{B}P^-). \arrow[l] \end{tikzcd}$$

It remains to compare both sides of (5.1). By construction,

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P})^{\mathrm{gen}} \underset{\mathrm{Bun}_G^{P^-\text{-gen}}}{\times} \mathrm{Bun}_P^{M\text{-gen}}$$

is isomorphic to the image of

$$P^-\backslash \mathrm{Vin}_{G,\geq C_P}/P \leftarrow \mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P}) \underset{\mathbb{B}P^-}{\times} \mathbb{B}M$$

under the functor $\mathbf{Maps}_{\mathrm{gen}}(X,-)$. Using Lemma D.1.5(1), the map

$$\mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P} \underset{P^-}{\times} M) \to \mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P}) \underset{\mathbb{B}P^-}{\times} \mathbb{B}M$$

is an isomorphism. Also, the LHS is just

$$\mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P} \underset{P^-}{\times} M) \simeq \mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P} \underset{G}{\times} P) \simeq P^-\backslash \mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}/P^-.$$

Hence we obtain a Z_M-equivariant isomorphism

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P})^{\mathrm{gen}} \underset{\mathrm{Bun}_G^{P^-\text{-gen}}}{\times} \mathrm{Bun}_P^{M\text{-gen}} \simeq Y^P_{\mathrm{rel}}.$$

It follows from construction that it is defined over $\mathrm{Bun}_M \times \mathrm{VinBun}_G \times \mathrm{Bun}_M$. Then we obtain the isomorphism (5.1) by taking quotients for the Z_M-actions.

□[Proposition-Construction 1.4.1 and Lemma 1.5.1]

5.2. **Proof of Proposition 1.4.2.**

Goal 5.2.1. *The objects* $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta') \circ \mathbf{K}(P)$ *and* $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta') \circ \mathbf{K}(G)$ *are both contained in the full subcategory*

$$\mathbf{I}(M \times G, M \times P^-) \subset \mathrm{D}(\mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}}).$$

We first prove the claim for the second object. Using the base-change isomorphisms, it is easy to see $\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta') \circ \mathbf{K}(G)$ is isomorphic to the image of $k_{\mathrm{Bun}_{P^-}}$ under the !-pushforward functor along

$$\mathrm{Bun}_{P^-} \to \mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}}.$$

This map has a factorization

$$\mathrm{Bun}_{P^-} \xrightarrow{f} \mathrm{Bun}_M \times \mathrm{Bun}_{P^-} \to \mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}}.$$

It is clear that $f_!(k_{\mathrm{Bun}_{P^-}})$ is $\mathbf{U}_P^-$-equivariant, i.e., is a $*$-pullback along

$$\mathrm{Bun}_M \times \mathrm{Bun}_{P^-} \to \mathrm{Bun}_M \times \mathrm{Bun}_M .$$

Then we are done by applying Proposition 0.2.11(1) to the reductive group $M\times G$ and the parabolic subgroup $M\times P^-$.

Now we prove the claim for the first object. Consider the restriction of β on the P-stratum:

$$\beta'_P : (\mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}} \leftarrow \mathrm{Bun}_{P^-}^{M\text{-gen}} \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \overline{\mathrm{Bun}}_{G,P} \to \overline{\mathrm{Bun}}_{G,P}).$$

Since $\mathbf{K}(P)$ is a !-extension along $i_P : \overline{\mathrm{Bun}}_{G,P} \to \overline{\mathrm{Bun}}_G$,

$$\mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta') \circ \mathbf{K}(P) \simeq \mathrm{Dmod}^{!,*}_{\mathrm{indhol}}(\beta'_P) \circ i_P^*(\mathbf{K}(P)).$$

It follows from construction that β'_P is isormophic to the composition of

$$\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}} \xleftarrow{\overline{\mathbf{\Delta}}_P^{\mathrm{enh}}} \overline{\mathrm{Bun}}_{G,P} \xrightarrow{=} \overline{\mathrm{Bun}}_{G,P}$$

and

$$\delta^- : (\mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}} \leftarrow \mathrm{Bun}_{P^-}^{M\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}}),$$

where the map $\overline{\mathbf{\Delta}}_P^{\mathrm{enh}}$ is provided by Proposition-Construction 1.1.2. Hence we only need to show

$$\mathrm{D}^{!,*}_{\mathrm{indhol}}(\delta^-) \circ \overline{\mathbf{\Delta}}_{P,!}^{\mathrm{enh}} \circ i_P^*(\mathbf{K}(P)) \tag{5.2}$$

is contained in $\mathbf{I}(M\times G, M\times P^-)$. In other words, we need to show its !-pullback along

$$\iota_{M\times P^-} : \mathrm{Bun}_M \times \mathrm{Bun}_{P^-} \to \mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}}$$

is $\mathbf{U}_P^-$-equivariant.

Consider the correspondence

$$\delta^+ : (\mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}} \leftarrow \mathrm{Bun}_P \times \mathrm{Bun}_G^{P^-\text{-gen}} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}}).$$

As before, $\mathrm{D}^{!,*}_{\mathrm{indhol}}(\delta^-)$ is isomorphic to the restriction of $\mathrm{D}^{\blacktriangle,!}(\delta^+)$. Hence we can rewrite (5.2) as $\mathrm{D}^{\blacktriangle,!}(\delta^+) \circ \overline{\mathbf{\Delta}}_{P,!}^{\mathrm{enh}} \circ i_P^*(\mathbf{K}(P))$.

Consider the correspondence

$$e : (\mathrm{Bun}_M \times \mathrm{Bun}_{P^-} \leftarrow \mathrm{Bun}_P \times \mathrm{Bun}_{P^-} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}}).$$

By the base-change isomorphisms, the functor $\iota^!_{M\times P^-} \circ \mathrm{D}^{\blacktriangle,!}(\delta^+)$ is just $\mathrm{D}^{\blacktriangle,!}(e)$. Hence we only need to show $\overline{\mathbf{\Delta}}_{P,!}^{\mathrm{enh}} \circ i_P^*(\mathbf{K}(P))$ is contained in the full subcategory

$$\mathbf{I}(G\times G, P\times P^-) \subset \mathrm{D}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}}).$$

Now this can be proved similarly to that in § 2.2. Namely, one only needs to replace Lemma 2.2.5 by the following lemma, whose proof is similar.

Lemma 5.2.2. *The functor*

$$\overline{\Delta}_{P,!}^{\mathrm{enh}} : \mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P}) \to \mathrm{D}_{\mathrm{indhol}}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}})$$

sends objects contained in $\mathrm{D}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P})^{\mathbf{U}_P\times\mathbf{U}_P^-}$ *into objects contained in* $\mathbf{I}(G\times G, P\times P^-)$.

□[Proposition 1.4.2]

6. Step 5

We have two results to prove in this step: Proposition-Construction 1.5.4 and Theorem 1.5.5 (Lemma 1.5.1 was proved in § 5.1). Each subsection corresponds to a result.

6.1. Proof of Proposition-Construction 1.5.4.

Goal 6.1.1. *The correspondence*

$$\psi_{P,\geq Q}:(H_{M,G\text{-}\mathrm{pos}}/Z_M \leftarrow (Y^P_{\mathrm{rel}}/Z_M)_{\geq Q} \to \overline{\mathrm{Bun}}_{G,\geq Q})$$

is isomorphic to the composition of

$$\psi_{Q,\geq Q}:(H_{L,G\text{-}\mathrm{pos}}/Z_L \leftarrow Y^Q_{\mathrm{rel}}/Z_L \to \overline{\mathrm{Bun}}_{G,\geq Q})$$

by a certain correspondence from $H_{L,G\text{-}\mathrm{pos}}/Z_L$ to $H_{M,G\text{-}\mathrm{pos}}/Z_M$.

Let us first recall the definition of the map $Y^P_{\mathrm{rel}} \to H_{M,G\text{-}\mathrm{pos}}$.

Construction 6.1.2. *By theorem [Wan17, Theorem 4.2.10], the closed subscheme $\overline{M} \hookrightarrow \mathrm{Vin}_{G,C_P}$ is equal to $\mathfrak{s}(C_P)\cdot \mathrm{Vin}_{G,C_P}\cdot\mathfrak{s}(C_P)$, where $\mathfrak{s}: T^+_{\mathrm{ad}} \to \mathrm{Vin}_G$ is the canonical section. Hence the map*

$$\mathrm{Vin}_{G,\geq C_P} \to \mathrm{Vin}_{G,C_P},\ x \mapsto \mathfrak{s}(C_P)\cdot x\cdot \mathfrak{s}(C_P)$$

factors through $\overline{M}$. It is easy to see the obtained map $\mathrm{Vin}_{G,\geq C_P} \to \overline{M}$ intertwines the actions of $P^- \times P \to M\times M$ and is Z_M-equivariant[43]*. Moreover, the map*

$$P^-\backslash \mathrm{Vin}_{G,\geq C_P}/P \to M\backslash \overline{M}/M$$

sends $P^-\backslash \mathrm{Vin}^{\mathrm{Bruhat}}_{G,C_P}/P$ into $M\backslash M/M$. Hence we obtain a map

$$\mathfrak{q}^-_{P,\mathrm{Vin}}: Y^P_{\mathrm{rel}}/Z_M \to H_{M,G\text{-}\mathrm{pos}}/Z_M.$$

Notation 6.1.3. *Recall L is the Levi subgroup of Q. We write Z_L for the center of L. Let $P_L = P\cap L$ and $P^-_L = P^-\cap L$ be the parabolic subgroups of L corresponding to P and P^-. Let $L^{P\text{-}\mathrm{Bruhat}}$ be the open Bruhat cell $P^-_L P_L$ in L.*

Notation 6.1.4. *The projection map*

$$Y^P_{\mathrm{rel}}/Z_M \to T^+_{\mathrm{ad},\geq C_P}/Z_M \simeq T^+_{\mathrm{ad},\geq P}/T$$

induces a stratification on Y^P_{rel}/Z_M labelled by the poset $\mathrm{Par}_{\geq P}$. As usual, for $Q \in \mathrm{Par}_{\geq P}$ we use the notation:

$$(Y^P_{\mathrm{rel}}/Z_M)_{\geq Q} \coloneqq (Y^P_{\mathrm{rel}}/Z_M)\underset{T^+_{\mathrm{ad},\geq P}/T}{\times}(T^+_{\mathrm{ad},\geq Q}/T).$$

The stack

$$Y^P_{\mathrm{rel},\geq C_Q} \coloneqq Y^P_{\mathrm{rel}}\underset{T^+_{\mathrm{ad},\geq C_P}}{\times} T^+_{\mathrm{ad},\geq C_Q}$$

inherits a Z_L-action from the Z_M-action on Y^P_{rel}. Note that we have an isomorphism

$$Y^P_{\mathrm{rel},\geq C_Q}/Z_L \simeq (Y^P_{\mathrm{rel}}/Z_M)_{\geq Q}.$$

Construction 6.1.5. *By construction, we have*

$$Y^P_{\mathrm{rel},\geq C_Q} \simeq \mathbf{Maps}_{\mathrm{gen}}(X, P^-\backslash \mathrm{Vin}_{G,\geq C_Q}/P \supset P^-\backslash \mathrm{Vin}^{P\text{-}\mathrm{Bruhat}}_{G,\geq C_Q}/P),$$

where

$$\mathrm{Vin}^{P\text{-}\mathrm{Bruhat}}_{G,\geq C_Q} \coloneqq \mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}\cap \mathrm{Vin}_{G,\geq C_Q}.$$

Note that the open locus $\mathrm{Vin}^{P\text{-}\mathrm{Bruhat}}_{G,\geq C_Q}$ is contained in $\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_Q}$. Indeed, the former is the $(P^-\times P)$-orbit of the canonical section, while the latter is the $(Q^-\times Q)$-orbit. Hence the map

$$P^-\backslash \mathrm{Vin}_{G,\geq C_Q}/P \to Q^-\backslash \mathrm{Vin}_{G,\geq C_Q}/Q$$

induces a Z_L-equivariant map $Y^P_{\mathrm{rel},\geq C_Q} \to Y^Q_{\mathrm{rel}}$. Hence we obtain a map

$$\pi_{P,Q}: (Y^P_{\mathrm{rel}}/Z_M)_{\geq Q} \simeq Y^P_{\mathrm{rel},\geq C_Q}/Z_L \to Y^Q_{\mathrm{rel}}/Z_L.$$

[43] This Z_M-action on $\mathrm{Vin}_{G,\geq C_P}$ is induced by the T-action on Vin_G.

By construction, we have the following commutative diagram

$$\begin{tikzcd} Y^P_{\mathrm{rel}}/Z_M \arrow[d, "\mathfrak{p}^-_{P,\mathrm{Vin}}"] & (Y^P_{\mathrm{rel}}/Z_M)_{\geq Q} \arrow[l, "\supset"'] \arrow[r, "\pi_{P,Q}"] \arrow[d] & Y^Q_{\mathrm{rel}}/Z_L \arrow[d, "\mathfrak{p}^-_{Q,\mathrm{Vin}}"] \\ \overline{\mathrm{Bun}}_{G,\geq P} & \overline{\mathrm{Bun}}_{G,\geq Q} \arrow[l, "\supset"'] & \overline{\mathrm{Bun}}_{G,\geq Q}, \arrow[l, "="'] \end{tikzcd} \tag{6.1}$$

where the left square is Cartesian.

Proposition-Construction 6.1.6. *Consider the lft algebraic stack*[44]

$$W_{P,Q} := \mathbf{Maps}_{\mathrm{gen}}(X, P_L^-\backslash\overline{L}/P_L \supset P_L^-\backslash L^{P\text{-}\mathrm{Bruhat}}/P_L).$$

Then there exists a canonical commutative diagram

$$\begin{tikzcd} Y^P_{\mathrm{rel}}/Z_M \arrow[d, "\mathfrak{q}^-_{P,\mathrm{Vin}}"] & (Y^P_{\mathrm{rel}}/Z_M)_{\geq Q} \arrow[l, "\supset"'] \arrow[r, "\pi_{P,Q}"] \arrow[d] & Y^Q_{\mathrm{rel}}/Z_L \arrow[d, "\mathfrak{q}^-_{Q,\mathrm{Vin}}"] \\ H_{M,G\text{-}\mathrm{pos}}/Z_M & W_{P,Q}/Z_L \arrow[l] \arrow[r] & H_{L,G\text{-}\mathrm{pos}}/Z_L \end{tikzcd} \tag{6.2}$$

such that the right square in it is Cartesian.

Proof. Via the identification

$$P_L^-\backslash\overline{L}/P_L \simeq \mathbb{B}P_L^- \underset{\mathbb{B}L}{\times} (L\backslash\overline{L}/L) \underset{\mathbb{B}L}{\times} \mathbb{B}P_L,$$

the open substack $P_L^-\backslash L^{\mathrm{Bruhat}}/P_L$ of the LHS is contained in the open substack $\mathbb{B}P_L^- \times_{\mathbb{B}L} \mathbb{B}L \times_{\mathbb{B}L} \mathbb{B}P_L$ of the RHS. Hence we obtain a Z_L-equivariant schematic open embedding

$$W_{P,Q} \to \mathrm{Bun}_{P_L^-} \underset{\mathrm{Bun}_L}{\times} H_{L,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_L}{\times} \mathrm{Bun}_{P_L}.$$

In particular, we obtain a map

$$W_{P,Q}/Z_L \to H_{L,G\text{-}\mathrm{pos}}/Z_L. \tag{6.3}$$

As explained in Construction 6.1.2, the map

$$\mathrm{Vin}_{G,\geq C_Q} \to \mathrm{Vin}_{G,C_Q},\ x \mapsto \mathfrak{s}(C_Q)\cdot x\cdot \mathfrak{s}(C_Q)$$

factors through $\overline{L}$. It is easy to see the obtained map $\mathrm{Vin}_{G,\geq C_Q} \to \overline{L}$ intertwines the actions of $Q^-\times Q \to L\times L$ and is Z_L-equivariant[45]. Moreover, the map

$$P^-\backslash \mathrm{Vin}_{G,\geq C_Q}/P \to P_L^-\backslash\overline{L}/P_L$$

sends the P-Bruhat cell to the P-Bruhat cell. Hence we obtain a Z_L-linear map $Y^P_{\mathrm{rel},\geq C_Q} \to W_{P,Q}$. By taking quotient, we obtain a map

$$(Y^P_{\mathrm{rel}}/Z_M)_{\geq Q} \simeq Y^P_{\mathrm{rel},\geq C_Q}/Z_L \to W_{P,Q}/Z_L. \tag{6.4}$$

Note that we have $\mathfrak{s}(C_P)\cdot x\cdot \mathfrak{s}(C_P) = \mathfrak{s}(C_P)\cdot \mathfrak{s}(C_Q)\cdot x\cdot \mathfrak{s}(C_Q)\cdot \mathfrak{s}(C_P)$ for $x \in \mathrm{Vin}_{G,\geq C_Q}$. Hence the composition

$$\mathrm{Vin}_{G,\geq C_Q} \to \mathrm{Vin}_{G,\geq C_P} \to \overline{M}$$

factors through $\overline{L}$. Since the above composition intertwines the action of $P^-\times P \to M\times M$ and is Z_L-equivariant, the obtained map $\overline{L} \to \overline{M}$ intertwines the actions of $P_L^-\times P_L \to M\times M$ and is Z_L-equivariant. Moreover, the map

$$P_L^-\backslash\overline{L}/P_L \to M\backslash\overline{M}/M$$

sends the P-Bruhat cell into $M\backslash M/M$. Hence we obtain a map

$$W_{P,Q}/Z_L \to H_{M,G\text{-}\mathrm{pos}}/Z_L \to H_{M,G\text{-}\mathrm{pos}}/Z_M. \tag{6.5}$$

---

[44] When $Q = G$, $W_{P,G}$ is just the open Zastava stack. When $Q = P$, $W_{P,P}$ is $H_{M,G\text{-}\mathrm{pos}}$.

[45] This Z_L-action on $\mathrm{Vin}_{G,\geq C_Q}$ is induced by the T-action on Vin_G.

It follows from constructions that the above maps (6.3), (6.4) and (6.5) fit into a commutative diagram (6.2). It remains to show its right square is Cartesian. We only need to show the maps

$$\begin{aligned} P^-\backslash \mathrm{Vin}_{G,\geq C_Q}/P &\to (Q^-\backslash \mathrm{Vin}_{G,\geq C_Q}/Q) \underset{(L\backslash \overline{L}/L)}{\times} (P_L^-\backslash \overline{L}/P_L),\\ P^-\backslash \mathrm{Vin}_{G,\geq C_Q}^{P\text{-Bruhat}}/P &\to (Q^-\backslash \mathrm{Vin}_{G,\geq C_Q}^{\mathrm{Bruhat}}/Q) \underset{(L\backslash L/L)}{\times} (P_L^-\backslash L^{P\text{-Bruhat}}/P_L) \end{aligned}$$

are isomorphisms. To prove the claim for the first map, we only need to show $\mathbb{B}P \simeq \mathbb{B}Q \times_{\mathbb{B}L} \mathbb{B}P_L$, but this follows from the fact that $Q \to L$ is surjective. The claim for the second map follows from the fact that the maps

$$\begin{aligned} P^-\backslash \mathrm{Vin}_{G,\geq C_Q}^{P\text{-Bruhat}}/P &\to M\backslash M/M \times T_{\mathrm{ad},\geq C_Q},\\ Q^-\backslash \mathrm{Vin}_{G,\geq C_Q}^{\mathrm{Bruhat}}/Q &\to L\backslash L/L \times T_{\mathrm{ad},\geq C_Q},\\ P_L^-\backslash L^{P\text{-Bruhat}}/P_L &\to M\backslash M/M. \end{aligned}$$

are all isomorphisms.

□[Proposition-Construction 6.1.6]

6.1.7. *Proof of Proposition-Construction 1.5.4.* The desired correspondence is

$$H_{M,G\text{-pos}}/Z_M \leftarrow W_{P,Q}/Z_L \to H_{L,G\text{-pos}}/Z_L.$$

It satisfies the requirement because of (6.1) and (6.2).

□[Proposition-Construction 1.5.4]

6.2. **Proof of Theorem 1.5.5.**

Goal 6.2.1. *Consider the diagram*

$$\begin{array}{ccccc} H_{M,G\text{-pos}}/Z_M & \xleftarrow{\mathfrak{q}^+_{P,\mathrm{Vin}}} & {}_{\mathrm{dfstr}}\overline{\mathrm{Bun}}_{G,P} & \xrightarrow{\mathfrak{p}^+_{P,\mathrm{Vin}}} & \overline{\mathrm{Bun}}_{G,\geq P},\\ H_{M,G\text{-pos}}/Z_M & \xleftarrow{\mathfrak{q}^-_{P,\mathrm{Vin}}} & Y^P_{\mathrm{rel}}/Z_M & \xrightarrow{\mathfrak{p}^-_{P,\mathrm{Vin}}} & \overline{\mathrm{Bun}}_{G,\geq P}. \end{array}$$

Then

$$\mathfrak{q}^{\mp}_{P,\mathrm{Vin},!} \circ \mathfrak{p}^{\mp,*}_{P,\mathrm{Vin}} \simeq \mathfrak{q}^{\pm}_{P,\mathrm{Vin},*} \circ \mathfrak{p}^{\pm,!}_{P,\mathrm{Vin}}$$

on ind-holonomic objects.

The proof is similar to that of Theorem 1.2.2 hence we omit some details.

Let γ and $\overline{\gamma}$ be as in § 3.3. Using the homomorphism

$$\mathbb{G}_m \xrightarrow{\gamma} Z_M \to T_{\mathrm{ad}} \xrightarrow{t\mapsto (t^{-1},t)} T_{\mathrm{ad}} \times T_{\mathrm{ad}},$$

we obtain a $\mathbb{G}_m$-action on $G\times G$, whose attractor, repellor and fixed loci are respectively given by $P^-\times P$, $P\times P^-$ and $M\times M$.

On the other hand, consider the action

$$\mathbb{G}_m \times \mathrm{Vin}_{G,\geq C_P},\ (s,x) \mapsto \mathfrak{s}(\overline{\gamma}(s))\cdot x \cdot \mathfrak{s}(\overline{\gamma}(s)).$$

This action can actually be extended to an $\mathbb{A}^1$-action using the same formula. Hence its attractor, repellor and fixed loci are respectively given by $\mathrm{Vin}_{G,\geq C_P}$, $\overline{M}$ and $\overline{M}$. Also, the attractor, repellor and fixed loci for the restricted action on $\mathrm{Vin}_{G,\geq C_P}^{\mathrm{Bruhat}}$ are respectively given by $\mathrm{Vin}_{G,\geq C_P}^{\mathrm{Bruhat}}$, M and M.

We claim the above $\mathbb{G}_m$-actions are compatible with the action $G\times G \curvearrowright \mathrm{Vin}_{G,\geq C_P}$. Indeed, one only need to prove this claim for the restricted actions on $\mathrm{Vin}_{G,\geq C_P} \times_{T^+_{\mathrm{ad}}} T_{\mathrm{ad}}$, which can be checked directly (see Lemma 6.2.3 below). As a corollary of this claim, we obtain an action (relative to $\mathbb{A}^1$) of the Drinfeld-Gaitsgory interpolation for $G\times G$ on that for $\mathrm{Vin}_{G,\geq C_P}$.

Let $(\mathrm{ActSch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{rel}}$ be the category defined similarly as $\mathrm{ActSch}^{\mathrm{aff}}_{\mathrm{ft}}$ (see Notation 3.3.14) but we replace "algebraic groups" by "affine group schemes over an affine base scheme". In other words, its objects are $(H \curvearrowright Y)_{/S}$, where S is an affine scheme, $H \to S$ is an affine group scheme and $Y \to S$ is an affine scheme equipped with an H-action. There is an obvious $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$-action on $(\mathrm{ActSch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{rel}}$. By the previous discussion,

$$(G\times G \curvearrowright \mathrm{Vin}_{G,\geq C_P})_{/\mathrm{pt}}.$$

is a $\mathbb{G}_m$-module object. Then Example 3.2.6 provides a weakly $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$-enriched functor

$$\Theta_{(G\times G\curvearrowright \mathrm{Vin}_{G,\geq C_P})}: \mathbf{P}_{\mathbb{A}^1} \to \mathrm{Corr}((\mathrm{ActSch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{rel}})_{\mathrm{all},\mathrm{all}},$$

sending α^+ and α^- respectively to

$$\begin{array}{ccccc} (G\times G \curvearrowright \mathrm{Vin}_{G,\geq C_P}) & \leftarrow & (P^-\times P \curvearrowright \mathrm{Vin}_{G,\geq C_P}) & \to & (M\times M \curvearrowright \overline{M}), \\ (M\times M \curvearrowright \overline{M}) & \leftarrow & (P\times P^- \curvearrowright \overline{M}) & \to & (G\times G \curvearrowright \mathrm{Vin}_{G,\geq C_P}). \end{array}$$

Passing to quotients, we obtain a weakly $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$-enriched right-lax functor

$$\Theta_{(G\backslash \mathrm{Vin}_{G,\geq C_P}/G)}: \mathbf{P}_{\mathbb{A}^1} \dashrightarrow \mathbf{Corr}(\mathrm{AlgStk}_{\mathrm{lft}})^{\mathrm{all},2\text{-op}}_{\mathrm{all},\mathrm{all}}.$$

It is easy to see it is strict at the composition $\alpha^+\circ\alpha^-$. Moreover, we claim it factors through $\mathbf{Corr}(\mathrm{AlgStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{all},\mathrm{all}}$. To prove the claim, one first proves Fact 6.2.2 below, then uses it to deduce the desired claim from Lemma 3.3.8.

In the previous construction, we ignored the open Bruhat cell. If we keep tracking it, we would obtain a certain weakly $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$-enriched right-lax functor

$$\mathbf{P}_{\mathbb{A}^1} \dashrightarrow \mathbf{Corr}(\mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}}))^{\mathrm{open},2\text{-op}}_{\mathrm{all},\mathrm{all}}.$$

By taking $\mathbf{Maps}_{\mathrm{gen}}(X,-)$ for it, we obtain a weakly $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$-enriched right-lax functor

$$\Theta: \mathbf{P}_{\mathbb{A}^1} \dashrightarrow \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{all},\mathrm{all}}$$

sending α^+ and α^- respectively to

$$\begin{array}{ccccc} \mathrm{VinBun}_{G,\geq C_P} & \leftarrow & Y^P_{\mathrm{rel}} & \to & H_{M,G\text{-}\mathrm{pos}}, \\ H_{M,G\text{-}\mathrm{pos}} & \leftarrow & {}_{\mathrm{dfstr}}\mathrm{VinBun}_{G,C_P} & \to & \mathrm{VinBun}_{G,\geq C_P}. \end{array}$$

Also, Θ is strict at the composition $\alpha^+\circ\alpha^-$.

As before, we can restrict to each connected component $H^{\lambda,\mu}_{M,G\text{-}\mathrm{pos}}$ of $H^{\lambda,\mu}_{M,G\text{-}\mathrm{pos}}$ and obtain a Drinfeld pre-input

$$\Theta_{\lambda,\mu}: \mathbf{P}_{\mathbb{A}^1} \dashrightarrow \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{safe},\mathrm{safe}}.$$

In fact, the right arms of the relevant correspondences are schematic.

Also, by taking quotients for the $\mathbb{G}_m$-actions, we can obtain a Drinfeld input sending α^+ and α^- respectively to

$$\begin{array}{ccccc} \mathrm{VinBun}_{G,\geq C_P}/\mathbb{G}_m & \leftarrow & Y^{P,\lambda,\mu}_{\mathrm{rel}}/\mathbb{G}_m & \to & H^{\lambda,\mu}_{M,G\text{-}\mathrm{pos}}/\mathbb{G}_m, \\ H^{\lambda,\mu}_{M,G\text{-}\mathrm{pos}}/\mathbb{G}_m & \leftarrow & {}_{\mathrm{dfstr}}\mathrm{VinBun}^{\lambda,\mu}_{G,C_P}/\mathbb{G}_m & \to & \mathrm{VinBun}_{G,\geq C_P}/\mathbb{G}_m. \end{array}$$

By Lemma 6.2.4 below, we see that the above $\mathbb{G}_m$-action on $\mathrm{VinBun}_{G,\geq C_P}$ can be obtained from the Z_M-actions by restriction along $2\gamma: \mathbb{G}_m \to Z_M$. Hence Theorem 3.2.8 implies $\mathfrak{q}^+_{P,\mathrm{Vin},*}\circ\mathfrak{p}^{+,!}_{P,\mathrm{Vin}}$ is left adjoint to

$$\prod_{\lambda,\mu} \mathfrak{p}^{-,\lambda,\mu}_{P,\mathrm{Vin},*}\circ\mathfrak{q}^{-,\lambda,\mu,!}_{P,\mathrm{Vin}}.$$

Note that the above functor is also the right adjoint of $\mathfrak{q}^-_{P,\mathrm{Vin},!}\circ\mathfrak{p}^{-,*}_{P,\mathrm{Vin}}$. Hence we obtain

$$\mathfrak{q}^+_{P,\mathrm{Vin},*}\circ\mathfrak{p}^{+,!}_{P,\mathrm{Vin}} \simeq \mathfrak{q}^-_{P,\mathrm{Vin},!}\circ\mathfrak{p}^{-,*}_{P,\mathrm{Vin}}.$$

The equivalence $\mathfrak{q}^-_{P,\mathrm{Vin},*}\circ\mathfrak{p}^{-,!}_{P,\mathrm{Vin}} \simeq \mathfrak{q}^+_{P,\mathrm{Vin},!}\circ\mathfrak{p}^{+,*}_{P,\mathrm{Vin}}$ can be obtained by exchanging the roles of α^+ and α^-.

□[Theorem 1.5.5]

Fact 6.2.2. *For a diagram*

$$(H_1 \curvearrowright Y_1)_{/S_1} \to (H_2 \curvearrowright Y_2)_{/S_2} \leftarrow (H_3 \curvearrowright Y_3)_{/S_3}$$

in $(\mathrm{ActSch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{rel}}$*, if* H_1*,* H_2*,* H_3 *and* $H_1 \times_{H_2} H_3$ *are all flat over their base schemes, then the following square is Cartesian*

$$\begin{tikzcd}
(Y_1 \times_{Y_2} Y_3)/(H_1 \times_{H_2} H_3) \arrow[r] \arrow[d] & (Y_1/H_1) \times_{(Y_2/H_2)} (Y_3/H_3) \arrow[d] \\
\mathbb{B}(H_1 \times_{H_2} H_3) \arrow[r] & \mathbb{B}H_1 \times_{\mathbb{B}H_2} \mathbb{B}H_3.
\end{tikzcd}$$

Lemma 6.2.3. *Consider the actions*

$$T_{\mathrm{ad}} \curvearrowright \mathrm{Vin}_G, \qquad t \cdot x := \mathfrak{s}(t) \cdot x \cdot \mathfrak{s}(t),$$
$$T_{\mathrm{ad}} \curvearrowright (G \times \mathrm{Vin}_G \times G), \quad t \cdot (g_1, x, g_2) := (\mathrm{Ad}_{t^{-1}}(g_1), \mathfrak{s}(t) \cdot x \cdot \mathfrak{s}(t), \mathrm{Ad}_t(g_2)).$$

The map

$$G \times \mathrm{Vin}_G \times G \to \mathrm{Vin}_G, (g_1, x, g_2) \mapsto g_1 \cdot x \cdot g_2^{-1}$$

is equivariant for these actions.

Proof. We only need to prove the lemma after restricting to the subgroup of invertible elements in Vin_G, which is given by $G_{\mathrm{enh}} := (G \times T)/Z_G$. Then we are done by a direct calculation. (Recall that the canonical section $T/Z_G \to (G \times T)/Z_G$ is given by $t \mapsto (t^{-1}, t)$).

□[Lemma 6.2.3]

Lemma 6.2.4. *Consider the following two* T*-actions on* $G\backslash \mathrm{Vin}_G /G$*:*

(i) *The action provided by Lemma 6.2.3 via the homomorphism* $T \to T_{\mathrm{ad}}$*.*
(ii) *The one obtained from the* T*-action on* Vin_G*, which commutes with the* $(G \times G)$*-action.*

The action in (i) is isomorphic to the square of the action in (ii).

Proof. Recall that the subgroup of invertible elements in Vin_G is isomorphic to $G_{\mathrm{enh}} := (G \times T)/Z_G$. We have a short exact sequence $1 \to G \to G_{\mathrm{enh}} \to T_{\mathrm{ad}} \to 1$. The canonical section $\mathfrak{s} : T_{\mathrm{ad}}^+ \to \mathrm{Vin}_G$ provides a splitting to the above sequence. Explicitly, this splitting is given by $t \mapsto (t^{-1}, t)$. Note that the corresponding T_{ad} on G is the *inverse* of the usual adjoint action.

Consider the sequence:

$$1 \to G \times G \to G_{\mathrm{enh}} \times G_{\mathrm{enh}} \to T_{\mathrm{ad}} \times T_{\mathrm{ad}} \to 1.$$

Recall that the $(G \times G)$-action on Vin_G is defined to be the restriction of the $(G_{\mathrm{enh}} \times G_{\mathrm{enh}})$-action on Vin_G. Hence the quotient stack $G\backslash \mathrm{Vin}_G /G$ inherits a $(T_{\mathrm{ad}} \times T_{\mathrm{ad}})$-action. By the last paragraph, the action in (i) is obtained from this $(T_{\mathrm{ad}} \times T_{\mathrm{ad}})$-action by restriction along the homomorphism

$$a : T \to T_{\mathrm{ad}} \times T_{\mathrm{ad}},\ t \mapsto (t, t^{-1}). \tag{6.6}$$

On the other hand, consider the certer $Z(G_{\mathrm{enh}}) \times Z(G_{\mathrm{enh}})$ of $G_{\mathrm{enh}} \times G_{\mathrm{enh}}$. Then $G_{\mathrm{enh}} \times G_{\mathrm{enh}}$-action on Vin_G induces a $Z(G_{\mathrm{enh}}) \times Z(G_{\mathrm{enh}})$-action on $G\backslash \mathrm{Vin}_G /G$. By construction, this action factors through the homomorphism

$$q : Z(G_{\mathrm{enh}}) \times Z(G_{\mathrm{enh}}) \to Z(G_{\mathrm{enh}}),\ (s_1, s_2) \mapsto s_1 s_2^{-1}.$$

In summary, we obtain compatible actions on $G\backslash \mathrm{Vin}_G /G$ by

$$Z(G_{\mathrm{enh}}) \xleftarrow{q} Z(G_{\mathrm{enh}}) \times Z(G_{\mathrm{enh}}) \xrightarrow{p} T_{\mathrm{ad}} \times T_{\mathrm{ad}},$$

where p is the composition $Z(G_{\mathrm{enh}}) \times Z(G_{\mathrm{enh}}) \to G_{\mathrm{enh}} \times G_{\mathrm{enh}} \to T_{\mathrm{ad}} \times T_{\mathrm{ad}}$.

Recall that the homomorphism $T \to (G \times T)/Z_G, t \mapsto (1, t)$ provides an isomorphism between $T \simeq Z(G_{\mathrm{enh}})$ and the T-action on Vin_G is defined by using this identification. Hence the square of the action in (ii) can be obtained from the $Z(G_{\mathrm{enh}}) \times Z(G_{\mathrm{enh}})$-action via the homomorphism

$$T \simeq Z(G_{\mathrm{enh}}) \xrightarrow{s \mapsto (s, s^{-1})} Z(G_{\mathrm{enh}}) \times Z(G_{\mathrm{enh}})$$

(because its composition with q is the square map). Then we are done because the composition of this map by p is equal to a.

$\square$[Lemma 6.2.4]

Appendix A. Theory of D-modules

We use the theory of 2-categories of correspondences developed in [GR17, Part III] to encode the theory of D-modules. We will use two types of this theory:

- We study all the D-modules and mainly work with the right (or standard) functors, i.e., !-pullbacks and $*$-pushforwards. See § A.1.1.
- We study ind-holonomic D-modules and mainly work with the left functors, i.e. $*$-pullbacks and !-pushforwards. See § A.1.2.

A.1.1. *Standard functors.* Consider the $(3,2)$-category

$$\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open}}_{\mathrm{QCAD,all}}$$

defined as follows:

- Its objects are lft prestacks;
- The $(2,1)$-category $\mathbf{Maps}_{\mathbf{Corr}}(Y_1,Y_2)$ is the 1-full subcategory of $(\mathrm{PreStk}_{\mathrm{lft}})_{/Y_1\times Y_2}$ where:
  - we restrict to those objects $Y_2 \leftarrow Z \to Y_1$ such that $Z \to Y_2$ is *QCAD* [46];
  - we restrict to those morphisms $Z_1 \to Z_2$ in $(\mathrm{PreStk}_{\mathrm{lft}})_{/Y_1\times Y_2}$ that are schematic open embeddings;
- the composition functor

$$\mathbf{Maps}_{\mathbf{Corr}}(Y_1,Y_2)\times\mathbf{Maps}_{\mathbf{Corr}}(Y_2,Y_3)\to\mathbf{Maps}_{\mathbf{Corr}}(Y_1,Y_3)$$

  sends $Y_2 \leftarrow U \to Y_1$ and $Y_3 \leftarrow V \to Y_2$ to

$$Y_3 \leftarrow V \underset{Y_2}{\times} U \to Y_1.$$

There exists[47] a canonical functor

$$\mathrm{Dmod}^{\blacktriangle,!}:\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD,all}}\to\mathbf{DGCat}_{\mathrm{cont}}; \tag{A.1}$$

$$Y\mapsto \mathrm{D}(Y),\ (Y_2 \xleftarrow{f} Z \xrightarrow{g} Y_1)\mapsto (f_\blacktriangle\circ g^!:\mathrm{D}(Y_1)\to\mathrm{D}(Y_2)).$$

The content of the claim is:

- for any lft prestack Y, there is a DG-category $\mathrm{D}(Y)$;
- for any morphism $f:Y_1\to Y_2$, there is a *!-pullback functor* $f^!$;
- for any QCAD morphism $f:Y_1\to Y_2$, there is a *renormalized pushforward functor* $f_\blacktriangle$ defined in [DG13];
- there are base-change isomorphisms for these !-pullback and $\blacktriangle$-pushforward functors;
- for any schematic open embedding $f:Y_1\to Y_2$, there is an adjoint pair $(f^!,f_\blacktriangle)$;
- there are certain higher compatibilities for the above data.

As shown in *loc.cit.*, for a *safe* map[48] $f:Y_1\to Y_2$, the renormalized pushforward functor $f_\blacktriangle$ can be identified with the usual de-Rham pushforward functor f_*. Therefore we keep the notation f_* and only use $f_\blacktriangle$ for non-safe map f.

---

[46] This means for any finite type affine test scheme $S\to Y_2$, the base-change $S\times_{Y_2} Z$ is a quasi-compact algebraic stack with affine diagonal map. Note that this condition is closed under compositions and base-changes. This condition is slightly stronger than that for *QCA maps* defined in [DG13].

[47] This claim was made in [DG13, Remark 9.3.13] even for QCA maps. A detailed construction is provided in the author's thesis, see [Che21, Construction C.2.13].

[48] A QCAD map f is safe if the automorphism groups in Footnote 46 are unipotent. For example, the map $\mathbb{B}B\to\mathbb{B}T$ is safe, while $\mathbb{B}T\to\mathrm{pt}$ is not safe.

A.1.2. *Holonomic D-modules.* For any finite type affine scheme $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, we write $\mathrm{D}_{\mathrm{indhol}}(S)$ for the full subcategory of $\mathrm{D}(S)$ generated by holonomic objects (under extensions and colimits).

For any lft prestack Y, we write $\mathrm{D}_{\mathrm{indhol}}(Y)$ for the full subcategory of $\mathrm{D}(Y)$ containing objects $\mathcal{F}$ such that $f^!(\mathcal{F}) \in \mathrm{D}_{\mathrm{indhol}}(S)$ for any map $f : S \to Y$ with $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. Equivalently, we define

$$\mathrm{D}_{\mathrm{indhol}}(Y) \coloneqq \lim_{S \in (\mathrm{AffSch}_{\mathrm{ft}})_{/Y}} \mathrm{D}_{\mathrm{indhol}}(S),$$

with the connecting functors given by !-pullbacks. An object in $\mathrm{D}_{\mathrm{indhol}}(Y)$ is called an *ind-holonomic* object in $\mathrm{D}(Y)$.

The following results are either well-known or formal[49]:

(1) For any map $f : Y_1 \to Y_2$ between lft prestacks, the functor $f^!$ preserves ind-holonomic objects. Also, the partially defined left adjoint $f_!$ of $f^!$ is well-defined on $\mathrm{D}_{\mathrm{indhol}}(Y_1)$ and sends it into $\mathrm{D}_{\mathrm{indhol}}(Y_2)$. Hence we have a functor

$$\mathrm{D}_{\mathrm{indhol}} : \mathrm{PreStk}_{\mathrm{lft}} \to \mathrm{DGCat} \tag{A.2}$$

sending morphisms to !-pushforward functors.

(2) For any lft prestack Y, we have an equivalence

$$\mathrm{D}_{\mathrm{indhol}}(Y) \coloneqq \operatorname*{colim}_{S \in (\mathrm{AffSch}_{\mathrm{ft}})_{/Y}} \mathrm{D}_{\mathrm{indhol}}(S),$$

with the connecting functors given by !-pushforwards. In particular, $\mathrm{D}_{\mathrm{indhol}}(Y)$ is compactly generated by objects of the form $g_!(\mathcal{F})$, where $g : S \to Y$ is an object in $(\mathrm{AffSch}_{\mathrm{ft}})_{/Y}$ and $\mathcal{F}$ is a compact object in $\mathrm{D}_{\mathrm{indhol}}(S)$.

(3) For any quasi-compact schematic map $f : Y_1 \to Y_2$ between lft prestacks, the functor f_* preserves ind-holonomic objects. Also, the partially defined left adjoint f^* of f_* is well defined on $\mathrm{D}_{\mathrm{indhol}}(Y_2)$ and sends it into $\mathrm{D}_{\mathrm{indhol}}(Y_1)$.

(4) For any lft *algebraic stacks*, there is an equivalence

$$\mathrm{D}_{\mathrm{indhol}}(Y) \coloneqq \lim_{S \in (\mathrm{AffSch}_{\mathrm{ft}})_{/Y}} \mathrm{D}_{\mathrm{indhol}}(S),$$

with the connecting functors given by $*$-pullbacks. This is implicit in [DG13, § 6.2.1-6.2.2].

(5) For any map $f : Y_1 \to Y_2$ between lft algebraic stacks, there is a functor

$$f^* : \mathrm{D}_{\mathrm{indhol}}(Y_2) \to \mathrm{D}_{\mathrm{indhol}}(Y_1)$$

uniquely characterized by its compatibility with (4). Moreover, there exists a functor

$$\begin{aligned} \mathrm{Dmod}^{!,*}_{\mathrm{indhol}} : \mathrm{Corr}(\mathrm{AlgStk}_{\mathrm{lft}})_{\mathrm{all,all}} \to \mathrm{DGCat}_{\mathrm{cont}},\ & Y \mapsto \mathrm{D}_{\mathrm{indhol}}(Y), \\ (Y_2 \xleftarrow{f} Z \xrightarrow{g} Y_1) \mapsto (f_! \circ g^* : \mathrm{D}_{\mathrm{indhol}}&(Y_1) \to \mathrm{D}_{\mathrm{indhol}}(Y_2)). \end{aligned} \tag{A.3}$$

We also have its $(\infty,2)$-categorical enrichment

$$\mathrm{Dmod}^{!,*}_{\mathrm{indhol}} : \mathbf{Corr}(\mathrm{AlgStk}_{\mathrm{lft}})^{\mathrm{open}}_{\mathrm{all,all}} \to \mathbf{DGCat}_{\mathrm{cont}} \tag{A.4}$$

obtained by using the "no cost" extension in [GR17, Chapter 7, § 4].

(6) For any stacky map $f : Y_1 \to Y_2$ between lft prestacks. (2) and (5) implies there is a functor $f^* : \mathrm{D}_{\mathrm{indhol}}(Y_2) \to \mathrm{D}_{\mathrm{indhol}}(Y_1)$ equipped with base-change isomorphisms against !-pushforwards. In fact, by left Kan extension along

$$\mathrm{Corr}(\mathrm{AlgStk}_{\mathrm{lft}})_{\mathrm{all,all}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{all,Stacky}},$$

we obtain from (A.3) a functor

$$\mathrm{Dmod}^{!,*}_{\mathrm{indhol}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{all,Stacky}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

[49]More details are provided in the author's thesis, see [Che21, § C.5].

It follows from (2) that its restriction on $\mathrm{PreStk}_{\mathrm{lft}} \simeq \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{all,iso}}$ can be identified with (A.2). We also have its "no cost" extension

$$\mathrm{Dmod}^{!,*}_{\mathrm{indhol}} : \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open}}_{\mathrm{all,Stacky}} \to \mathbf{DGCat}_{\mathrm{cont}} \tag{A.5}$$

Appendix B. Well-definedness results in [Gai15]

B.1.1. *Proof of Proposition 0.2.7.* Let $\widetilde{\mathrm{Bun}}_P$ be the Drinfeld's compactification constructed in [BG02]. Recall it is defined as

$$\widetilde{\mathrm{Bun}}_P := \mathbf{Maps}_{\mathrm{gen}}(G\backslash\overline{G/U}/M \supset G\backslash(G/U)/M),$$

where $\overline{G/U}$ is the affine closure of G/U. By [Bar14, Remark 4.1.9], the map $\mathrm{Bun}_P \to \mathrm{Bun}_G^{P\text{-gen}}$ factors as

$$\mathrm{Bun}_P \xrightarrow{j} \widetilde{\mathrm{Bun}}_P \xrightarrow{\widetilde{\iota}_P} \mathrm{Bun}_G^{P\text{-gen}},$$

and the restriction of the map $\widetilde{\iota}_P$ on each connected component of $\widetilde{\mathrm{Bun}}_P$ is proper. Also, the map $\widetilde{\iota}_P$ is obtained by applying $\mathbf{Maps}_{\mathrm{gen}}(X,-)$ to the morphism

$$(G\backslash\overline{G/U}/M \supset G\backslash(G/U)/M) \to (\mathbb{B}G \leftarrow \mathbb{B}P).$$

The above properness implies $\widetilde{\iota}_{P,!}$ is well-defined. On the other hand, it was proved in [DG16, § 1.1.6] that the composition

$$\mathrm{D}(\widetilde{\mathrm{Bun}}_P) \xrightarrow{j^!} \mathrm{D}(\mathrm{Bun}_P) \xrightarrow{\mathfrak{q}_{P,*}} \mathrm{D}(\mathrm{Bun}_M)$$

has a left adjoint isomorphic to

$$j_! \circ \mathfrak{q}_P^*(-) \simeq j_!(k_{\mathrm{Bun}_P}) \overset{!}{\otimes} \mathfrak{q}_P^!(-)[\mathrm{shift}], \tag{B.1}$$

where [shift] is a cohomological shift locally constant on Bun_M. Combining the above two results, we obtain the well-definedness of $\iota_{P,!} \circ \mathfrak{q}_P^*$.

To prove the second claim, we need to calculate $\iota_P^! \circ \iota_{P,!} \circ \mathfrak{q}_P^*$. Consider the diagram

$$\begin{tikzcd} \mathrm{Bun}_P \times_{\mathrm{Bun}_G^{P\text{-gen}}} \widetilde{\mathrm{Bun}}_P \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & \mathrm{Bun}_P \arrow[d, "\iota_P"] \\ \widetilde{\mathrm{Bun}}_P \arrow[r, "\widetilde{\iota}_P"] & \mathrm{Bun}_G^{P\text{-gen}}. \end{tikzcd}$$

By the base-change isomorphism, we have

$$\iota_P^! \circ \widetilde{\iota}_{P,!} \simeq \mathrm{pr}_{1,!} \circ \mathrm{pr}_2^!.$$

A direct calculation shows

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \widetilde{\mathrm{Bun}}_P \simeq \mathbf{Maps}_{\mathrm{gen}}(X, P\backslash\overline{G/U}/M \leftarrow P\backslash(P/U)/M).$$

Let $\overline{M}$ be the closure of P/U in $\overline{G/U}$, then we have

$$\mathbf{Maps}_{\mathrm{gen}}(X, P\backslash\overline{G/U}/M \leftarrow P\backslash(P/U)/M) \simeq \mathbf{Maps}_{\mathrm{gen}}(X, P\backslash\overline{M}/M \leftarrow P\backslash(P/U)/M).$$

Now the RHS is isomorphic to $\mathrm{Bun}_P \times_{\mathrm{Bun}_M} H_{M,G\text{-pos}}$, where

$$H_{M,G\text{-pos}} := \mathbf{Maps}_{\mathrm{gen}}(X, M\backslash\overline{M}/M \supset M\backslash M/M)$$

is the G-positive Hecke stack for M-tosors (see [Sch16, § 3.1.5]). Recall that the map

$$i : \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-pos}} \to \widetilde{\mathrm{Bun}}_P$$

is bijective on geometric points, and the connected components of the source provide a stratification on $\widetilde{\mathrm{Bun}}_P$ (known as the *defect stratification*).

We obtain

$$\iota_P^! \circ \iota_{P,!} \circ \mathfrak{q}_P^* \simeq \mathrm{pr}_{1,!} \circ i^! \circ j_! \circ \mathfrak{q}_P^*.$$

Hence it remains to show the functor $i^! \circ j_! \circ \mathfrak{q}_P^*$ factors through

$$\mathrm{D}(H_{M,G\text{-pos}}) \xrightarrow{*\text{-pull}} \mathrm{D}(\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-pos}}). \tag{B.2}$$

By (B.1), we only need to show $i^! \circ j_!(k_{\mathrm{Bun}_P})$ is contained in the image of (B.2). However, this is well-known and can be proved using the Hecke actions in [BG02, § 6.2].

□[Proposition 0.2.7]

B.1.2. *Proof of Proposition 0.2.11.* Let M (resp. L) be the Levi quotient group of P (resp. Q). Let P_L be the image of P in L, which is a parabolic subgroup of L. Consider the correspondence

$$\mathrm{Bun}_L \leftarrow \mathrm{Bun}_{P_L} \rightarrow \mathrm{Bun}_M$$

and the corresponding geometric Eisenstein series functor

$$\mathrm{Eis}_{P_L,!} : \mathrm{D}(\mathrm{Bun}_M) \rightarrow \mathrm{D}(\mathrm{Bun}_L)$$

defined in [BG02]. Recall that it is defined as the $*$-pull-!-push along the above correspondence.

It is easy to check the composition[50] of the correspondences

$$\begin{array}{ccccc} \mathrm{Bun}_L & \leftarrow & \mathrm{Bun}_{P_L} & \rightarrow & \mathrm{Bun}_M, \\ \mathrm{Bun}_G^{Q\text{-gen}} & \xleftarrow{\iota_Q} & \mathrm{Bun}_Q & \xrightarrow{\mathfrak{q}_Q} & \mathrm{Bun}_L \end{array}$$

is isomorphic to the composition of the correpondences

$$\begin{array}{ccccc} \mathrm{Bun}_G^{P\text{-gen}} & \xleftarrow{\iota_P} & \mathrm{Bun}_P & \xrightarrow{\mathfrak{q}_P} & \mathrm{Bun}_M, \\ \mathrm{Bun}_G^{Q\text{-gen}} & \xleftarrow{\mathfrak{p}^{\mathrm{enh}}_{P\to Q}} & \mathrm{Bun}_G^{P\text{-gen}} & \xrightarrow{=} & \mathrm{Bun}_G^{P\text{-gen}} \end{array}$$

Hence by the base-change isomorphisms, we have

$$\mathfrak{p}^{\mathrm{enh}}_{P\to Q,!} \circ \iota_{P,!} \circ \mathfrak{q}_P^* \simeq \iota_{Q,!} \circ \mathfrak{q}_Q^* \circ \mathrm{Eis}_{P_L,!}\,. \tag{B.3}$$

In particular, the LHS is well-defined. Hence by Remark 0.2.9, $\mathfrak{p}^{\mathrm{enh}}_{P\to Q,!}$ is well-defined. This proves (1).

To prove (2), since $\mathbf{I}(G,P)$ is compactly generated (see Remark 0.2.9), we only need to prove $\mathrm{Eis}^{\mathrm{enh}}_{P\to Q}$ preserve compact objects. By Remark 0.2.9 again, it suffices to prove $\mathrm{Eis}^{\mathrm{enh}}_{P\to Q} \circ \iota_{M,!}$ preserves compact objects. By (B.3), we have

$$\mathrm{Eis}^{\mathrm{enh}}_{P\to Q} \circ \iota_{M,!} \simeq \iota_{L,!} \circ \mathrm{Eis}_{P_L,!}\,.$$

Then we are done because both $\iota_{L,!}$ and $\mathrm{Eis}_{P_L,!}$ preserves compact objects.

□[Proposition 0.2.11]

Appendix C. D-modules on stacks stratified by power sets

We begin with the following definition.

Definition C.1.1. *Let Y be an algebraic stack and I be a finite set. A* stratification of Y labelled by the power poset $P(I)$ *is an assignment of open substacks $U_i \subset Y$ for any $i \in I$.*

The above definition coincides with the usual one in the literature because of the following construction.

Construction C.1.2. *Let Y be as in Definition C.1.1. For any object $J \in P(I)$, we define $i_J : Y_J \to Y$ to be the reduced locally closed substack of Y given by*

$$\Big(\bigcup_{j\in J} U_j\Big) - \Big(\bigcup_{i\notin J} U_i\Big).$$

We call Y_J the stratum labelled by J. *Note that every geometric point of Y is contained in exactly one stratum.*

For any object $J \in P(I)$, there is a unique open substack $Y_{\geq J} \subset Y$ whose geometric points are exactly those contained in $\bigcup_{K\supset J} Y_K$. Similarly, we define the reduced closed substack $Y_{\leq J}$.

[50] See Appendix A for the definition of compositions of correspondences.

Note that the stratum Y_I is an open substack of Y. Hence we also write $j_I := i_I$ for this open embedding.

Also note that U_i can be recovered as $Y_{\geq\{i\}}$.

Example C.1.3. Let Y be a finite type scheme and $\{f_i\}_{i\in I}$ be regular functions on Y. Then we obtain a stratification of Y labelled by $P(I)$ with U_i given by the non-vanishing locus of f_i. In particular, the coordinate functions induce a stratification of the affine space $\mathbb{A}^I$ labelled by $P(I)$. This stratification is known as the *coordinate stratification*.

Remark C.1.4. Since the theory of D-modules is insensitive to non-reduced structures, in this paper, we also use the notations Y_J and $Y_{\leq J}$ for certain infinitesimal thickening of the stacks defined above. For example, if we have a map $Y \to Z$ and a stratification of Z labelled by $P(I)$, then we obtain a stratification of Y labelled by $P(I)$ by pulling back the open substacks. We often write $Y_J := Y \times_Z Z_J$ although it is not necessarily reduced.

Definition C.1.5. *Let Y be an algebraic stack stratified by a power poset $P(I)$. We define*

$$\mathrm{Funct}(P(I), \mathrm{D}_{\mathrm{indhol}}(Y))_! \subset \mathrm{Funct}(P(I), \mathrm{D}_{\mathrm{indhol}}(Y))$$

to be the full subcategory consisting of those functors $F : P(I) \to \mathrm{D}(Y)$ such that $F(J)$ is !-extended from the stratum Y_J.

Lemma C.1.6. *Let Y be an algebraic stack stratified by a power poset $P(I)$. The functor*

$$\mathbf{C}_Y : \mathrm{Funct}(P(I), \mathrm{D}_{\mathrm{indhol}}(Y))_! \to \mathrm{D}_{\mathrm{indhol}}(Y),$$
$$F \mapsto \mathrm{coFib}(\operatorname*{colim}_{J\subsetneq I} F(J) \to F(I))$$

is an equivalence. Also, its inverse sends an object $\mathcal{F} \in \mathrm{D}_{\mathrm{indhol}}(Y)$ to a certain functor

$$P(I) \to \mathrm{D}_{\mathrm{indhol}}(Y),\ J \mapsto i_{J,!} \circ i_J^*(\mathcal{F})[|J| - |I|].$$

Proof. First note that the second claim follows from the first one because

$$i_K^*(\mathrm{coFib}(\operatorname*{colim}_{J\subsetneq I} F(J) \to F(I))) \simeq i_K^*(F(K))[|I| - |K|].$$

It remains to show $\mathbf{C}_Y$ is an equivalence. The case $I = \{*\}$ is well-known. The general case can be proved by induction as follows. Suppose $I = I^\flat \bigsqcup \{a\}$ and I is nonempty. Note that apart from the embedding $P(I^\flat) \subset P(I)$, we also have a map

$$P(I^\flat) \to P(I),\ J \mapsto J^\sharp := J \bigsqcup \{a\}.$$

The open substacks $\{U_i\}_{i\in I^\flat}$ provide a stratification of Y labelled by $P(I^\flat)$. We use the notation Z to denote the same stack Y equipped with this new stratification. For any $K \in P(I^\flat)$, the stratum Z_K inherits a stratification by $P(\{a\})$, whose big stratum is isomorphic to $Y_{K^\sharp}$ and small stratum is isomorphic to Y_K. Consider the functor

$$A : \mathrm{Funct}(P(I), \mathrm{D}_{\mathrm{indhol}}(Y))_! \to \mathrm{Funct}(P(I^\flat), \mathrm{D}_{\mathrm{indhol}}(Z))_!, F \mapsto A(F),$$

where $A(F)(K) := \mathrm{coFib}(F(K) \to F(K^\sharp))$. Note that this is well-defined, i.e. $A(F)(K)$ is indeed a !-extension from Z_K. Moreover, A is an equivalence by the $I = \{*\}$ case of the lemma (applying to each Z_K).

Hence by induction hypothesis, $\mathbf{C}_Z \circ A$ is also an equivalence. It remains to show $\mathbf{C}_Y \simeq \mathbf{C}_Z \circ A$. Note that we have the following pushout diagram

$$\begin{tikzcd} \operatorname{colim}_{K\subsetneq I^\flat} F(K) \arrow[r] \arrow[d] & \operatorname{colim}_{J\subsetneq I, J\neq I^\flat} F(J) \arrow[d] \\ \operatorname{colim}_{K\subset I^\flat} F(K) \arrow[r] & \operatorname{colim}_{J\subsetneq I} F(J), \end{tikzcd}$$

which is obtained by writing the simplicial nerve of $P(I)-\{I\}$ as a pushout. By cofinality, the above diagram is equivalent to

$$\begin{tikzcd} \operatorname{colim}_{K\subsetneq I^\flat} F(K) \arrow[r] \arrow[d] & \operatorname{colim}_{K\subsetneq I^\flat} F(K^\sharp) \arrow[d] \\ F(I^\flat) \arrow[r] & \operatorname{colim}_{J\subsetneq I} F(J). \end{tikzcd}$$

Then we have

$$\begin{aligned} & \operatorname{coFib}(\operatorname*{colim}_{J\subsetneq I} F(J) \to F(I)) \\ \simeq\ & \operatorname{coFib}(\operatorname{coFib}(\operatorname*{colim}_{K\subsetneq I^\flat} F(K) \to \operatorname*{colim}_{K\subsetneq I^\flat} F(K^\sharp)) \to \operatorname{coFib}(F(I^\flat) \to F(I))) \\ \simeq\ & \operatorname{coFib}(\operatorname*{colim}_{K\subsetneq I^\flat} (\operatorname{coFib}(F(K) \to F(K^\sharp))) \to \operatorname{coFib}(F(I^\flat) \to F(I))) \\ \simeq\ & \operatorname{coFib}(\operatorname*{colim}_{K\subsetneq I^\flat} A(F)(K) \to A(F)(I^\flat)) \end{aligned}$$

as desired. This proves the claim.

□[Lemma C.1.6]

The above lemma implies

Corollary C.1.7. *Let Y be an algebraic stack stratified by a power poset $P(I)$. The functor*

$$\mathbf{J}_Y : \operatorname{Funct}(P(I), \mathrm{D}_{\mathrm{indhol}}(Y))_! \to \mathrm{D}_{\mathrm{indhol}}(Y_I),\ F \mapsto j_I^* \circ F(I)$$

has a right adjoint sending an object $\mathcal{F} \in \mathrm{D}_{\mathrm{indhol}}(Y_I)$ to a certain functor

$$\mathbf{G}^*_{\mathcal{F},Y} : P(I) \to \mathrm{D}_{\mathrm{indhol}}(Y),\ J \mapsto i_{J,!} \circ i_J^* \circ j_{I,*}(\mathcal{F})[|J| - |I|].$$

Proof. Follows from the fact that $\mathbf{J}_Y \simeq j_I^* \circ \mathbf{C}_Y$.

□[Corollary C.1.7]

Remark C.1.8. Note that the functor $\mathbf{G}^*_{\mathcal{F},Y}$ sends the arrow $J \subset I$ to a morphism

$$i_{J,!} \circ i_J^* \circ j_{I,*}(\mathcal{F})[|J| - |I|] \to i_{I,!}(\mathcal{F}).$$

Applying $i_J^!$ to this map, we obtain a map

$$i_J^* \circ j_{I,*}(\mathcal{F})[|J| - |I|] \to i_J^! \circ i_{I,!}(\mathcal{F}).$$

Note that this map is invertible if $|J| = |I| - 1$, but not for general J.

Lemma C.1.9. *Let Y be an algebraic stack stratified by a power poset $P(I)$ and $J \in P(I)$. Consider the maps*

$$Y_I \xrightarrow{j_{I,\geq J}} Y_{\geq J} \xrightarrow{j_{\geq J}} Y.$$

For any $\mathcal{F} \in \mathrm{D}_{\mathrm{indhol}}(Y_I)$, we have

$$\operatorname{coFib}(\operatorname*{colim}_{J\subset K\subsetneq I} \mathbf{G}^*_{\mathcal{F},Y}(K) \to \mathbf{G}^*_{\mathcal{F},Y}(I)) \simeq (j_{\geq J})_! \circ (j_{I,\geq J})_*(\mathcal{F}).$$

Proof. The case $J = I$ follows from definition. Indeed, the LHS is given by

$$\mathbf{C}_Y \circ (\mathbf{J}_Y)^R \simeq \mathbf{C}_Y \circ (j_I^* \circ \mathbf{C}_Y)^R \simeq (j_I^*)^R \simeq j_{I,*}.$$

In the general case, note that both sides are contained in the image of the functor $(j_{\geq J})_!$. Hence we only need to show

$$\operatorname{coFib}(\operatorname*{colim}_{J\subset K\subsetneq I} j_{\geq J}^* \circ \mathbf{G}^*_{\mathcal{F},Y}(K) \to j_{\geq J}^* \circ \mathbf{G}^*_{\mathcal{F},Y}(I)) \simeq (j_{I,\geq J})_*(\mathcal{F}).$$

Consider the open substack $Y_{\geq J}$. It inherits a stratification by the poset $P(I-J)$ with $(Y_{\geq J})_K \simeq Y_{J\sqcup K}$. Hence we also have a functor

$$\mathbf{G}^*_{\mathcal{F},Y_{\geq J}} : P(I-J) \to \mathrm{D}_{\mathrm{indhol}}(Y_{\geq J}).$$

It follows from construction that this functor is isomorphic to

$$P(I-J) \xrightarrow{-\sqcup J} P(I) \xrightarrow{\mathbf{G}^*_{\mathcal{F},Y}} \mathrm{D}_{\mathrm{indhol}}(Y) \xrightarrow{j^*_{\geq J}} \mathrm{D}_{\mathrm{indhol}}(Y_{\geq J}).$$

Hence we only need to show

$$\mathrm{coFib}(\operatorname*{colim}_{K \subsetneq I-J} \mathbf{G}^*_{\mathcal{F},Y_{\geq J}}(K) \to \mathbf{G}^*_{\mathcal{F},Y_{\geq J}}(I-J)) \simeq (j_{I,\geq J})_*(\mathcal{F}).$$

In other words, we have reduced the lemma to the case $J = I$.

□[Lemma C.1.9]

Appendix D. The group scheme $\widetilde{G}$

Consider the $(G\times G)$-action on Vin_G. Note that it preserves the fibers of $\mathrm{Vin}_G \to T^+_{\mathrm{ad}}$. We write $\widetilde{G}$ for the corresponding stabilizer of the canonical section $\mathfrak{s} : T^+_{\mathrm{ad}} \to \mathrm{Vin}_G$. In this appendix, we review some results about $\widetilde{G}$.

We begin by reviewing some facts:

Fact D.1.1. *We have the following facts*[51]*:*

(1) *$\widetilde{G}$ is a closed subgroup of $G\times G\times T^+_{\mathrm{ad}}$ (relative to T^+_{ad}), whose fiber at C_P is*
$$\widetilde{G}_{C_P} \simeq P \underset{M}{\times} P^-.$$

(2) *By [DG16, Corollary D.5.4], $\widetilde{G}$ is smooth over T^+_{ad}, and we have*
$$ {}_0\mathrm{Vin}_G \simeq (G\times G\times T^+_{\mathrm{ad}})/\widetilde{G},\ G\backslash {}_0\mathrm{Vin}_G/G \simeq \mathbb{B}\widetilde{G}, \tag{D.1}$$
where $\mathbb{B}$ means taking relative classifying stack.

(3) *By (2), the T-action on Vin_G (which commutes with the $G\times G$-action) induces a diagram between group actions:*
$$(\mathrm{pt} \curvearrowright \mathbb{B}G\times\mathbb{B}G) \leftarrow (T \curvearrowright \mathbb{B}\widetilde{G}) \to (T \curvearrowright T^+_{\mathrm{ad}}).$$

(4) *$\widetilde{G}$ contains the locally closed subscheme*
$$\Gamma : G\times T_{\mathrm{ad}} \to G\times G\times T^+_{\mathrm{ad}},\ (g,t)\mapsto (g,\mathrm{Ad}_t(g),t)$$

(5) *$\widetilde{G}$ is preserved by the action*
$$(T_{\mathrm{ad}}\times T_{\mathrm{ad}}) \curvearrowright (G\times G\times T^+_{\mathrm{ad}}),\ (t_1,t_2)\cdot(g_1,g_2,s)\mapsto (\mathrm{Ad}_{t_1^{-1}}(g_1),\mathrm{Ad}_{t_2^{-1}}(g_2),t_1\cdot s\cdot t_2^{-1}).$$

Warning D.1.2. *The T-action on Vin_G does not induce a T-action on $\widetilde{G}$ because this action does not preserve the canonical section $\mathfrak{s} : T^+_{\mathrm{ad}} \to \widetilde{G}$.*

The following result generalizes [DG16, Proposition D.6.4]:

Lemma D.1.3. *$\widetilde{G}$ is isomorphic to the closure of the locally closed embedding*
$$\Gamma : G\times T_{\mathrm{ad}} \to G\times G\times T^+_{\mathrm{ad}},\ (g,t)\mapsto (g,\mathrm{Ad}_t(g),t).$$

Proof. Let $\overline{\Gamma}$ be the desired closure. Hence we obtain a closed embedding $\overline{\Gamma}\to\widetilde{G}$. Since $\widetilde{G}$ is reduced, it remains to show $\overline{\Gamma}\to\widetilde{G}$ is surjective. Note that $\overline{\Gamma}$ is also preserved by the action in Fact D.1.1(5). Hence we only need to check the fiber of $\overline{\Gamma}\to\widetilde{G}$ at $C_P\in T^+_{\mathrm{ad}}$ is surjective. Then we are done by [DG16, Proposition D.6.4].

□[Lemma D.1.3]

Lemma D.1.4. *The closed subscheme*
$$\widetilde{G}_{\leq P} \coloneqq \widetilde{G} \underset{T^+_{\mathrm{ad}}}{\times} T^+_{\mathrm{ad},\leq P} \hookrightarrow G\times G\times T^+_{\mathrm{ad},\leq P}$$
is contained in $P\times P^-\times T^+_{\mathrm{ad},\leq P}$.

[51] (1) and (2) are well-known, (4) and (5) follow from the identification $\mathrm{Vin}_G\times_{T^+_{\mathrm{ad}}} T_{\mathrm{ad}} \simeq (G\times T)/Z_G$.

Proof. Using the action in Fact D.1.1(5), we only need to show $\widetilde{G}_{C_Q}$ is contained in $P\times P^-$ for any $Q\subset P$. But this is obvious.

□[Lemma D.1.4]

Lemma D.1.5. *We write* $\widetilde{G}_{\geq C_P} := \widetilde{G}\times_{T^+_{\mathrm{ad}}} T^+_{\mathrm{ad},\geq C_P}$. *We have:*

(1) *The closed subscheme*

$$P^-\underset{G}{\times}\widetilde{G}_{\geq C_P}\hookrightarrow P^-\times G\times T^+_{\mathrm{ad},\geq C_P}$$

is contained in $P^-\times P^-\times T^+_{\mathrm{ad},\geq C_P}$.

(2) *The composition*

$$P^-\underset{G}{\times}\widetilde{G}\underset{T^+_{\mathrm{ad}}}{\times}T^+_{\mathrm{ad},\geq C_P}\to P^-\times P^-\times T^+_{\mathrm{ad},\geq C_P}\xrightarrow{\mathrm{pr}_{23}} P^-\times T^+_{\mathrm{ad},\geq C_P} \tag{D.2}$$

is an isomorphism, where the first map is obtained by (1).

Warning D.1.6. *The similar statement for* pr_{13} *is false.*

Proof. We first prove (1). Using the action in Fact D.1.1(5), we only need to check the similar claim at any $C_{P'}\in T^+_{\mathrm{ad},\geq C_P}$. But this is obvious.

Similarly, it is easy to see (D.2) induces isomorphisms between fibers at any closed point of $T^+_{\mathrm{ad},\geq C_P}$. To prove (2), we only need to show $P^-\times_G\widetilde{G}\times_{T^+_{\mathrm{ad}}}T^+_{\mathrm{ad},\geq C_P}$ is smooth over $T^+_{\mathrm{ad},\geq C_P}$.

We claim $P^-\times G\times T^+_{\mathrm{ad},\geq C_P}$ and $\widetilde{G}_{\geq C_P}$ are transversal in $G\times G\times T^+_{\mathrm{ad},\geq C_P}$. Indeed, by the last paragraph, the dimension of any irreducible component of their intersection is at most $\dim(P^-)+\dim(T^+_{\mathrm{ad},\geq C_P})$. But this number is equal to

$$\dim(P^-\times G\times T^+_{\mathrm{ad},\geq C_P})+\dim(\widetilde{G}_{\geq C_P})-\dim(G\times G\times T^+_{\mathrm{ad},\geq C_P}).$$

This proves the transversity. In particular, we obtain that $P^-\times_G\widetilde{G}\times_{T^+_{\mathrm{ad}}}T^+_{\mathrm{ad},\geq C_P}$ is smooth.

It remains to show $f:P^-\times_G\widetilde{G}\times_{T^+_{\mathrm{ad}}}T^+_{\mathrm{ad},\geq C_P}\to T^+_{\mathrm{ad},\geq C_P}$ induces surjections between tangent spaces. Note that the fibers of this map is smooth and of dimension $\dim(P^-)$. Hence at any closed point x of the source, we have

$$\dim(\ker(df_x))=\dim(P^-)=\dim(P^-\underset{G}{\times}\widetilde{G}\underset{T^+_{\mathrm{ad}}}{\times}T^+_{\mathrm{ad},\geq C_P})-\dim(T^+_{\mathrm{ad},\geq C_P}).$$

This implies df_x is surjective.

□[Lemma D.1.5]

Lemma D.1.7. *Consider the* $(P^-\times G)$*-action on* $\mathrm{Vin}_{G,\geq C_P}$. *Its stablizer for the canonical section is* $P^-\times_G\widetilde{G}_{\geq C_P}$. *Then the map*

$$(P^-\times G\times T^+_{\mathrm{ad},\geq C_P})/(P^-\underset{G}{\times}\widetilde{G}_{\geq C_P})\to {}_0\mathrm{Vin}_{G,\geq C_P} \tag{D.3}$$

induced by this action is an open embedding.

Proof. We claim the LHS is a smooth scheme. By Lemma D.1.5, there is an isomorphism

$$P^-\underset{G}{\times}\widetilde{G}_{\geq C_P}\simeq P^-\times T^+_{\mathrm{ad},\geq C_P} \tag{D.4}$$

between group schemes over $T^+_{\mathrm{ad},\geq C_P}$. Moreover, the projection map $P^-\times G\times T^+_{\mathrm{ad},\geq C_P}\to G\times T^+_{\mathrm{ad},\geq C_P}$ intertwines the actions of (D.4). Hence we obtain a map

$$(P^-\times G\times T^+_{\mathrm{ad},\geq C_P})/(P^-\underset{G}{\times}\widetilde{G}_{\geq C_P})\to(G\times T^+_{\mathrm{ad},\geq C_P})/(P^-\times T^+_{\mathrm{ad},\geq C_P})\simeq G/P^-\times T^+_{\mathrm{ad},\geq C_P}.$$

Since $P^-\times G\times T^+_{\mathrm{ad},\geq C_P}\to G\times T^+_{\mathrm{ad},\geq C_P}$ is affine and smooth, the above map is also affine and smooth. This proves the claim on smoothness. Then the lemma follows from the fact that both sides of (D.3) have the same dimenison and that this map is injective on the level of closed points.

□[Lemma D.1.7]

Corollary D.1.8. *The map*

$$\mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P}) \to \mathbb{B}P^- \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}_{\geq C_P}$$

is a schematic open embedding

Proof. Follows from Lemma D.1.7 by taking quotients for the $(P^- \times G)$-actions. □[Corollary D.1.8]

References

[Bar14] Jonathan Barlev. D-modules on spaces of rational maps. *Compositio Mathematica*, 150(5):835–876, 2014.

[BBK18] Joseph Bernstein, Roman Bezrukavnikov, and David Kazhdan. Deligne–lusztig duality and wonderful compactification. *Selecta Mathematica*, 24(1):7–20, 2018.

[Ber19] Dario Beraldo. Deligne-lusztig duality on the stack of local systems. *arXiv preprint arXiv:1906.00934*, 2019.

[BG02] A Braverman and D Gaitsgory. Geometric eisenstein series. *Inventiones mathematicae*, 2(150):287–384, 2002.

[Bra03] Tom Braden. Hyperbolic localization of intersection cohomology. *Transformation groups*, 8(3):209–216, 2003.

[BYD18] Roman Bezrukavnikov and Alexander Yom Din. On parabolic restriction of perverse sheaves. *arXiv preprint arXiv:1810.03297*, 2018.

[Che21] Lin Chen. Nearby cycles and dualities in geometric langlands program. *PhD Thesis*, 2021.

[DG13] Vladimir Drinfeld and Dennis Gaitsgory. On some finiteness questions for algebraic stacks. *Geometric and Functional Analysis*, 23(1):149–294, 2013.

[DG14] Vladimir Drinfeld and Dennis Gaitsgory. On a theorem of braden. *Transformation groups*, 19(2):313–358, 2014.

[DG15] V Drinfeld and D Gaitsgory. Compact generation of the category of d-modules on the stack of g-bundles on a curve. *Cambridge Journal of Mathematics*, 3(1):19–125, 2015.

[DG16] Vladimir Drinfeld and Dennis Gaitsgory. Geometric constant term functor (s). *Selecta Mathematica*, 22(4):1881–1951, 2016.

[DL82] Pierre Deligne and George Lusztig. Duality for representations of a reductive group over a finite field. *Journal of Algebra*, 74(1):284–291, 1982.

[Dri13] Vladimir Drinfeld. On algebraic spaces with an action of g_m. *arXiv preprint arXiv:1308.2604*, 2013.

[DW16] Vladimir Drinfeld and Jonathan Wang. On a strange invariant bilinear form on the space of automorphic forms. *Selecta Mathematica*, 22(4):1825–1880, 2016.

[FKM20] Michael Finkelberg, Vasily Krylov, and Ivan Mirković. Drinfeld–gaitsgory–vinberg interpolation grassmannian and geometric satake equivalence. *Journal of Topology*, 13(2):683–729, 2020.

[Gai11] Dennis Gaitsgory. What acts on geometric eisenstein series, 2011.

[Gai15] Dennis Gaitsgory. Outline of the proof of the geometric langlands conjecture for gl (2). *Astérisque*, 2015.

[Gai17] Dennis Gaitsgory. A "strange" functional equation for eisenstein series and miraculous duality on the moduli stack of bundles. In *Annales Scientifiques de l'École Normale Supérieure*. Societe Mathematique de France, 2017.

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry*, volume 1. American Mathematical Soc., 2017.

[Lur09] Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009.

[Lur12] Jacob Lurie. Higher algebra, 2012.

[Sch16] Simon Schieder. Geometric bernstein asymptotics and the drinfeld-lafforgue-vinberg degeneration for arbitrary reductive groups. *arXiv preprint arXiv:1607.00586*, 2016.

[Wan17] Jonathan Wang. On the reductive monoid associated to a parabolic subgroup. *Journal of Lie Theory*, 27(3):637–655, 2017.

[Wan18] Jonathan Wang. On an invariant bilinear form on the space of automorphic forms via asymptotics. *Duke Mathematical Journal*, 167(16):2965–3057, 2018.

[YD19] Alexander Yom Din. On the deligne–lusztig involution for character sheaves. *Selecta Mathematica*, 25(3):49, 2019.