PERIODIC APPROXIMATIONS OF IRRATIONAL PSEUDO-ROTATIONS USING PSEUDEHOLOMORPHIC CURVES

BARNEY BRAMHAM

Abstract. We prove that every $C^\infty$-smooth, area preserving diffeomorphism of the closed 2-disk having not more than one periodic point is the uniform limit of periodic $C^\infty$-smooth diffeomorphisms. In particular every smooth irrational pseudo-rotation can be $C^0$-approximated by integrable systems. This partially answers a long standing question of A. Katok regarding zero entropy Hamiltonian systems in low dimensions. Our approach uses pseudoholomorphic curve techniques from symplectic geometry.

1. Introduction

1.1. The main result. In this paper we prove a statement which is a significant first step towards answering the following general question of Katok.

In low dimensions is every conservative dynamical system with zero topological entropy a limit of integrable systems?

This is stated as problem 1 in [27], but relates also to the much older [1]. Low dimensions means maps on surfaces or flows on 3-dimensional manifolds. Arguably one of the main obstacles to answering this question affirmatively is the existence of ergodic components of positive measure. Indeed, ergodic maps (with respect to Lebesgue measure) exhibit strongly different dynamical behavior to integrable ones. In particular, almost every point is the initial condition for a dense orbit.
Using finite energy foliations by pseudoholomorphic curves we obtain a result of this nature for the class of area preserving diffeomorphisms of the 2-disk known as irrational pseudo-rotations. Irrational pseudo-rotations are of particular interest with regard to this question because they include all known ergodic disk maps with zero topological entropy.

For each \( t \in \mathbb{R} \) let \( R_{2\pi t} : D \to D \) denote the rigid rotation through angle \( 2\pi t \) on the disk \( D = \{(x, y) \mid x^2 + y^2 \leq 1\} \). Let \( \text{Diff}_\infty^+ (D) \) and \( \text{Diff}_\infty^+ (D, \omega_0) \) be the spaces of \( C^\infty \)-smooth diffeomorphisms of the disk which preserve orientation and the area form \( \omega_0 = dx \wedge dy \) respectively.

**Theorem 1** (Main result). Suppose \( \varphi \in \text{Diff}_\infty^+ (D, \omega_0) \) fixes the origin and has no other periodic points. Then it is the \( C^0 \)-limit of a sequence of maps of the form \( \varphi_k = g_k^{-1} R_{2\pi p_k/q_k} g_k \), for a sequence of conjugating maps \( g_k \in \text{Diff}_\infty^+ (D) \) which fix the origin, and a sequence of rationals \( p_k/q_k \) converging to an irrational number.

Elements \( \varphi \in \text{Diff}_\infty^+ (D, \omega_0) \) having precisely one periodic point are also known as (smooth) **irrational pseudo-rotations**. See definition 3.

Concerning the convergence of the approximation maps in theorem 1 it is natural to ask whether \( C^0 \)-convergence is so weak as to allow “almost anything” to be obtained in the limit? A more natural topology to consider Katok’s question, as discussed in [27], is at least a \( C^{1,\varepsilon} \)-topology, \( \varepsilon > 0 \), in which the topological entropy is lower semi-continuous. The author would therefore like to thank Patrice Le Calvez for pointing out the following and the idea of its proof using work of Franks. A slightly more general statement is proven in appendix A.1.

**Proposition 2.** Suppose \( \varphi \in \text{Diff}_\infty^+ (D, \omega_0) \) is the \( C^0 \)-limit of a sequence of maps of the form \( \varphi_k = g_k^{-1} R_{2\pi p_k/q_k} g_k \), for a sequence of conjugating maps \( g_k \in \text{Diff}_\infty^+ (D) \) fixing the origin, and a sequence of rationals \( p_k/q_k \) converging to an irrational number. Then \( \varphi \) necessarily has precisely one periodic point. In particular it is an irrational pseudo-rotation.

Thus \( C^0 \)-convergence is still strong enough to guarantee that the limit object is an irrational pseudo-rotation. In particular that it has zero entropy. (Due, for example, to Katok’s theorem for \( C^{1,\varepsilon} \) surface diffeomorphisms [26] that bounds entropy from above by the exponential growth rate of periodic points.)

The existence of ergodic disk maps with zero entropy was established back in 1970 by Anosov and Katok [11]. Previous to their constructions it was even an open question in the non-conservative setting: Shnirelman 1930 [33] found a (non-area preserving) diffeomorphism of the disk with a dense orbit, further discussion of which can be found in [10].

It is interesting to compare the Anosov-Katok construction to the statement of theorem 1. They construct an ergodic map \( \varphi : D \to D \) as the \( C^\infty \)-limit of a sequence of maps \( \varphi_k : D \to D \) which are inductively constructed with the following form. For each \( k \in \mathbb{N} \) there exists \((p_k, q_k) \in \mathbb{Z} \times \mathbb{N}\)
relatively prime, and \( g_k \in \text{Diff}^\infty(D, \omega_0) \), also fixing the origin, so that

\[
\varphi_k = g_k^{-1} \circ R_{2\pi p_k/q_k} \circ g_k.
\]

The maps \( g_k \) are arranged so that the orbits of \( \varphi_k \) increasingly spread out over the disk as \( k \to \infty \). Consequently, the sequence \( \{g_k\} \) blows up in every \( C^r \) topology. But by interatively choosing \( q_k+1 - q_k \) sufficiently large depending on the size of \( \|g_k\|_{C^k} \), the \( C^k \) norm of \( \varphi_k \) can be controlled. A limiting subsequence converges to a map \( \varphi \) with the desired “pathological” behavior such as a dense orbit, or ergodicity, or even weak mixing. More details of this method, other results and questions, are in Fayad-Katok [10]. See also Fayad-Saprykina [11].

In some sense then, theorem 1 reverses the limiting process just described above. However our conclusions are in two respects weaker than a word for word converse to the Anosov-Katok construction. Firstly, the convergence \( \varphi_k \to \varphi \) in (1) is in the \( C^{\infty} \)-sense. Secondly, each \( \varphi_k \) in (1) preserves the standard area form. We do not show this for the approximation maps in theorem 1. This raises natural questions for further investigation.

A remark on integrability: The notion of integrability for a map on a surface that appears to be referred to in [27] is that the map should admit a “first integral”, that is, a continuous or smooth real valued function on the surface that is not constant on any open set but is constant on the orbits of the given map. It is obviously in this sense that each of our approximation maps in theorem 1 is integrable. A natural question is whether approximation maps can be found that are integrable in the Liouville-Arnold sense. This would follow if they were area preserving.

1.2. Idea of the proof. Pick a closed loop of Hamiltonians \( H_t : D \to D \), over \( t \in \mathbb{R}/\mathbb{Z} \), which generate a symplectic isotopy whose time-one map is \( \varphi \). Denote the 1-periodic path of Hamiltonian vector fields on the disk by \( X_{H_t} \). For each \( n \in \mathbb{N} \) equip \( Z_n = \mathbb{R}/n\mathbb{Z} \times D \) with coordinates \((\tau, z)\). Then the vector field \( R_n(\tau,z) = \partial_\tau + X_{H_t}(z) \) defines a flow on \( Z_n \) with time-one map \( \varphi \) and first return map \( \varphi^n \).

Consider the 4-manifold \( W_n := \mathbb{R} \times Z_n \) with the unique almost complex structure satisfying

\[
\begin{cases}
J_n \partial_{\mathbb{R}} = R_n \\
J_n|_{TD} = i
\end{cases}
\]

where \( \partial_{\mathbb{R}} \) is the vector field dual to the \( \mathbb{R} \)-coordinate on \( W_n \). Then \( (W_n, J_n) \) is a so called cylindrical, symmetric, almost complex manifold. That is, it is compatible in a precise way with the necessary symplectic structures for the compactness framework from symplectic field theory [4] to apply to \( J_n \)-holomorphic curves.

In section 7 we adapt techniques developed by Hofer, Wysocki, and Zehnder [20, 21, 22, 24, 25] to construct finite energy foliations of \( (W_n, J_n) \). That is, a foliation \( \mathcal{F} \) by embedded surfaces that are the images of finite energy
$J_n$-holomorphic curves, and where the set $\mathcal{F}$ is invariant under translations in the $\mathbb{R}$-direction on $W_n$.

![Figure 1. A leaf $F$ in a finite energy foliation $\mathcal{F}$ of $W_n = \mathbb{R} \times Z_n$ intersects the hypersurface $\{0\} \times Z_n$ in a closed 1-dimensional curve. We use this to define a disk map $\varphi_F$.](image)

In this setting a finite energy foliation $\mathcal{F}$ of $(W_n, J_n)$ can be used to define a disk map $\varphi_F : D \to D$ in the following fairly natural way: identify the mapping torus $Z_n$ with the hypersurface $\{0\} \times Z_n \subset W_n$. Then each leaf $F \in \mathcal{F}$ is either disjoint from $Z_n$ or intersects it transversally in an embedded closed curve as in figure 1. The closed curve will then intersect each disk slice $\{\tau\} \times D$ in $Z_n$ transversally and precisely once (each such disk is also $J_n$-holomorphic). If the curve intersects the disk $\{0\} \times D$ at a point $(0, \xi)$ and intersects the disk $\{1\} \times D$ at a point $(1, \xi')$, then we set $\varphi_F(\xi) = \xi'$.

For each $n \in \mathbb{N}$ there is also a “trivial” $\mathbb{R}$-invariant foliation of $(W_n, J_n)$ by $J_n$-holomorphic curves we could refer to as the vertical foliation $\mathcal{F}^{\text{vert}}(W_n, J_n)$. The leaves of the vertical foliation are all of the form $\mathbb{R} \times \gamma(\mathbb{R})$ for each trajectory $\gamma : \mathbb{R} \to Z_n$ of the vector field $R_n$. Usually this foliation is of little interest due to its “instability” and each leaf besides one has infinite so called $\lambda$-energy. However if we use the vertical foliation to define a disk map as above, we obtain the map $\varphi$ of interest. The leaves of the vertical foliation are characterized by having vanishing so called $\omega$-energy.

It turns out that there exists a sequence of finite energy foliations $\mathcal{F}_n$ of $(W_n, J_n)$ with the following property: each leaf in $\mathcal{F}_n$ has $\omega$-energy zero or $\{n\alpha\} \in [0, 1)$ where $\alpha \in \mathbb{R}$ can be identified with the rotation number of the circle map $\varphi|_{\partial D}$. In particular $\alpha$ is irrational. $\{x\}$ denotes the fractional part of $x$. 
Restricting to a subsequence \( n_j \) for which \( \{n_j \alpha\} \to 0 \) as \( j \to \infty \), the leaves in \( \mathcal{F}_{n_j} \) converge in some sense to the vertical foliation \( \mathcal{F}_{\text{vert}} \). Correspondingly the disk maps \( \varphi_{\mathcal{F}_{n_j}} \) converge pointwise to the disk map \( \varphi_{\mathcal{F}_{\text{vert}}} = \varphi \).

By being slightly more careful one obtains uniform convergence. Finally, each foliation \( \mathcal{F}_n \) has a certain symmetry that ensures that the induced map \( \varphi_{\mathcal{F}_n} \) has \( n \)-th iterate \( (\varphi_{\mathcal{F}_n})^n = \text{id}_D \). This is equivalent to \( \varphi_{\mathcal{F}_n} \) being conjugate to a rigid rotation through an angle \( 2\pi p/n \) for some \( p \in \{0, 1, \ldots, n-1\} \).

Remark 1. We only use that \( \varphi \) is an irrational pseudo-rotation at two points in this argument. First, to find the nice expression for the \( \omega \)-energy of the leaves. Secondly to show that the maps \( \varphi_{\mathcal{F}_n} \) are roots of unity. Indeed the finite energy foliations themselves exist for generic area preserving disk maps, not just pseudo-rotations. This will be shown in [5].

1.3. Results in the literature of related interest. Using quite different techniques, Le Calvez proved in 2004, see Theorem 1.9 in [30], that every minimal \( C^1 \)-diffeomorphism of the 2-torus that is homotopic to the identity can be \( C^0 \)-approximated by periodic \( C^1 \)-diffeomorphisms. Recall that a diffeomorphism is minimal if every point is the initial condition for a dense orbit. Thus, in this result also strongly non-integrable maps are approximated by, in some sense, integrable ones.

An interesting result about irrational pseudo-rotations in the class of homeomorphisms of the open and closed annulus homotopic to the identity, was obtained by Béguin-Crovisier-LeRoux-Patou [2] and Béguin-Crovisier-LeRoux [3]. Stated for maps on the closed disk this is as follows. Let \( \varphi \) be an orientation preserving, measure preserving, homeomorphism of the disk with a single periodic point and boundary rotation number \( \alpha \in \mathbb{R}/\mathbb{Z} \). Then the rigid rotation \( R_{2\pi \alpha} \) is the \( C^0 \)-limit of maps (not necessarily area preserving) conjugate to \( \varphi \). The authors of these papers note that one does not know from their approach that \( \varphi \) is in the closure of the set of maps conjugate to \( R_{2\pi \alpha} \).

1.4. Acknowledgements. I would particularly like to thank Helmut Hofer for his interest, encouragement, and many valuable discussions and suggestions. Also for constructive comments on an earlier version of this paper. I thank Patrice LeCalvez for pointing out that proposition 2 should hold (also as stated in appendix A.1) and for explaining the idea of the proof, and Richard Siefring and Chris Wendl for many helpful discussions about pseudoholomorphic curves. I also wish to thank Anatole Katok for his interest.

This work is based upon work supported by the National Science Foundation under agreement No. DMS-0635607. Any opinions, findings and conclusions or recommendations in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.
2. IRRATIONAL PSEUDO-ROTATIONS

Definition 3. A (smooth) irrational, pseudo-rotation is a $C^\infty$-diffeomorphism $\varphi : D \to D$ of the closed 2-disk $D$ with the following properties: (1) $\varphi$ preserves the volume form $dx \wedge dy$. (2) $\varphi(0) = 0$. (3) $\varphi$ has no periodic points in $D \setminus \{0\}$.

There are equivalent definitions which admit generalizations to rational pseudo-rotations which we will not need. See for example [31] and [2].

If $\varphi : D \to D$ is an irrational pseudo-rotation, then the restriction of $\varphi$ to the boundary is an orientation preserving circle diffeomorphism without periodic points. It therefore has irrational rotation number on the boundary.

More precisely, let $\pi : \mathbb{R} \to \partial D$ be the projection map $x \mapsto e^{2\pi ix}$. Then for any lift $f : \mathbb{R} \to \mathbb{R}$ of $\varphi|_{\partial D}$ via $\pi$, the limit

$$(2) \quad \tau(f) := \lim_{n \to \infty} \frac{f^n(x) - x}{n} \in \mathbb{R}$$

exists and is independent of $x$, see for example [28], and is called the translation number of $f$. Furthermore, the element $[\tau(f)] \in \mathbb{R}/\mathbb{Z}$ in the quotient space, is even independent of the choice of lift $f$, and is called the rotation number of $\varphi|_{\partial D}$.

Definition 4. Let $\varphi : D \to D$ be an irrational pseudo-rotation. Then we define the rotation number of $\varphi$ to be the value on the circle

$$\text{Rot}(\varphi) := [\tau(f)] \in \mathbb{R}/\mathbb{Z}$$

for any lift $f : \mathbb{R} \to \mathbb{R}$ of the restriction $\varphi : \partial D \to \partial D$.

A preferred homotopy $\{\varphi_t\}_{t \in [0,1]}$ from $\varphi_0 = \text{id}_{\partial D}$ to $\varphi_1 = \varphi|_{\partial D}$ gives us a preferred lift of $\text{id}_{\partial D}$. Namely the terminal map of the unique lift to a homotopy in the universal covering space which begins at $\text{id}_\mathbb{R}$. In particular, any Hamiltonian generating $\varphi$ as its time-one map, restricts to a homotopy on the boundary of the disk from $\text{id}_{\partial D}$ to $\varphi|_{\partial D}$ and thus determines a canonical lift of the latter. Using this we define:

Definition 5. Let $\varphi : D \to D$ be an irrational pseudo-rotation. Let $H_t \in C^\infty(D,\mathbb{R})$ be a path of Hamiltonians on $(D,\omega_0 = dx \wedge dy)$ generating $\varphi$ as its time-one map. Then we define the rotation number of $\varphi$ with respect to $H$ to be the real number

$$\text{Rot}(\varphi; H) := \tau(f) \in \mathbb{R}$$

where $f : \mathbb{R} \to \mathbb{R}$ is the canonical lift of $\varphi : \partial D \to \partial D$ determined by $H$.

If $\varphi$ is an irrational pseudo-rotation, then the unique periodic point is non-degenerate in the sense that for all $n \in \mathbb{N}$, the linearization $D\varphi^{(n)}(0)$ does not have eigenvalue 1. The proof is a well known application of the Poincaré-Birkhoff fixed point theorem. See appendix A.2.
3. Finite energy foliations

For any \( H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R}) \) with \( H_t := H(t, \cdot) \) constant on the boundary of \( D \) for each \( t \in \mathbb{R}/\mathbb{Z} \), the smooth time-dependent vector field \( X_H(t, \cdot) := X_{H_t} \) on \( D \) defined by

\[
\omega_0(X_{H_t}(z), \cdot) = -dH_t(z)
\]

for all \( z = (x, y) \in D \) is tangent to \( \partial D \) and therefore generates a 1-parameter family of diffeomorphisms \( \phi^t : D \to D \) over \( t \in \mathbb{R} \). Using that the disk is simply connected it is well known that one may find an \( H \) for any element \( \varphi \in \text{Diff}^\infty(D, \omega_0) \) so that \( \varphi = \phi^1 \). Then \( H \) is said to generate \( \varphi \).

From now on let \( \varphi : D \to D \) be a fixed irrational pseudo-rotation. Unless stated otherwise \( H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R}) \) is a 1-periodic time-dependent Hamiltonian generating \( \varphi \).

Remark 2. By precomposing \( H \) with a suitable closed loop in \( \text{Diff}^\infty(D, \omega_0) \) based at the identity, we may assume that the unique 1-periodic orbit of \( X_{H_t} \) corresponding to the fixed point \( 0 \in D \) of \( \varphi \) is the constant trajectory \( t \mapsto 0 \in D \) for all \( t \in \mathbb{R} \). This is not necessary, but makes the proof of theorem slightly easier.

Define a smooth vector field \( R_H \) on the solid torus \( Z := \mathbb{R}/\mathbb{Z} \times D \) by

\[
R_H(\tau, z) := \partial_\tau + X_H(\tau, z)
\]

for all \( \tau \in \mathbb{R}/\mathbb{Z}, z \in D \). The first return map on \( \{0\} \times D \) is canonically identified with the pseudo-rotation \( \varphi \).

For each \( n \in \mathbb{N} \) let \( Z_n \) be the 3-manifold-with-boundary \( \mathbb{R}/n\mathbb{Z} \times D \), and \( R_n \) the vector field on \( Z_n \) that projects down to \( R_H \) under the natural projection \( \pi_n : Z_n \to Z \). Clearly the first return map of the flow generated by \( R_n \) is the \( n \)-th iterate \( \varphi^n : D \to D \). We will refer to the pair \((Z_n, R_n)\) as the mapping torus of length-\( n \) associated to \( H \). It will also be useful to denote by \( Z_\infty := \mathbb{R} \times D \) the universal covering of each \( Z_n \).

All the dynamical information on \((Z_n, R_n)\) can be captured by an almost complex structure on the 4-manifold \( \mathbb{R} \times Z_n \) as follows. For each \( n \in \mathbb{N} \cup \{\infty\} \) define \( J_n \) on \( \mathbb{R} \times Z_n \) by

\[
\begin{align*}
J_n(a, \tau, z)\partial_{\mathbb{R}} &= R_n \\
J_n(a, \tau, z)|_{T_z D} &= i
\end{align*}
\]

for all \((a, \tau, z) \in \mathbb{R} \times Z_n\). Here, \( \partial_{\mathbb{R}} \) is the vector field dual to the \( \mathbb{R} \)-coordinate on \( \mathbb{R} \times Z_n \), and \( i \) denotes the constant almost complex structure on the disk coming from the standard integrable complex structure on \( \mathbb{C} \). In other words \( i\partial_x = \partial_y \) and \( i\partial_y = -\partial_x \). Observe that \( J_n \) is independent of the \( \mathbb{R} \)-coordinate on \( \mathbb{R} \times Z_n \), referred to as \( \mathbb{R} \)-invariance. This idea of coupling

*For convenience we use this “non-generic” almost complex structure, although it is not necessary. The pseudoholomorphhic curves we encounter are either orbit cylinders or embedded with genus zero, one boundary component, and Fredholm index 2. Such curves are automatically regular.
a suitable conservative vector field in an odd-dimensional manifold with the $\mathbb{R}$-direction in the product 4-manifold by an almost complex structure is due to Hofer [17].

There is a 1-parameter family of 2-tori

$$L_c := \{ c \} \times \partial Z_n$$

for $c \in \mathbb{R}$, that fill the boundary $\mathbb{R} \times \partial Z_n$ of the 4-manifold. Each $L_n$ is totally real with respect to the almost complex structure $J_n$, that is

$$TL_c \oplus J_n T(L_c)$$

is the full 4-dimensional tangent space at each point of $L_c$. These will form the boundary conditions for our pseudoholomorphic curves with boundary.

Let us describe the $J_n$-holomorphic half infinite cylinders with totally real boundary conditions that we are interested in. Let $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^- = (-\infty, 0)$. For $n \in \mathbb{N}$ we consider maps $\bar{u} = (a, \tau, z) \in C^\infty(\mathbb{R}^\pm \times \mathbb{R}^n Z, \mathbb{R} \times Z_n = \mathbb{R} \times \mathbb{R}^n Z \times D)$ for which there exists $c \in \mathbb{R}$ such that

$$\begin{cases} 
\partial_s \bar{u}(s, t) + J_n(\bar{u}(s, t)) \partial_t \bar{u}(s, t) = 0 & \text{for all } (s, t) \in \mathbb{R}^\pm \times \mathbb{R}^n Z \\
\bar{u}(0, t) \in L_c & \text{for all } t \in \mathbb{R}^n Z \\
\tau(0, \cdot) : \mathbb{R}^n Z \to \mathbb{R}^n Z & \text{has degree } +1,
\end{cases}$$

having so called finite total energy, which we define in a moment.

This setting is a special case of that described in [4]. In particular $(\mathbb{R} \times Z_n, J_n)$ is a cylindrical symmetric almost complex manifold, and the almost complex structure $J_n$ is compatible with the stable Hamiltonian structure $(\omega_n, \lambda_n)$ on $Z_n$ given by

$$\omega_n = dx \wedge dy + d\tau \wedge dH$$

in coordinates $(\tau, (x, y))$ on $\mathbb{R}/\mathbb{R}^n Z \times D$. Recall that this means that $\lambda_n \wedge \omega_n > 0$ and $\ker(\omega_n) \subset \ker(d\lambda_n)$, see for example [4] or [8]. The compactness theory in [4] leads us to consider the following two quantities for a solution to (5) which we will refer to as the $\omega$-energy, the $\lambda$-energy, and the sum of them as the total energy. In our context the $\lambda$-energy of a solution $\bar{u} = (a, \tau, z) \in \mathbb{R} \times \mathbb{R}/\mathbb{R}^n Z \times D$ to (5) is the quantity

$$E_\lambda(\bar{u}) := \sup_{\psi \in \mathcal{C}} \int_{\mathbb{R}^+ \times \mathbb{R}/\mathbb{R}^n Z} \bar{u}^* \left( \psi(a) da \wedge d\tau \right) \in [0, +\infty]$$

where $\mathcal{C}$ is the set of smooth functions $\psi : \mathbb{R} \to [0, \infty)$ for which $\int_{\mathbb{R}} \psi(s) ds = 1$. The second energy, that which in the more general context of [4] is called the $\omega$-energy, is

$$E_\omega(\bar{u}) := \int_{\mathbb{R}^+ \times \mathbb{R}/\mathbb{R}^n Z} \bar{u}^* \omega_n \in [0, +\infty].$$

In section 6 we will prove the following.
Lemma 6. Let $n \in \mathbb{N}$. Suppose $\tilde{u} \in C^\infty(\mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}, \mathbb{R} \times Z_n)$ is a solution to (9) with $E_\lambda(\tilde{u}) < \infty$. Then there exists $(s_0, t_0) \in \mathbb{R} \times \mathbb{R}/n\mathbb{Z}$ so that

$$(9) \quad \tilde{u}(s + s_0, t + t_0) = (s, t, z(s, t))$$

for all $(s, t) \in [-s_0, \infty) \times \mathbb{R}/n\mathbb{Z}$, where $z \in C^\infty([-s_0, \infty) \times \mathbb{R}/n\mathbb{Z}, D)$ satisfies the Floer equation

$$(10) \quad \partial_s z(s, t) + i(\partial_t z(s, t) - X_H(t, z(s, t))) = 0$$

for all $(s, t) \in [-s_0, \infty) \times \mathbb{R}/n\mathbb{Z}$.

This is a converse to “Gromov’s trick” [15]. It follows that if a solution $\tilde{u} = (a, \tau, z)$ to (9) has finite $\lambda$-energy then $E_\lambda(\tilde{u}) = n$, and the $\omega$-energy of $\tilde{u}$ is equal to the Floer energy of $z$,

$$E_\omega(\tilde{u}) = \frac{1}{2} \int_{s=0}^{+\infty} \int_{t=0}^{n} \left| \partial_s z(s, t) \right|^2 + \left| \partial_t z(s, t) - X_H(t, z(s, t)) \right|^2 ds dt.$$

Definition 7. For a solution $\tilde{u} = (a, \tau, z) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}, \mathbb{R} \times Z_n)$ to (9) we refer to the degree of the circle map $z(0, \cdot) : \mathbb{R}/n\mathbb{Z} \to \partial D$ as the boundary index of $\tilde{u}$.

Theorem 8. Let $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ be a Hamiltonian generating an irrational pseudo-rotation $\varphi$. Let $(Z_1, R_1), (Z_2, R_2), \ldots$ be the corresponding sequence of mapping tori. For each $n \in \mathbb{N}$ let $\gamma_n : \mathbb{R}/n\mathbb{Z} \to Z_n$ be the unique $n$-periodic orbit of $R_n$ for which $\gamma_n(0) \in \{0\} \times D$. Assume $H$ was chosen so that $\gamma(t) = (t, 0)$ for all $t \in \mathbb{R}/\mathbb{Z}$ (see remark [2]). Let $\alpha := \operatorname{Rot}(\varphi; H) \in \mathbb{R}$, which is necessarily irrational.

Then for each $n \in \mathbb{N}$ there exist two foliations $\mathcal{F}^+_n, \mathcal{F}^-_n$ of $\mathbb{R} \times Z_n$ by smoothly embedded surfaces, with the following properties:

- **Cylinder leaf:** The cylinder $C_n := \mathbb{R} \times \gamma_n(\mathbb{R}/n\mathbb{Z}) \subset \mathbb{R} \times Z_n$ is a leaf in both $\mathcal{F}^+_n$ and $\mathcal{F}^-_n$.

- **Pseudo-holomorphic:** If $F \in \mathcal{F}^+_n$ (resp. $F \in \mathcal{F}^-_n$) is not $C_n$, then $F$ is parameterized by a solution $\tilde{u}$ to (9) with $J_n$ as in [3], with $E_\lambda(\tilde{u}) + E_\omega(\tilde{u}) < \infty$ and boundary index $\lfloor n\alpha \rfloor$ (resp. $\lceil n\alpha \rceil$).

- **$\mathbb{R}$-invariance:** If $F \in \mathcal{F}^+_n$ (resp. $F \in \mathcal{F}^-_n$) is a leaf and $c \in \mathbb{R}$, the set $F + c := \{(a + c, \tau, z) | (a, \tau, z) \in F\}$ is also a leaf in $\mathcal{F}^+_n$ (resp. in $\mathcal{F}^-_n$).

- **Uniqueness:** $\mathcal{F}^+_n$ and $\mathcal{F}^-_n$ are uniquely determined by the above properties.

- **Smooth foliation:** $\mathcal{F}^+_n$ and $\mathcal{F}^-_n$ are $C^\infty$-smooth foliations at each point on the complement of $C_n$.

The proof is postponed to section [4]. This a special case of a much more general result to appear in [5].
Remark 3. For each leaf $F \in \mathcal{F}^+_n$ (resp. $F \in \mathcal{F}^-_n$) that is not the cylinder, any parameterization $\tilde{u}$ satisfying (5) has domain $\mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$ (resp. $\mathbb{R}^- \times \mathbb{R}/n\mathbb{Z}$). Hence the superscripts in $\mathcal{F}^\pm_n$. In either case, as the unique $n$-periodic orbit $\gamma_n$ is non-degenerate the finite energy of $\tilde{u}$ implies that the $Z_n$ component $u_\pm(s, \cdot) : \mathbb{R}/n\mathbb{Z} \to Z_n$ converges to $\gamma_n$ uniformly in $C^\infty(\mathbb{R}/n\mathbb{Z}, Z_n)$ as $s \to +\infty$ (resp. $s \to -\infty$). This can be seen in two ways. Either as a consequence of the compactness results in [17] applied to $\tilde{u}$ (as generalized in [18, 4]); or, via lemma [9] the original work of Floer [12] applied to the disk component $z$.

Remark 4. Let $n \in \mathbb{N}$. The foliations $\mathcal{F}^\pm_n$ of $\mathbb{R} \times Z_n$ can be visualized as follows. Under the projection map : $\mathbb{R} \times Z_n \to Z_n$, $\mathcal{F}^+_n$ and $\mathcal{F}^-_n$ project to smooth foliations of $Z_n \setminus \{\gamma_n\}$ by smoothly embedded surfaces diffeomorphic to $\mathbb{R}/n\mathbb{Z} \times (0, 1]$. The vector field $R_n = \partial_t + X_{H_\tau}$ on $Z_n$ is transverse to these leaves coming from $\mathcal{F}^+_n$ and $\mathcal{F}^-_n$ in opposite directions. A typical transverse disk slice to either of these looks something like in figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Figure 2.}
\end{figure}

The following formula will be crucial to our application. Recall that $\alpha := \text{Rot}(\varphi; H) \in \mathbb{R}$. In particular $\alpha$ is irrational.

Lemma 9. Let $n \in \mathbb{N}$. For any half cylinder leaf $F \in \mathcal{F}^+_n$,

$$E_\omega(F) = \{n\alpha\}\pi$$

where $\{\cdot\}$ denotes the fractional part of a real number.

This is proven in section [8].

4. Proof of theorem [1]

We use the finite energy foliations of theorem [8] to define new disk maps.

Definition 10. For each $n \in \mathbb{N}$ define $\varphi_n : D \to D$ as follows. For $\xi \in D$ there is a unique leaf $F \in \mathcal{F}^+_n$ containing $(0, 0, \xi) \in \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D$. Define $\varphi_n(\xi) = \xi'$ where $\xi' \in D$ is unique such that $(0, 1, \xi') \in F$. 
Remark 5. \( \varphi_n \) is well defined: by lemma \( \square \) if a leaf \( F \in \mathcal{F}_n^+ \) intersects the hypersurface
\[
\hat{Z}_n := \{0\} \times Z_n
\]
then it does so transversally, and for each \( \tau \in \mathbb{R}/n\mathbb{Z} \) it will intersect the disk slice \( \{\tau\} \times D \subset \hat{Z}_n \) in a unique point.

Remark 6. We could as easily define maps in terms of the foliations \( \mathcal{F}_n^- \).

Remark 7. The inverse map \( \varphi_n^{-1} \) exists and can be defined similarly in terms of \( \mathcal{F}_n^+ \).

**Lemma 11.** Each map \( \varphi_n : D \to D \) is \( n \)-periodic, that is \( (\varphi_n)^n = \text{id}_D \).

**Proof.** We could define \( n \) many disk maps using \( \mathcal{F}_n^+ \); say \( \varphi_{n,i} \) for \( i = 0, 1, \ldots, n-1 \), by requiring that \( \varphi_{n,i} \) takes the point \( \xi \in D \) to \( \xi' \in D \) if \( (0, i, \xi) \) and \( (0, i + 1, \xi') \) lie on the same leaf in \( \mathcal{F}_n^+ \). Since each leaf in the foliation closes up after going once around in the \( \mathbb{R}/n\mathbb{Z} \) direction, it follows that the composition \( \varphi_{n,n-1} \circ \ldots \circ \varphi_{n,0} \) is the identity map.

We now exploit a symmetry in \( \mathcal{F}_n^+ \) to see that each of the maps \( \varphi_{n,i} \) is equal to \( \varphi_n \). Consider the \( \hat{Z}_n \) action on \( \mathbb{R} \times Z_n \) generated by the deck transformation
\[
T : \mathbb{R} \times Z_n \to \mathbb{R} \times Z_n
\]
which preserves the almost complex structure; \( T^* J_n = J_n \). From the uniqueness part of theorem \( \square \) we conclude that the foliation \( T(\mathcal{F}_n^+) := \{T(F) \mid F \in \mathcal{F}_n^+\} \) is equal to \( \mathcal{F}_n^+ \). Hence \( \varphi_{n,i} = \varphi_n \) for each \( i \).

Note that \( \varphi_n(0) = 0 \) as the cylinder \( C_n \) passes through the center of the disk slices \( \{0\} \times D \) and \( \{1\} \times D \) in \( \hat{Z}_n = \{0\} \times Z_n \).

**Lemma 12.** \( \varphi_n : D \to D \) is \( C^\infty \)-smooth on \( D \setminus \{0\} \).

**Proof.** Recall that by theorem \( \square \) \( \mathcal{F}_n^+ \) is a \( C^\infty \)-smooth foliation on the complement of the leaf \( C_n \). From lemma \( \square \) the leaves of \( \mathcal{F}_n^+ \) are uniformly transverse to the hypersurface \( \hat{Z}_n = \{0\} \times Z_n \), and within this to all the disk slices, in particular to \( \{0\} \times D \) and \( \{1\} \times D \). Now consider any point \( \xi \in D \setminus \{0\} \). Then \( \xi' := \varphi_n(\xi) \in D \setminus \{0\} \). Therefore there exist local smooth foliation charts about \((0, 0, \xi)\) and \((0, 1, \xi')\). In these charts the map \( \varphi_n \) from its definition has a smooth expression.

**Lemma 13.** \( \varphi_n : D \to D \) is continuous at the origin.

**Proof.** Let \( \xi_j \in D \setminus \{0\} \) be a sequence of points converging to \( 0 \in D \), and \( p_j := (0, 0, \xi_j) \in \mathbb{R} \times Z_n \). By lemma \( \square \) there exists a sequence of parameterizations \( \tilde{u}_j \) of the unique leaf \( F_j \in \mathcal{F}_n^+ \) containing \( p_j \) that converges in the \( C^\text{loc} \)-topology to \( C_n \). By lemma \( \square \) we may choose each \( \tilde{u}_j \) to take the form \( \tilde{u}_j : [S_j, \infty) \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times Z_n \) for some \( S_j \in \mathbb{R} \), with
\[
\tilde{u}_j(s, t) = (s, t, z_j(s, t))
\]
for some sequence $z_j$. It follows that the sequence of points
\[ F_j \cap \{(0, 1)\} \times D = (0, 1, z_j(0, 1)) \]
converges to $(0, 1, 0)$. Which means that $\varphi_n(\xi_j) = z_j(0, 1) \to 0$ as $j \to \infty$.

Fix a subsequence $\{n_j\}_{j \in \mathbb{N}}$ for which the sequence of fractional parts
(11) \[ \lim_{j \to \infty} \{n_j\alpha\} = 0. \]
From lemma 9 this implies that the $\omega$-energies of the leaves in the foliations $\mathcal{F}_{n_j}^+$ tends to zero, as $j \to \infty$, uniformly over all leaves.

For maps $f, g : D \to D$, define $d_{C^0}(f, g)$ using the linear structure and Euclidean norm on $\mathbb{R}^2$ by
\[ d_{C^0}(f, g) := \sup_{\xi \in D} |f(\xi) - g(\xi)|. \]

**Proposition 14.** The subsequence $\varphi_{n_j}$ converges to the pseudo-rotation $\varphi$ in the following sense:
(12) \[ d_{C^0}(\varphi_{n_j}, \varphi) + d_{C^0}(\varphi_{n_j}^{-1}, \varphi^{-1}) \to 0 \]
as $j \to \infty$.

**Proof.** We show that $d_{C^0}(\varphi_{n_j}, \varphi) \to 0$ as $j \to \infty$, as the same argument will work for the inverses.

Arguing indirectly, there exists a sequence of points $\xi_j \in D$ and $\delta > 0$ such that $|\varphi_{n_j}(\xi_j) - \varphi(\xi_j)| \geq \delta$ for all $j \in \mathbb{N}$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^2$. Restricting to a subsequence we may assume that $\xi_j \to \xi$ for some $\xi \in D$, and $\delta \leq |\varphi_{n_j}(\xi_j) - \varphi(\xi_j)| \leq |\varphi_{n_j}(\xi_j) - \varphi_j(\xi)| + |\varphi_j(\xi_j) - \varphi(\xi)|$ for all $j \in \mathbb{N}$. Therefore as $\varphi$ is continuous,

\[ \frac{1}{2} \delta \leq |\varphi_{n_j}(\xi_j) - \varphi(\xi)| \]
for all $j$ sufficiently large.

For each $j \in \mathbb{N}$ let $F_j \in \mathcal{F}_{n_j}^+$ be the unique leaf containing the point $(0, 0, \xi_j) \in \mathbb{R} \times \mathbb{R}/n_j\mathbb{Z} \times D$. Let us assume that each $F_j$ is a half cylinder, otherwise the argument is even easier. There exists a solution $\tilde{u}_j$ to (10) parameterizing $F_j$. After a holomorphic reparameterization we may assume that
\[ \tilde{u}_j : [S_j, \infty) \times \mathbb{R}/n_j\mathbb{Z} \to \mathbb{R} \times Z_{n_j} \]
\[ \tilde{u}_j(0, 0) = (0, 0, \xi_j) \]
for some $S_j \leq 0$. For each $j$, $E_\lambda(\tilde{u}_j) < \infty$, so by lemma 10, $\tilde{u}_j$ takes the form
(13) \[ \tilde{u}_j(s, t) = (s, t, z_j(s, t)) \]
some $z_j : [S_j, \infty) \times \mathbb{R}/n_j\mathbb{Z} \to D$. Moreover, the sequence $\{\tilde{u}_j\}_{j \in \mathbb{N}}$ satisfies all the criterion for the compactness result theorem 25. In particular
\[ \lim_{j \to \infty} E_\omega(\tilde{u}_j) = 0 \]
due to our choice of subsequence satisfying (11). We conclude that the sequence \( \tilde{u}_j \) converges in the following sense: for each \( j \) let \( \tilde{u}_j : [S_j, \infty) \times \mathbb{R} \to \mathbb{R} \times Z_\infty \) be the unique lift of \( \tilde{u}_j \) to the universal covering, satisfying

\[
\tilde{u}_j(0, 0) = (0, 0, \xi_j).
\]

After restricting to a further subsequence we can assume that \( \tilde{u}_j \to \tilde{u}_\infty \) in the \( C^\infty_{\text{loc}}(\mathbb{R}^2, \mathbb{R} \times Z_\infty) \) topology, that is, uniformly on compact sets, where \( \tilde{u}_\infty \) takes the form

\[
\tilde{u}_\infty(s, t) = (s, t, \gamma(t))
\]

for some \( \gamma \in C^\infty(\mathbb{R}, D) \) solving \( \dot{\gamma}(t) = X_H(t, \gamma(t)) \) for all \( t \in \mathbb{R} \). From (14) we have \( \tilde{u}_\infty(0, 0) = (0, 0, \xi) \). We conclude that for all \( j \),

\[
(0, 1, \phi_{n_j}(\xi_j)) = \tilde{u}_j(0, 1).
\]

The right hand side converges to

\[
\tilde{u}_\infty(0, 1) = (0, 1, \gamma(1))
= (0, 1, \phi(\gamma(0)))
= (0, 1, \phi(\xi)).
\]

This contradicts (12) and we are done. \( \blacksquare \)

Combining the results of this section we have proven the following statement which is almost theorem 1.

**Theorem 15.** Suppose \( \varphi \in \text{Diff}^\infty(D, \omega_0) \) fixes the origin and has no other periodic points. Then there exists a sequence of maps \( \varphi_j \in \text{Homeo}^+(D) \cap \text{Diff}^\infty(D \setminus \{0\}) \) over \( j \in \mathbb{N} \), with the following properties. For each \( j \in \mathbb{N} \), \( \varphi_j(0) = 0 \), there exists \( n_j \in \mathbb{N} \) such that \( \varphi_j^{n_j} = \text{id}_D \), and \( d_{C^0}(\varphi_j, \varphi) + d_{C^0}(\varphi_j^{-1}, \varphi^{-1}) \to 0 \) as \( j \to \infty \).

There are presumably nicer ways to go from this conclusion to the final statement. For example using changes of coordinates from the pseudoholomorphic curves themselves. This will presumably follow from a more serious analysis of the asymptotic properties of the curves.

**Proof of theorem 15.** It is a classical result \([7, 9, 29]\) that if \( f \in \text{Homeo}^+(D) \) satisfies \( f^n = \text{id}_D \) for some \( n \in \mathbb{N} \), then there exists \( g \in \text{Homeo}^+(D) \) and \( q \in \{0, 1, \ldots, n - 1\} \) so that

\[
g \circ f \circ g^{-1} = R_{2\pi q/n}.
\]

If \( q \neq 0 \) then \( g \) must fix the origin, and if \( q = 0 \) then \( f = \text{id}_D \) anyway, so we may assume \( g \) fixes the origin. Applying this to each \( \varphi_j \) we find \( g_j \in \text{Homeo}^+(D) \), fixing the origin, and \( p_j \in \mathbb{Z} \) such that

\[
\varphi_j = g_j^{-1} \circ R_{2\pi p_j/n_j} \circ g_j.
\]

Now we replace \( g_j \) by a \( C^0 \)-close smooth approximation. More precisely, let \( \hat{g}_j \) be a sequence in \( \text{Diff}^\infty(D) \), each fixing the origin, with \( d_{C^0}(g_j, g_j) + \)
\[ d_{C^0}(\hat{g}_j^{-1}, g_j^{-1}) \to 0 \text{ as } j \to \infty. \] Then the maps \( \hat{\varphi}_j := \hat{g}_j^{-1} \circ R_{2\pi p_j/n_j} \circ \hat{g}_j \) are \( C^\infty \)-diffeomorphisms which converge in the \( C^0 \)-sense to the irrational pseudo-rotation \( \varphi \). The maps \( \hat{\varphi}_j \) satisfy the conditions of theorem 16. \( \blacksquare \)

5. Calculation of the \( \omega \)-energy

The aim of this section is to prove lemma 9, recalled as lemma 18.

Choose a 1-form \( \lambda_0 \) on the disk so that \( d\lambda_0 = \omega_0 = dx \wedge dy \). For each \( n \in \mathbb{N} \) define the action functional \( \mathbf{A}_n : C^\infty(\mathbb{R}/n\mathbb{Z}, Z_n) \to \mathbb{R} \) (associated to \( \lambda_0 \)) by

\[
\mathbf{A}_n(\sigma) := \int_{\mathbb{R}/n\mathbb{Z}} \sigma^* \lambda_0 - \int_0^1 H(\sigma(t))dt.
\]

We may rewrite this as

\[
\mathbf{A}_n(\sigma) := \int_{\mathbb{R}/n\mathbb{Z}} \sigma^* \eta_n
\]

where \( \eta_n := \lambda_0 - Hdt \) is a primitive of \( \omega_n \) the 2-form used to define the \( \omega \)-energy. Note that \( \eta_n \) restricts to a closed 1-form on \( \partial Z_n \) since \( R_n \) is tangent to \( \partial Z_n \) and \( dt\eta_n(R_n, \cdot) = \omega_n(R_n, \cdot) = 0 \). Hence \( \mathbf{A}_n \) restricted to \( C^\infty(\mathbb{R}/n\mathbb{Z}, \partial Z_n) \) descends to a map on homology.

Lemma 16. For each \( n \in \mathbb{N} \)

\[
\mathbf{A}_n(1_{\mathbb{R}/n\mathbb{Z}}) + [n\alpha] \mathbf{A}_n(1_{\partial D}) \leq \mathbf{A}_n(\gamma_n) \leq \mathbf{A}_n(1_{\mathbb{R}/n\mathbb{Z}}) + [n\alpha] \mathbf{A}_n(1_{\partial D})
\]

where \( 1_{\mathbb{R}/n\mathbb{Z}} \) and \( 1_{\partial D} \) are the closed loops in \( \partial Z_n = \mathbb{R}/n\mathbb{Z} \times \partial D \) given by \( \mathbb{R}/n\mathbb{Z} \ni t \mapsto (t, pt) \) and \( \mathbb{R}/n\mathbb{Z} \ni t \mapsto (pt, e^{2\pi it/n}) \) respectively.

Proof. Let \( \tilde{u}_\pm = (a_\pm, u_\pm) : \mathbb{R}^\pm \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times Z_n \) be parameterizations of leaves \( F^\pm \in \mathcal{F}_n^\pm \) respectively which satisfy \( \mathbf{9} \). In either case \( u_\pm(s, \cdot) \) converges uniformly in \( C^\infty(\mathbb{R}/n\mathbb{Z}, Z_n) \) to a parameterization \( \gamma_n(\text{const}_\pm + \cdot) \) as \( s \to \pm\infty \) respectively. Applying Stokes theorem,

\[
E_\omega(\tilde{u}_+) = \int_{\mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}} u_+^* \omega_n = \mathbf{A}_n(\gamma_n) - \mathbf{A}_n(u_+(0, \cdot)),
\]

and

\[
E_\omega(\tilde{u}_-) = \int_{\mathbb{R}^- \times \mathbb{R}/n\mathbb{Z}} u_-^* \omega_n = \mathbf{A}_n(u_-(0, \cdot)) - \mathbf{A}_n(\gamma_n).
\]

Therefore, as the energies are non-negative,\( \mathbf{A}_n(u_+(0, \cdot)) \leq \mathbf{A}_n(\gamma_n) \leq \mathbf{A}_n(u_-(0, \cdot)) \).

We observed that the action \( \mathbf{A}_n \) of a closed loop in \( \partial Z_n \) depends only on its homology class. From theorem \( \mathbf{3} \) \( u_+(0, \cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z} \times \partial D \) and \( u_-(0, \cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z} \times \partial D \) are homologous to \( t \mapsto (t, e^{2\pi i(\alpha_\pm)/n}) \) and \( t \mapsto (t, e^{2\pi i((-\alpha_\pm)/n)}) \) respectively. Therefore we get the inequalities in \( \mathbf{17} \). \( \blacksquare \)
Corollary 17. The unique 1-periodic orbit $\gamma_1 : \mathbb{R}/\mathbb{Z} \to Z_1$ has action
\begin{equation}
A_1(\gamma_1) = A_1(1_{\mathbb{R}/\mathbb{Z}}) + \alpha A_1(1_{\partial D}).
\end{equation}

Proof. From the definition of $A_n$,
\begin{align*}
A_n(\gamma_n) &= nA_1(\gamma_1) \\
A_n(1_{\mathbb{R}/n\mathbb{Z}}) &= nA_1(1_{\mathbb{R}/\mathbb{Z}}) \\
A_n(1_{\partial D}) &= A_1(1_{\partial D})
\end{align*}
for all $n \in \mathbb{N}$. Substituting these into the inequalities in lemma (16), and dividing through by $n$ and letting $n \to +\infty$ gives (18).

Lemma 18. Let $n \in \mathbb{N}$. Every leaf $F \in F^+_n$ with boundary has $\omega$-energy
\[ E_\omega(F) = \{n\alpha\} \pi \]
where $\{\cdot\}$ applied to any real number denotes its fractional part.

Proof. By Stokes theorem as in the last lemma,
\[ E_\omega(F) = A_n(\gamma_n) - A_n(u(0, \cdot)) \]
where $\tilde{u} = (a, u) : \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times Z_n$ is a parameterization of $F$. Using corollary 17 and that $u(0, \cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z} \times \partial D$ is homologous to $t \mapsto (t, e^{2\pi i (\{n\alpha\}/n)t})$, this becomes
\begin{align*}
E_\omega(F) &= n \left( A_1(1_{\mathbb{R}/\mathbb{Z}}) + \alpha A_1(1_{\partial D}) \right) - \\
&\quad \left( A_n(1_{\mathbb{R}/n\mathbb{Z}}) + \lfloor n\alpha \rfloor A_n(1_{\partial D}) \right) \\
&= \left( nA_1(1_{\mathbb{R}/\mathbb{Z}}) - \lfloor n\alpha \rfloor A_1(1_{\partial D}) \right) - \\
&\quad \left( nA_1(1_{\mathbb{R}/\mathbb{Z}}) + \lfloor n\alpha \rfloor A_1(1_{\partial D}) \right) \\
&= (n\alpha - \lfloor n\alpha \rfloor) A_1(1_{\partial D}) \\
&= \{n\alpha\} \cdot \int_D dx \wedge dy.
\end{align*}

6. Compactness

We prove some compactness statements that were used in the proof of theorem 11. These are for sequences of pseudoholomorphic curves $\tilde{u}_j$ that map into the spaces $(\mathbb{R} \times Z_{n_j}, J_{n_j})$ that were described in section 3. In 6.1 we consider the situation when $n_j = n$ is fixed and the total energy of the curves is uniformly bounded. In section 6.2 we consider situations when $n_j \to +\infty$ and the total energy of the curves is also unbounded.

Remark 8. Norms, such as $\|\nabla \tilde{u}\|_{L^\infty}$, are implicitly with respect to the Riemannian metric $dx^2 + dy^2 + d\tau^2 + da^2$ on $\mathbb{R} \times Z_n$, where $(x, y)$ are the standard Euclidean coordinates on the disk, $\tau$ is the “coordinate” on $\mathbb{R}/n\mathbb{Z}$ and $a$ is the $\mathbb{R}$-coordinate. This metric is $J_n$-invariant.
6.1. Compactness when $n$ is uniformly bounded. The aim of this section is to prove Lemma 19 stated earlier. We break this into the following two lemmas.

Lemma 19 (The Floer equation from the Cauchy-Riemann equations). Let $\bar{u} = (a, \tau, z) : \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D$ be a solution to (5) for which $\|\nabla \bar{u}\|_{L^\infty} < \infty$. Then there exist $(a_0, \tau_0) \in \mathbb{R} \times \mathbb{R}/n\mathbb{Z}$ such that

\begin{equation}
\begin{cases}
a(s, t) = s + a_0 \\
\tau(s, t) = t + \tau_0
\end{cases}
\end{equation}

for all $(s, t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$, and moreover $z : \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z} \to D$ satisfies the following Floer equation:

\begin{equation}
\partial_s z(s, t) + i(\partial_\tau z(s, t) - X_H(t + \tau_0, z(s, t))) = 0
\end{equation}

for all $(s, t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$.

Proof. Writing out in coordinates what it means for $\bar{u}$ to satisfy (5) gives us

\begin{equation}
(a_s - \tau_t) \partial_a + (a_t + \tau_s) \partial_\tau + \left(a_t X_H(\tau, z) + z_s + i(z_t - \tau_s X_H(\tau, z))\right) = 0.
\end{equation}

The boundary condition on $\bar{u}$ in (5) implies $a_t(0, t) = 0$ for all $t \in \mathbb{R}/n\mathbb{Z}$. From (21),

\begin{align}
a_s(s, t) &= \tau_t(s, t) \\
a_t(s, t) &= -\tau_s(s, t)
\end{align}

for all $(s, t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$. In particular both functions $a, \tau$ lift to harmonic functions on the upper half plane with gradient bounded in $L^\infty$. The boundary conditions on $a$ allow a smooth extension by reflection to the whole plane, still with gradient in $L^\infty$, and therefore by Liouville the partial derivatives of $a$ are constant. So there exists $b, c, a_0 \in \mathbb{R}$ so that $a(s, t) = cs + bt + a_0$ for all $(s, t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$. Putting this into (22) there exists $\tau_0 \in \mathbb{R}$ so that $\tau(s, t) = ct - bs + \tau_0$ for all $(s, t) \in \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z}$. The $n$-periodicity of $a$ in the $t$ variable implies $b = 0$. Also $\tau(s, t + n) = \tau(s, t) + n$ for all $(s, t) \in \mathbb{R}^+ \times \mathbb{R}$ because we are assuming that $\text{deg}(\bar{u}) = 1$. Therefore $c = 1$. This proves (6).

From (21) we also have $a_t X_H(\tau, z) + z_s + i(z_t - \tau_s X_H(\tau, z)) = 0$. But we have shown that $\tau_t \equiv 1$ and $a_t \equiv 0$. Substituting these in we obtain

\begin{equation}z_s + i(z_t - X_H(t + \tau_0, z)) = 0\end{equation}

as required. ■

The next statement says that we can use the above relation between the Cauchy-Riemann and Floer equations if (and only if) the $\lambda$-energy is finite.

Lemma 20. Let $\bar{u} = (a, \tau, z)$ be a solution to (2). Then $E_\lambda(\bar{u}) < \infty$ implies $\|\nabla a\|_{L^\infty} < \infty$ (equivalently $\|\nabla \tau\|_{L^\infty} < \infty$).
Proof. The equations (6.2) in the last lemma did not require the gradient bounds, and so the map \( f : \mathbb{R}^+ \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \) given by \( f(s,t) := (a(s,t), \tau(s,t)) \) in terms of the \( a \) and \( \tau \) components of \( \tilde{u} \), is holomorphic. Also \( f(0,t) \in \{c\} \times \mathbb{R}/n\mathbb{Z} \) for all \( t \in \mathbb{R}/n\mathbb{Z} \).

Arguing indirectly suppose that the gradient of \( a \) is unbounded. Then the gradient must blow up along a sequence of points that leaves every compact subset of the domain, in particular does not converge to the boundary. Therefore standard rescaling arguments applied to \( f \) yield a holomorphic plane \( g : \mathbb{C} \to \mathbb{R} \times \mathbb{R}/n\mathbb{Z} \) with the following properties:

\[
\begin{align*}
|\nabla g(0)| &= 1 \\
|\nabla g(\xi)| &\leq 2 \quad \text{for all } \xi \in \mathbb{C} \\
E_\lambda(g) &< \infty.
\end{align*}
\]

Indeed, it is easily checked that \( E_\lambda(g) \leq E_\lambda(\tilde{u}) \). The first two properties imply that \( g \) has constant, non-zero, gradient from Liouville’s theorem. But this implies the contradiction \( E_\lambda(g) = +\infty \). \( \blacksquare \)

In our proof that the maps \( \varphi_n \) are continuous we used the following.

**Lemma 21.** For each fixed \( n \in \mathbb{N} \), \( \mathcal{F}_n^+ \) is compact in the following sense. Suppose that \( F_j \in \mathcal{F}_n^+ \) is a sequence of leaves over \( j \in \mathbb{N} \), and \( p_j \in F_j \) is a sequence of points. Suppose that \( p_j \to p \) some \( p \in \mathbb{R} \times n\mathbb{Z} \). Then there exists a sequence of parameterizations of \( F_{jk} \) which converge in a \( C^\infty \)-sense to a parameterization of the unique leaf in \( \mathcal{F}_n^+ \) containing \( p \).

**Proof.** This is a standard property of finite energy foliations from positivity of intersections, used many times in [24] as the \( \lambda \)-energy and \( \omega \)-energy are uniformly bounded in \( j \). \( \blacksquare \)

### 6.2. Compactness as \( n \to \infty \)

In our proof of convergence of the disk maps \( \varphi_n \) in proposition [14] we used a compactness statement for a sequence of \( J_n \)-holomorphic maps \( \tilde{u}_n : [S_n, \infty) \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times n\mathbb{Z} \) for which

\[
\begin{align*}
E_\lambda(\tilde{u}_n) &= n \to +\infty \\
E_\omega(\tilde{u}_n) &= \{n\alpha\} \pi
\end{align*}
\]

for some irrational real number \( \alpha \). Hence the total energy \( E(\tilde{u}_n) = E_\lambda(\tilde{u}_n) + E_\omega(\tilde{u}_n) \) diverges to \( +\infty \). In general, for a sequence of maps \( \{\tilde{u}_n\} \) for which the total energy is unbounded one cannot expect uniform bounds on the gradient in \( L^\infty \). However if the \( \lambda \)-energy grows at most linearly with \( n \), and the \( \omega \)-energy is bounded then indeed uniform bounds on \( \|\nabla \tilde{u}_n\|_{L^\infty} \) can be achieved. (Actually much weaker assumptions suffice, but we will not need to explore these here.) Our arguments will be further simplified since we restricted to a subsequence for which the \( \omega \)-energy of the sequence decays to zero.
Consider a sequence \( \{\tilde{u}_n\}_{n \in \mathbb{N}} \) of smooth \( J_n \)-holomorphic maps, for \( J_n \) as in \( \{\mathbb{4}\} \), with numbers \( c_n, S_n \in \mathbb{R}, S_n \leq 0 \), satisfying for each \( n \),
\[
\begin{align*}
\tilde{u}_n &= (a_n, \tau_n, z_n) : [S_n, \infty) \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R} \times Z_n, \\
\partial_s \tilde{u}_n(s, t) + J_n(\tilde{u}_n(s, t)) \partial_t \tilde{u}_n(s, t) &= 0 \\
\tilde{u}_n(S_n, t) &\in L_{c_n} \\
\tau_n : (S_n, \cdot) : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/n\mathbb{Z} &\text{ has degree 1}
\end{align*}
\]
for all \( (s, t) \in [S_n, \infty) \times \mathbb{R}/n\mathbb{Z} \).

**Proposition 22.** Suppose that \( E_\lambda(\tilde{u}_n) < \infty \) for each \( n \in \mathbb{N} \), and \( \lim_{n \to \infty} E_\omega(\tilde{u}_n) = 0 \). Then there exists \( C \in (0, \infty) \) such that
\[
\| \nabla \tilde{u}_n \|_{L^\infty([S_n, \infty) \times \mathbb{R}/n\mathbb{Z})} \leq C
\]
for all \( n \in \mathbb{N} \).

Note that we do not assume uniform bounds on the \( \lambda \)-energy.

**Proof.** Since \( E_\lambda(\tilde{u}_n) < \infty \) for each \( n \), lemma \( \{\mathbb{20}\} \) implies \( \| \nabla a_n \|_{L^\infty} < \infty \) (for each \( n \)). Therefore, since also each \( \tau_n \) has degree 1, lemma \( \{\mathbb{19}\} \) applies so
\[
\begin{align*}
a_n(s, t) &= s + a_n \\
\tau_n(s, t) &= t + \tau_n
\end{align*}
\]
for all \( (s, t) \in [S_n, \infty) \times \mathbb{R}/n\mathbb{Z} \), some constants \( (a_n, \tau_n) \in [S_n, \infty) \times \mathbb{R}/n\mathbb{Z} \).
Thus
\[
\| \nabla a_n \|_{L^\infty} \leq 1 \quad \text{and} \quad \| \nabla \tau_n \|_{L^\infty} \leq 1
\]
for all \( n \in \mathbb{N} \).

It therefore remains to show that the gradients of the \( z_n \) are uniformly bounded. Arguing indirectly we find a sequence \( \xi_n \in [S_n, \infty) \times \mathbb{R}/n\mathbb{Z} \) for which \( |\nabla \tilde{u}_n(\xi_n)| \geq |\nabla z_n(\xi_n)| \to \infty \) as \( n \to \infty \). A standard rescaling argument produces a \( J_{\infty} \)-holomorphic plane or half plane in \( \mathbb{R} \times Z_{\infty} \). That is, a map \( \tilde{v} : \mathbb{C} \to \mathbb{R} \times Z_{\infty} \), or \( \tilde{v} : \mathbb{H} \to \mathbb{R} \times Z_{\infty} \) with totally real boundary conditions \( \tilde{v}(\partial \mathbb{H}) \subset \{ c \} \times \partial Z_{\infty} \) for some \( c \in \mathbb{R} \). In either case, as a result of the rescaling process, \( \tilde{v} = (a, \tau, z) \) has the following properties:
\[
\nabla a \equiv 0 \\
\nabla \tau \equiv 0 \\
|\nabla \tilde{v}(0)| > 0, \\
E_\omega(\tilde{v}) = 0.
\]
The first two properties are because of the uniform bounds \( \| \nabla a_j \|_{L^\infty} \leq 1 \) and \( \| \nabla \tau_j \|_{L^\infty} \leq 1 \) respectively even before rescaling, so that rescaling “kills” these terms in the limit. The vanishing \( \omega \)-energy is because \( E_\omega(\tilde{v}) \leq \lim_{n \to \infty} E_\omega(\tilde{u}_n) = 0 \) by Fatou’s lemma. Thus there exist constants \( a_0, \tau_0 \in \mathbb{R} \) such that
\[
\tilde{v}(s, t) = (a_0, \tau_0, z(s, t)) \in \mathbb{R} \times \mathbb{R} \times D
\]
for all \((s, t) \in \mathbb{C}\) (resp. all \((s, t) \in \mathbb{H}\)). That \(\tilde{\nu}\) is \(J_{\infty}\)-holomorphic translates into \(z: \mathbb{C} \to D\) or \(z: \mathbb{H} \to D\) satisfying the equation \(a_tX_H(\tau, z) + z_t + i(z_t - \tau_tX_H(\tau, z)) = 0\), see [21]. So \(a\) and \(\tau\) constant implies \(z_t + iz_t = 0\). (We could alternatively have just rescaled the sequence \(\{z_n\}\) as in Floer theory, to get the same conclusion.) Therefore,

\[
0 = E_\omega(\tilde{\nu}) = \int \tilde{\nu}^* \omega = \int \frac{1}{2} (|z_s|^2 + |z_t|^2) \, dsdt
\]

and so \(z\) is also constant. Thus we have shown that \(\tilde{\nu}\) is constant, contradicting \(|\nabla \tilde{\nu}(0)| > 0\).

Now standard arguments can convert these uniform bounds on the gradient in \(C^0\) to uniform \(C^k\)-bounds on the gradient for all \(k \in \mathbb{N}\). The key result is the following local statement which is proven using the \(W^{k,p}\)-elliptic estimates for the linear Cauchy-Riemann operator.

For each \(r \geq 0\) let \(D_r := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq r\}\).

**Theorem 23** (Hofer [17]). Let \(J\) be a fixed, smooth, almost complex structure on \(\mathbb{R}^{2d}\), \(d \in \mathbb{N}\). Let \(C' \in (0, \infty)\). Consider the set of maps \(B(J, C') \subset C^\infty(D_1, \mathbb{R}^{2d})\), consisting of all \(f\) satisfying,

\[
\partial_s f + J(f) \partial_t f = 0
\]

\[
|f(0)| \leq 2
\]

\[
\|\nabla f\|_{C^0(D_1)} < C.
\]

Then for all \(r \in (0, 1)\) there exists a sequence \(c_k \in (0, \infty)\) over \(k \in \mathbb{N}\), such that for all \(f \in B(J, C')\),

\[
\|\nabla f\|_{C^k(D_r)} < c_k
\]

for all \(k \in \mathbb{N}\).

Replacing \(D_1, D_r\) for half disks \(D_1^+, D_r^+\), where \(D_r^+ := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq r, y \geq 0\}\), the same statement holds for maps that take the boundary points \([-1, 1] \times \{0\}\) into a smooth path of \(J\)-totally real subspaces in \(\mathbb{R}^{2d}\). This is by a straightforward reflection argument.

**Corollary 24.** Suppose \(c_0 \in (0, \infty)\). Then there exists a sequence \(c_k \in (0, \infty)\) over \(k \in \mathbb{N}\), with the following property. If \(\{\tilde{u}_n\}_{n \in \mathbb{N}}\) is a sequence of solutions to (24) such that \(\sup_{n \in \mathbb{N}} \|\nabla \tilde{u}_n\|_{L^\infty(S_n, \infty) \times \mathbb{R}/n\mathbb{Z}} \leq c_0\), then

\[
(24) \quad \|\nabla \tilde{u}_n\|_{C^k(S_n, \infty) \times \mathbb{R}/n\mathbb{Z}} < c_k
\]

for all \(k \in \mathbb{N}\) and \(n \in \mathbb{N}\).

**Proof.** This follows easily from the local result, theorem 23 using that the almost complex structures \(J_n\) satisfy: (1) they are invariant under the \(\mathbb{R}\) and \(\mathbb{Z}_n\) actions on \(\mathbb{R} \times \mathbb{R}/n\mathbb{Z} \times D\), and (2) they each lift to the same almost complex structure \(J_\infty\) on the universal covering \(\mathbb{R} \times \mathbb{R} \times D\).

Combining the last three statements we can prove the following.
Theorem 25. Let $\bar{u}_n : [S_n, \infty) \times \mathbb{R} \rightarrow \mathbb{R} \times Z_\infty$ be a sequence of $J_\infty$-holomorphic maps over $n \in \mathbb{N}$, where each $\bar{u}_n$ is a lift of a solution $u_n$ to (26). Suppose that $\bar{u}_n(0,0)$ is uniformly bounded in $n$ and that $E_\lambda(\bar{u}_n) < \infty$ for each $n \in \mathbb{N}$, and that $\lim_{n \to \infty} E_\omega(\bar{u}_n) = 0$. Then there exists a subsequence \{$\bar{u}_{n_j}$\}$_{j \in \mathbb{N}}$ such that $\bar{u}_{n_j}$ converges in $C_{\text{loc}}^\infty(\mathbb{C}, \mathbb{R} \times Z_\infty)$ to a $J_\infty$-holomorphic map $\bar{u}_\infty$ having domain either $\Sigma = \mathbb{C}$ or $\Sigma = [S, \infty) \times \mathbb{R} \subset \mathbb{C}$ for some $S \in (-\infty, 0]$. Moreover $\bar{u}_\infty$ takes the following form: there exist constants $a_0, \tau_0 \in \mathbb{R}$ such that

$$\bar{u}_\infty : \Sigma \to \mathbb{R} \times \mathbb{R} \times D$$

(25)

$$\bar{u}_\infty(s,t) = (s + a_0, t + \tau_0, \gamma(t))$$

for all $(s,t) \in \Sigma$, where $\gamma \in C^\infty(\mathbb{R}, D)$ satisfies

$$\dot{\gamma}(t) = X_{H_t + \tau_0}(\gamma(t))$$

(26)

for all $t \in \mathbb{R}$.

Proof. Taking a subsequence we may assume that $\bar{u}_{n_j}(0,0)$ converges, and that $S_{n_j}$ converges to some $S \in (-\infty, 0] \cup \{ -\infty \}$. From proposition 22 and corollary 23 we obtain uniform bounds on $\| \nabla \bar{u}_n \|_{C_k}$ for each $k \in \mathbb{N}$, and therefore also $C^0$-bounds on $\bar{u}_n$ on compact subsets, uniform in $n$. Repeated use of the Arzela-Ascoli theorem yields a subsequence converging uniformly with all derivatives on each compact subset of $\mathbb{C}$ to a smooth map $\bar{u}_\infty : \Sigma \to \mathbb{R} \times Z_\infty$ where $\Sigma = [S, \infty) \times \mathbb{R}$ if $S$ is finite and $\Sigma = \mathbb{C}$ otherwise. From lemma 19 each map in the sequence $\bar{u}_{n_j}$ takes the form

$$\bar{u}_{n_j}(s,t) = (s + a_j, t + \tau_j, z_j(s,t))$$

for constants $a_j, \tau_j \in \mathbb{R}$, with $z_j : [S_{n_j}, \infty) \times \mathbb{R} \to D$ satisfying

$$\partial_s z_j(s,t) + i \left( \partial_t z_j(s,t) - X_H(t + \tau_j, z_j(s,t)) \right) = 0$$

for all $(s,t) \in [S_{n_j}, \infty) \times \mathbb{R}$. Therefore $\bar{u}_\infty$ takes the form

$$\bar{u}_\infty(s,t) = (s + a_\infty, t + \tau_\infty, z_\infty(s,t))$$

for constants $a_\infty, \tau_\infty \in \mathbb{R}$ and some $z_\infty : \Sigma \to D$ satisfying

$$\partial_s z_\infty + i \left( \partial_t z_\infty - X_H(t + \tau_\infty, z_\infty) \right) = 0.$$ 

Let $\omega_\infty := dx \wedge dy + d\tau \wedge dH$ on $Z_\infty$. Then

$$0 \leq \int_{\mathbb{R}^2} \bar{u}_\infty^* \omega_\infty \leq \lim_{j \to \infty} E_\omega(\bar{u}_{n_j}) = 0.$$

Thus

$$\frac{1}{2} \int_{\mathbb{R}^2} \left| \frac{\partial z_\infty}{\partial s}(s,t) \right|^2 + \left| \frac{\partial z_\infty}{\partial t}(s,t) - X_H(t + \tau_\infty, z_\infty(s,t)) \right|^2 dsdt =$$

$$\int_{\mathbb{R}^2} \bar{u}_\infty^* \omega_\infty = 0.$$

Hence $z_\infty(s,t) = \gamma(t)$ for some solution $\gamma : \mathbb{R} \to D$ to (26). $\blacksquare$
7. Construction of the finite energy foliations

In this final section we give a terse proof of theorem 8. The approach is along by now fairly standard lines, the only minor complication arises from the presence of the boundary of the almost complex manifold. A more general construction will appear in [5].

We will assume more familiarity with terminology from [4] than elsewhere in this article, and with the intersection theory in [35].

Recall the statement of theorem 8 when \( n = 1 \). This is what we will prove as the general case is identical.*

**Theorem 26.** Let \( H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R}) \) be a Hamiltonian generating an irrational pseudo-rotation \( \varphi \). Let \((Z = \mathbb{R}/\mathbb{Z} \times D, R)\) be the corresponding Hamiltonian mapping torus. Let \( J \) be the the associated \( \mathbb{R} \)-invariant almost complex structure characterized by

\[
\begin{align*}
J \partial_R &= R \\
J|_{TD} &= i.
\end{align*}
\]

Denote by \( \gamma : \mathbb{R}/\mathbb{Z} \to Z \) the unique 1-periodic orbit of \( R \) for which \( \gamma(0) \in \{0\} \times D \), and assume \( H \) was chosen so that \( \gamma(t) = (t, 0) \) for all \( t \in \mathbb{R}/\mathbb{Z} \) (see remark 3). Let \( \alpha := \text{Rot}(\varphi; H) \in \mathbb{R} \), which is necessarily irrational.

Then there exist two foliations \( \mathcal{F}^+, \mathcal{F}^- \) of \( \mathbb{R} \times Z \) by \( J \)-holomorphic curves with the following properties:

- **Cylinder leaf:** The cylinder \( C := \mathbb{R} \times \gamma(\mathbb{R}/\mathbb{Z}) \subset \mathbb{R} \times Z \) is a leaf in both \( \mathcal{F}^+ \) and \( \mathcal{F}^- \).
- **Pseudo-holomorphic:** If \( F \in \mathcal{F}^+ \) (resp. \( F \in \mathcal{F}^- \)) is not \( C \), then \( F \) is parameterized by a solution \( \tilde{u} \) to \( (27) \), with \( E_\lambda(\tilde{u}) + E_\omega(\tilde{u}) < \infty \) and boundary index \( \lfloor \alpha \rfloor \) (resp. \( \lceil \alpha \rceil \)).
- **\( \mathbb{R} \)-invariance:** If \( F \in \mathcal{F}^+ \) (resp. \( F \in \mathcal{F}^- \)) is a leaf and \( c \in \mathbb{R} \), the set \( F + c := \{(a + c, \tau, z) \mid (a, \tau, z) \in F\} \) is also a leaf in \( \mathcal{F}^+ \) (resp. in \( \mathcal{F}^- \)).
- **Uniqueness:** \( \mathcal{F}^+ \) and \( \mathcal{F}^- \) are uniquely determined by the above properties.
- **Smooth foliation:** \( \mathcal{F}^+ \) and \( \mathcal{F}^- \) are \( C^\infty \)-smooth foliations at each point on the complement of \( C \).

In section 7.1 we prove the statement for \( \mathcal{F}^+ \) assuming restrictions on the boundary. Section 7.2 removes the boundary restrictions. In section 7.3 we explain how the construction of \( \mathcal{F}^- \) differs from \( \mathcal{F}^+ \). We defer to the end two technical points: section 7.4 fills in a detail regarding compactness. In section 7.5 we explain how to find the filling of the boundary used in section 7.1.

*For maps more general than irrational pseudo-rotations the proof for \( n > 1 \) is a little more involved as the almost complex structure has additional symmetry which makes transversality less obvious. But for irrational pseudo-rotations automatic transversality suffices and the same proof works for all \( n \geq 1 \).
Remark 9. In our proof of theorem we do not actually use that $\varphi$ is a pseudo-rotation. We only use the much weaker assumption that $\varphi$ has only a single fixed point, and that it is non-degenerate. We will exploit this in section 7.2.

7.1. Construction of $\mathcal{F}^+$ when $\varphi|_{\partial D}$ is conjugate to a rigid rotation.

In this section suppose that there exists $g \in \text{Diff}^C_+(\partial D)$ so that

$$\sigma \circ \varphi|_{\partial D} \circ \sigma^{-1} = R_{2\pi\alpha}$$

where $R_{2\pi\alpha} : \partial D \to \partial D$ is the rigid rotation $z \mapsto ze^{2\pi\alpha}$.

Let $H_-, H_+ \in C^\infty(Z, \mathbb{R})$ be as follows. $H_-$ is as in the statement of theorem 26. Define $H_+$ by

$$H_+(\tau, z) := \pi\alpha|z|^2 + C$$

for some constant $C \leq 0$ chosen so that $\max H_+ < \min H_-$.

$H_-$ and $H_+$ define a closed loop of Hamiltonians on the disk $H_{\pm} := H_{\pm}(t, \cdot) : D \to \mathbb{R}$, over $t \in \mathbb{R}/\mathbb{Z}$. Let $X_{H_{\pm}}(z)$ be the corresponding time-dependent Hamiltonian vector fields on the disk $D$, with respect to the symplectic form $\omega_0 = dx \wedge dy$. The time-one map generated by $X_{H_+}^t(z)$ is the rigid rotation $R_{2\pi\alpha}$, while the time-one map generated by $X_{H_-}^t(z)$ is the pseudo-rotation $\varphi$. From the statement of theorem 26, $\text{Rot}(R_{2\pi\alpha}, H_+) = \alpha$, and therefore $\text{Rot}(R_{2\pi\alpha}, H_-) = \text{Rot}(\varphi, H_-)$.

The two pairs of differential forms $\mathcal{H}_{\pm} = (\omega_{\pm}, \lambda_{\pm})$, on $Z$, given by

$$\omega_{\pm} = dx \wedge dy + d\tau \wedge dH_{\pm}, \quad \lambda_{\pm} = d\tau$$

define stable Hamiltonian structures on $Z$. Their associated (stable Hamiltonian) Reeb vector fields $R_+$ and $R_-$ are defined by the conditions $\omega_\pm (R_\pm, \cdot) = 0$ and $\lambda_\pm (R_\pm) = 1$. These are calculated to be

$$R_\pm(t, z) = \partial_t + X_{H_\pm}(z).$$

Thus, the first return map of $R_+$ on the initial disk slice $\{0\} \times D \subset Z$ is the rigid rotation $R_{2\pi\alpha}$, while the first return map of $R_-$ on this disk slice is the pseudo-rotation $\varphi$. The flows generated by $R_+$ and $R_-$ each have a unique 1-periodic orbit. Let $\gamma_{\pm} : \mathbb{R}/\mathbb{Z} \to Z$ be the parameterization of each, uniquely satisfying $\gamma_{\pm}(0) \in \{0\} \times D$. These have equal Conley-Zehnder indices (using any conventions).

Choose a function $H \in C^\infty(\mathbb{R} \times Z, \mathbb{R})$ interpolating between $H_-$ and $H_+$ in the following way:

$$
\begin{cases}
H(a, m) = H_+(m) & \text{for all } a \geq 1 \\
\partial_a H(a, m) < 0 & \text{for } -1 < a < 1 \\
H(a, m) = H_-(m) & \text{for all } a \leq -1.
\end{cases}
$$

For example, let us assume we chose $H(a, m) = \chi(a)H_+(m)+(1-\chi(a))H_-(m)$ for some $\chi \in C^\infty[0, 1]$ with $\chi \equiv 0$ on $(-\infty, -1]$ and $\chi \equiv 1$ on $[1, \infty)$ and such that $\chi'(a) > 0$ for all $a \in (-1, 1)$. 
Define an almost complex structure $\hat{J}$ on $W := \mathbb{R} \times Z$ by
\[
\begin{align*}
\hat{J}(a, \tau, z)\partial_{\mathbb{R}} &= \partial_{\tau} + X_{H^+}(z) \\
\hat{J}|_{TD} &= i
\end{align*}
\]
where $X_{H^+}$ is the Hamiltonian vector field of $H^+_a := H(a, \tau, \cdot) : D \to \mathbb{R}$.

Then $(W, \hat{J})$ is an almost complex manifold with cylindrical ends $E_+ = [1, \infty) \times Z$ and $E_- = (-\infty, -1] \times Z$, adjusted to the stable Hamiltonian structures $\mathcal{H}_\pm = (\omega_\pm, \lambda_\pm)$ on these ends. In the “cobordim region” $(-1, 1) \times Z$, $\hat{J}$ tames the exact symplectic 2-form
\[
\Omega := dx \wedge dy + d\tau \wedge dH.
\]
Although $\hat{J}$ is not $\mathbb{R}$-invariant, the following cylinder
\[
C := \{(a, \tau, 0) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times D \mid a \in \mathbb{R}, \tau \in \mathbb{R}/\mathbb{Z} \}
\]
is a pseudoholomorphic curve in $(W, \hat{J})$. Its positive puncture is asymptotic to $\gamma_+$ and its negative puncture is asymptotic to $\gamma_-$. Although $C$ is not an orbit cylinder, it has the following properties in common with one:
\[
C \cdot C = -1,
\]
where $\cdot$ refers to the generalized intersection number in the sense of Siefring [35], $C$ is embedded, has Fredholm index $\text{ind}(C) = 0$, and is Fredholm regular. Indeed [34] follows from the adjuntion formula, theorem 4.6, in Siefring [35].

We will show in section 7.5 that after a small perturbation of $\hat{J}$ on a neighborhood of the boundary points $[-1, 1] \times \partial Z$ in $W$, that $\hat{J}$ is Levi-flat on the boundary of $W$. More precisely, $\mathbb{R} \times \partial Z$ is filled by a set $\mathcal{S}$ of immersed $\hat{J}$-holomorphic planes which in the ends $E_+ \cap (\mathbb{R} \times \partial Z)$ coincide with the product of the $\mathbb{R}$-component and a Reeb trajectory of $R_\pm$. Let us assume that such a perturbation has been made.

We proceed to construct $\mathcal{F}^+$ in four steps.

**Step 1:** The almost complex structure $\hat{J}$ on $W$ satisfies $\hat{J}|_{E_+} = J_+|_{E_+}$ where $J_+$ is the cylindrical almost complex structure
\[
\begin{align*}
J_+ \partial_{\mathbb{R}} &= \partial_{\tau} + 2\pi \alpha \partial_{\theta} \\
J_+|_{TD} &= i
\end{align*}
\]
on $\mathbb{R} \times Z$, in standard polar coordinates $(r, \theta)$ on the disk. For each $c \in \mathbb{R}$ and $z \in \partial D$, the map
\[
\tilde{u}_{c, z} : \mathbb{R}^+ \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times Z \\
\tilde{u}_{c, z}(s, t) = (s + c, t, ze^{2\pi([\alpha]-\alpha)s}e^{2\pi i[\alpha]t})
\]
is $J_+$-holomorphic. The combined images of these maps along with the cylinder
\[
C_+ := \{(a, \tau, 0) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times D \mid a \in \mathbb{R}, \tau \in \mathbb{R}/\mathbb{Z} \}
\]
defines an \( \mathbb{R} \)-invariant finite energy foliation for \((\mathbb{R} \times \mathbb{Z}, J_+)\) with boundary index \(\lfloor \alpha \rfloor\). A direct calculation shows that if \(F_1\) and \(F_2\) are the images of two curves in \(\mathbb{R}\), then
\[
F_1 \cdot F_2 = 0 \quad \text{and} \quad F_1 \cdot C_+ = 0.
\]
See appendix A.3 for a discussion of the generalized intersection number for pairs of curves with boundary.

**Step 2:** We return to the manifold \((W = \mathbb{R} \times \mathbb{Z}, \hat{J})\) with cylindrical ends. Let \(\mathcal{M}\) denote the moduli space of all finite energy \(\hat{J}\)-holomorphic curves \(F \subset W\) which admit a \(\hat{J}\)-holomorphic parameterization by a map \(\hat{u} = (a, \tau, z) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}, \mathbb{R} \times \mathbb{Z})\) satisfying
\[
\begin{align*}
\hat{u}(0, \cdot) &\in L_c \\
\tau(0, \cdot) &: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \text{ has degree } +1 \\
z(0, \cdot) &: \mathbb{R}/\mathbb{Z} \to \partial D \text{ has degree } \lfloor \alpha \rfloor
\end{align*}
\]
for some \(c \in \mathbb{R}\). Equip \(\mathcal{M}\) with the topology coming from convergence in \(C^\infty_{\text{loc}} \cap C^0([0, \infty) \times \mathbb{R}/\mathbb{Z}, W)\). \(\mathcal{M}\) is non-empty as it contains the image of each curve \(\hat{u}_{c, z}\) from \(\mathbb{R}\) for which \(c \geq 1\), as \(\hat{J} = J_+\) on the positive end \(E_+ = [1, \infty) \times \mathbb{Z}\). Let \(\mathcal{M}_0 \subset \mathcal{M}\) be the connected component containing the curves \(\hat{u}_{c, z}\) from \(\mathbb{R}\) for which \(c \geq 1\).

Recall that the boundary of \(W\) has a filling \(S\) by \(\hat{J}\)-holomorphic planes. Positivity of intersections between the curves in \(\mathcal{M}_0\) and those in \(S\) imply the following.

**Lemma 27.** Suppose \(F \in \mathcal{M}_0\). Then
\begin{enumerate}
\item \(F\) meets \(\partial W\) transversely along \(\partial F\).
\item \(F \setminus \partial F\) lies in \(W \setminus \partial W\).
\item \(\partial F\) is an embedded closed loop.
\end{enumerate}

**Proof.** These conditions are open in the topology on \(\mathcal{M}\). It suffices to show then that these conditions are also closed. That is, suppose that \(F_k \in \mathcal{M}_0\) is a sequence converging to \(F \in \mathcal{M}_0\), and that each \(F_k\) satisfies conditions (1), (2), and (3). We wish to show that \(F\) also satisfies these properties.

Take any point \(q \in \partial F\). This is a limit of points \(q_k \in \partial F_k\). By assumption each \(q_k\) is an isolated transverse intersection point between \(F_k\) and a leaf \(S_k \in S\). The local intersection index at \(q_k\) is \(+1\) (or \(+1/2\) depending on conventions) as transverse intersections between immersed curves.

Let \(S \in S\) be the unique leaf containing \(q\). It is not too hard to show from this that \(q\) must also be an isolated intersection point of \(F\) with \(S\), and that the local intersection index is \(+1\). Indeed, as each leaf in \(S\) is locally embedded, one can get away with using the similarity principle after viewing each \(F_k\) as a graph over \(S_k\) near \(q_k\). The only slightly tricky point
is to show that other intersections between $F_k$ and $S_k$ do not accumulate at $q$. In the target they will indeed accumulate, but in the domains of the curves they will remain isolated uniformly in $k$. This latter can be seen by considering the pairs of 1-dimensional parameterized curves $f_k$ and $s_k$ in $\partial Z$ where $F_k$ and $S_k$ intersect $L_{c_k}$ respectively. The curves $f_k$ and $s_k$ intersect transversely and all intersections have the same sign, because each $F_k$ is assumed to be in $M_0$. For topological reasons this makes it impossible for intersections between $f_k$ and $s_k$ to accumulate, from the point of view of their domains.

The local intersection index of $+1$ at $q \in \partial F$, between $F$ and $S$, then implies that $F$, viewed locally as a graph over $S$, corresponds to a zero of the graph of order $1$. This implies that $F$ is immersed at $q$ and is transverse to $S$ at $q$, and therefore also transverse to $\partial W$ at $q$.

Similar arguments, again using positivity of intersections between the curves in $M_0$ and those in $S$ allow to conclude that $F$ can have no boundary-boundary double points, and that $F \setminus \partial F$ is disjoint from $\partial W$. This shows that $F$ satisfies conditions (1),(2), and (3).

Now that our curves have these nice properties in relation to the boundary of $W$, we can use the homotopy invariant intersection number and adjunction formula from the appendix A.3.

The explicit curves $\tilde{u}_{c,z}$ from step 1 have Fredholm index 2 and are clearly embedded. Therefore by the adjunction formula $\tilde{u}_{c,z} \cdot \tilde{u}_{c,z} = 0$. For all $F \in M_0$, $F$ is homotopic to the image of $\tilde{u}_{c,z}$, for any $c \geq 1$. More precisely homotopic through curves in the space $C^\infty(\gamma_+, \partial W)$ described in appendix A.3. Thus $F \cdot F = 0$

for each $F \in M_0$.

**Lemma 28.** The set $E := \{w \in W \mid w \in F \text{ for some } F \in M_0\}$ is a non-empty open and closed subset of $W \setminus C$.

**Proof.** $M_0$ is non-empty by definition.

Openness: each curve $F$ in $M_0$ has Fredholm index $\text{ind}(F) = 2$. The last lemma showed that $F$ is embedded on its boundary and meets $\partial W$ transversely. Moreover we just saw that $F \cdot F = 0$. Thus proposition 39 applies, and the curves in $M_0$ fill up an open neighborhood of $F$ in $W$.

Closedness: suppose that $p_k \in E$ is a sequence of points converging to a point $p_\infty \in W \setminus C$. Let $F_k \in M_0$ be a curve containing $p_k$, and view $p_k$ as the image of a marked point. The total energy of $F_k$ is uniformly bounded in $k$ and we find a converging subsequence in the sense of 4. In proposition 32 the limit is found to be a height-1 building $F_\infty$ say, that is also a curve in $M_0$. $F_\infty$ must contain $p_\infty$, so $p_\infty \in E$.

Thus each point in $W \setminus C$ contains a curve in $M_0$.

**Step 3:** Let $p_k \in W \setminus C$ be a sequence of points which converge to a point $p_\infty \in C$. Let $F_k \in M_0$ be a sequence of curves with $p_k \in F_k$. As in step 2
these curves have uniformly bounded total energy because the asymptotic data $\gamma_+$ is fixed, as is the homology class in $\partial W$ in which the boundaries of the curves lie in. View each $p_k$ as the image of a marked point of $F_k$. Applying the compactness theory in [4] we get a convergent subsequence to a stable nodal holomorphic building $\bar{F}$ say. By proposition 32 $\bar{F}$ is a height-2 building, with non-empty lower level a half cylinder $F_-$ in the cylindrical manifold $(\mathbb{R} \times Z, J_-)$. $F_-$ is asymptotic to $\gamma_-$ as a positive puncture and has boundary index $\lfloor \alpha \rfloor$. Moreover,

$$F_- \cdot F_- = 0 \quad \text{and} \quad F_- \cdot C_- = 0$$

where $C_-$ is the orbit cylinder in $(\mathbb{R} \times Z, J_-)$ over $\gamma_-$. From lemma 6 $F_-$ is embedded and transverse to $\mathbb{R} \times \partial Z$.

**Step 4:** We may now argue in the manner of step 2, and conclude that $F_-$ lies in a non-empty moduli space of half cylinders $\mathcal{M}_-$ which fills an open and closed subset of $(\mathbb{R} \times Z) \setminus C_-$. By lemma 6 each curve in $\mathcal{M}_-$ is embedded and meets the boundary $\mathbb{R} \times \partial Z$ transversely. By the homotopy invariance of the generalized intersection number, the properties

$$(37) \quad F_0 \cdot F_1 = 0 \quad \text{and} \quad F_0 \cdot C_- = 0$$

for all $F_0, F_1 \in \mathcal{M}_-$, are inherited from $F_-$. By proposition 39 then, $\mathcal{M}_-$ is a smooth foliation of $(\mathbb{R} \times Z) \setminus C_-$ by embedded curves. Now we set $\mathcal{F}^+ := \mathcal{M}_- \cup C_-$. Both properties follow from (37).

The moduli space $\mathcal{M}_-$ is by definition $\mathbb{R}$-invariant. To prove uniqueness it suffices to show that if $G$ is any $J_-$-holomorphic half cylinder in $\mathbb{R} \times Z$ that is asymptotic to $\gamma_-$ as a positive puncture, and has boundary behavior as in (36), then $G$ is in the moduli space $\mathcal{M}_-$. First, by lemma 6 $G$ is embedded and transverse to $\mathbb{R} \times \partial Z$. Therefore it is homotopic to any leaf $F$ in $\mathcal{M}_-$ through curves along which the intersection number remains constant. Thus,

$$F \cdot G = F \cdot F = 0$$

for all $F \in \mathcal{M}_-$. In particular choose $F \in \mathcal{M}_-$ sharing a point with $G$. Then $F$ and $G$ are not disjoint but have $F \cdot G = 0$. Therefore they are equal, and so $G \in \mathcal{F}^+$.

### 7.2. Construction of $\mathcal{F}^+$ without boundary restrictions

This completes the construction for any irrational pseudo-rotation $\varphi : D \to D$ which restricts to a circle diffeomorphism $\varphi|_{\partial D}$ on the boundary that is smoothly conjugate to a rigid rotation. A deep result of Herman implies that the set of such boundary conditions is dense, and a further limiting step can remove these restrictions. We explain this now.
The set of Diophantine numbers $D \subset \mathbb{R}$ is defined as follows. Say that $\alpha \in D$ if there exists $n \geq 2$ and $C \in (0, \infty)$ such that for all $(p, q) \in \mathbb{Z} \times \mathbb{N}$
\[
\left| \alpha - \frac{p}{q} \right| \geq C \frac{1}{q^n}.
\]
It is easy to show that $D$ is dense in $\mathbb{R}$ (indeed of full measure). Denote by $\text{Diff}_D(\partial D)$ those orientation preserving $C^\infty$-smooth circle diffeomorphisms that have rotation number in $D$.

**Theorem 29** (Herman [16]). If $f \in \text{Diff}_D(\partial D)$ then there exists $\sigma \in \text{Diff}_\infty^+ (\partial D)$ such that $\sigma^{-1} f \sigma = R$ where $R : \partial D \to \partial D$ is a rigid rotation.

**Lemma 30.** $\text{Diff}_D(\partial D)$ is dense in $\text{Diff}_\infty^+ (\partial D)$ with the $C^\infty$-topology.

**Proof.** Fix any $f \in \text{Diff}_D(\partial D)$. Consider the continuous path $f_t := R_{2\pi t} \circ f \in \text{Diff}_+^\infty (\partial D)$ over $t \in [0, 1/2]$. The rotation numbers $\text{Rot}(f_t)$ vary continuously with $t$. Moreover, there is a continuous family of lifts $\tilde{f}_t : \mathbb{R} \to \mathbb{R}$ with the monotonicity property that for $t > 0$, $\tilde{f}_t(x) > f_0(x)$ for all $x \in \mathbb{R}$. Therefore, since $\text{Rot}(f_0)$ is irrational, it follows that for all $t \in (0, 1/2]$, $\text{Rot}(f_t) > \text{Rot}(f_0)$. See for example proposition 11.1.9 in [28]. As $D$ is dense in $\mathbb{R}/\mathbb{Z}$, we find a sequence $t_j \in (0, 1/2]$ converging to zero such that $\text{Rot}(f_{t_j}) \in D$. 

Now suppose that $\varphi \in \text{Diff}^\infty(D, \omega_0)$ is any smooth irrational pseudo-rotation. Let $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ be a Hamiltonian with time-one map $\varphi$. Using the lemma one can find smooth perturbations $H_j \in C^\infty(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R})$ of $H$ near the boundary $\mathbb{R}/\mathbb{Z} \times \partial D$, so that
\[
H_j \to H \text{ in } C^\infty \ni \text{Rot}(\varphi_j|_{\partial D}) \in D
\]
for all $j$, where $\varphi_j$ is the time-one map of $H_j$. For $j$ sufficiently large $\varphi_j$ has no fixed points besides the origin, which of course remains non-degenerate.

**Remark 10.** We do not claim that we can find perturbed disk maps which are also irrational pseudo-rotations. In general a perturbation will create new periodic points with high period.

By Herman’s theorem $\varphi_j|_{\partial D}$ is smoothly conjugate to a rigid rotation. Therefore, by steps 1 to 4 in section 7.1 we can find a finite energy foliation $\mathcal{F}_{H_j}$ of the cylindrical almost complex manifold $(\mathbb{R} \times Z, J_{H_j})$.

**Remark 11.** Note that we are applying remark 8 here. To know that each foliation $\mathcal{F}_{H_j}$ exists even though $\varphi_j$ is not exactly an irrational pseudo-rotation.

The almost complex structures $J_{H_j}$ converge uniformly to $J_H$ as $j \to \infty$, and we may pass to a limit of foliations and obtain a finite energy foliation for $(\mathbb{R} \times Z, J_H)$. For example we can take a limit of a single sequence of half cylinder leaves $F_j \in \mathcal{F}_{H_j}$ to obtain a single $J_H$-holomorphic half cylinder
$F_{\infty}$ disjoint from the orbit cylinder $C$ and with vanishing self intersection number. The same argument as in step 4 shows that a moduli space containing $F_{\infty}$ fills up the complement of $C$ by embedded curves that combine to form a smooth $\mathbb{R}$-invariant foliation. Uniqueness is also as in step 4.

7.3. Constructing $\mathcal{F}^-$. The construction of $\mathcal{F}^-$ is along exactly the same lines as for $\mathcal{F}^+$, but beginning with a different model foliation in step 1. Indeed, in place of the curves in (33), each of which is bounded from below, we use curves of the form

$$v_{c,z} : \mathbb{R}^- \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times \mathbb{Z}$$

over $c \in \mathbb{R}$ and $z \in \partial D$. (These maps are also pseudoholomorphic with respect to the almost complex structure in (32).) Each $v_{c,z}$ has image bounded from above, so we have to insert one of them into the negative end of our almost complex manifold ($W = \mathbb{R} \times \mathbb{Z}$). So the main difference is that from the start we reverse the roles of $H_-$ and $H_+$, this time picking a positive constant $C$ in (29) so that $\min H_- > \max H_+$ still holds. Then the remaining steps are exactly analogous, and in the final foliation the curves with boundary have boundary index $\lceil \alpha \rceil$ instead, as the curves in (38) do.

7.4. Compactness. In section 7.1 we applied the compactness theory in [4] in the form of proposition 32 below. We state and justify this now.

Recall that $(W = \mathbb{R} \times \mathbb{Z}, \tilde{J})$ from section 7.1 has cylindrical ends; that the Reeb flow on the positive and negative ends each has a unique simply covered periodic orbit $\gamma_+$ and $\gamma_-$ respectively. Although $\tilde{J}$ is not $\mathbb{R}$-invariant, the cylinder

$$C = \{(a, \tau, 0) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{D} \mid a \in \mathbb{R}, \tau \in \mathbb{R}/\mathbb{Z}\}$$

is a pseudoholomorphic curve in $(W, \tilde{J})$. Its positive puncture is asymptotic to $\gamma_+$ and its negative puncture is asymptotic to $\gamma_-$. We observed in (31) that

$$C \cdot C = -1.$$ 

Consider a sequence $F_k$ of $\tilde{J}$-holomorphic curves in $(W, \tilde{J})$ with a single marked point whose image we denote by $p_k \in W$. We suppose that each $F_k$ is the image of a $\tilde{J}$-holomorphic embedding $\tilde{u}_k = (a_k, \tau_k, z_k) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}, \mathbb{R} \times \mathbb{Z})$ that is asymptotic to $\gamma_+$ and has the boundary conditions described by (36). Furthermore, we assume that

$$F_k \cdot F_k = 0 \quad \text{and} \quad F_k \cdot C = 0$$

for each $k$.

Suppose that $F_k$ converges to a generalized nodal holomorphic building $\tilde{F}$ in the sense of [4].

**Lemma 31.** The building $\tilde{F}$ has no disk components.
Proof. Arguing indirectly suppose that $\bar{F}$ has a disk component $D$. Suppose first that $D$ is in the middle level $(W, \hat{\mathcal{J}})$. Then $\partial D$ must be a closed loop in one of the totally real surfaces $L_c \simeq \partial Z$ say. As $\partial D$ is contractible in $W$ it must lie in a homology class $m \partial D \in H_1(\partial Z)$ for some $m \in \mathbb{Z}$. The topological count of intersections between $D$ and the cylinder $C$ is $m$. The building is stable with at most one marked point implies that the component $D$ is not just a point. Therefore it has non-zero energy, which implies that $m \neq 0$. Therefore $D$ and $C$ are distinct holomorphic curves with interior intersections. Therefore $F_k$ and $C$ intersect for large $k$, contradicting that they are in fact disjoint for all $k$.

If instead $D$ is in an upper or lower level a similar argument proves that $D$ has isolated interior intersections with the unique orbit cylinder in the relevant level. This again implies that $F_k$ and $C$ intersect for large $k$. ■

Proposition 32. Suppose the marked points $p_k$ converge to $p_\infty \in W$.

1. If $p_\infty \notin C$, then $\bar{F}$ is a height-1 building (i.e. there is no breaking). This building has a single component, $C$. $C$ is a $\hat{\mathcal{J}}$-holomorphic half cylinder with positive puncture asymptotic to $\gamma_+$ and the same boundary index as each $F_k$.

2. If $p_\infty \in C$, then $\bar{F}$ has precisely two non-empty levels. The middle level is equal to the cylinder $C$, the top levels are empty, and the lower level is a single component curve $F_\sim$. $F_\sim$ is a $\mathcal{J}_-\sim$-holomorphic half cylinder with positive puncture asymptotic to $\gamma_-$ and the same boundary index as each $F_k$. Moreover, $F_- \cdot C_- = 0$ where $C_-$ is the orbit cylinder over $\gamma_-$, and $F_- \cdot F_- = 0$.

Proof. Any bubbling off on the boundary would result in $\bar{F}$ having a disk component, which is ruled out by lemma \[31\]. Any interior bubbling would result in a component of $\bar{F}$ that is a finite energy plane, which is impossible as there are no contractible periodic orbits. Any nodes in $\bar{F}$ would imply $\bar{F}$ has a component that is a plane or a closed curve. But any closed curve would be constant as the symplectic form $\Omega$ in the cobordism region is exact. However, there are not enough marked points for $\bar{F}$ to have any constant components and yet be stable.

We are therefore reduced to the following two scenarios.

Case 1: Suppose no breaking occurs. Then $\bar{F}$ is a height-1 holomorphic building, with a single component, and this component must be a half cylinder asymptotic to $\gamma_+$. The boundary index must be the same for each $F_k$ as no disks bubbled off.

It follows that each $F_k$ is homotopic to the limiting curve $F$ (through curves asymptotic to $\gamma_+$). Thus $F \cdot C = F_k \cdot C = 0$. Similarly $F \cdot F = F_k \cdot F_k = 0$ for all $k$.

Case 2: Suppose breaking does occur. Then, as there is only one periodic orbit in the negative end, the middle level $F_0$ of $\bar{F}$ must be a single component that is a cylinder with punctures of opposite sign. The positive puncture is asymptotic to $\gamma_+$ and the negative puncture asymptotic to $\gamma_-$. 


Therefore $F_0$ is homotopic to $C$ through curves with fixed end points, and so $F_0 \cdot C = C \cdot C = -1$ by (39), which implies that $F_0$ is a covering of $C$ and therefore equals $C$.

The marked point must be on $F_0$. Therefore there are no orbit cylinders in the lower or upper levels. Thus, $\tilde{F}$ is a height-2 holomorphic building, with middle level $F_0$ as described and lower level $F_-$ a single component that is a half cylinder asymptotic to connecting orbit $\gamma_-$ as a positive puncture. Again the boundary index of $F_-$ must be the same as for each $F_k$ as no disks bubbled off.

By assumption $F_k$ converges to the building $\tilde{F}$ in the sense of [4]. The constant sequence $C_k := C$ also converges to the (unstable) building $C$ which consists of a single middle level equal to the cylinder $C$ and a single lower level equal to the orbit cylinder $C_-$ and empty upper level. In particular both buildings have a unique connecting orbit $\gamma_-$, and this is simply covered.


$$\lim_{k \to \infty} F_k \cdot C_k = C \cdot C + F_- \cdot C_- + p(\gamma_-)$$

where $p(\gamma_-) \in \{0, 1\}$ denotes the parity of the Conley-Zehnder index of $\gamma_-$. Due to lemma 36 $\gamma_-$ is elliptic, so $p(\gamma_-) = 1$. By assumption $F_k \cdot C_k = 0$ for each $k$, and we observed in (39) that $C \cdot C = -1$. We conclude then from (40) that $F_- \cdot C_- = 0$.

The formula that we applied to $F_k$ and $C_k$ to obtain (39) holds equally well for self intersections. Applied to $F_k$ this yields

$$0 = \lim_{k \to \infty} F_k \cdot F_k = C \cdot C + F_- \cdot F_- + p(\gamma_-).$$

The left hand side vanishes, $C \cdot C = -1$ [39], and $p(\gamma_-) = 1$. So $F_- \cdot F_- = 0$.

7.5. **Foliating the boundary.** Finally we prove the existence of the foliation $\mathcal{S}$ of the boundary of $\mathbb{R} \times Z$ that was used in section 7.1.

For each $a \in \mathbb{R}$, consider for a fixed value of $a$ the resulting time-dependent Hamiltonian on the disk $H_a$ given by

$$H_a := H(a, t, \cdot) : D \to \mathbb{R}$$

over $t \in \mathbb{R}/\mathbb{Z}$. By modifying $H_a$ on any arbitrarily small neighborhood of the boundary of the disk we may arrange that the time-one map of the path of generated Hamiltonian disk maps is any prescribed orientation preserving diffeomorphism on the boundary of the disk. By extension, given any smooth path $a \mapsto f_a \in \text{Diff}_+^\infty(\partial D)$ over $a \in \mathbb{R}$, satisfying

$$f_a = \begin{cases} 
R_{2\pi a} & \text{if } a \geq 1 \\
\varphi & \text{if } a \leq -1,
\end{cases}$$

the function $H : \mathbb{R} \times Z \to \mathbb{R}$ may be modified on any small neighborhood of $[-1, 1] \times \partial Z$ so that for each $a \in \mathbb{R}$ the time-one map of the modified Hamiltonian $H_a$ now coincides with $f_a$ on the boundary of the disk.
Suppose that we can find a smooth path \( a \mapsto f_a \in \text{Diff}^\infty_+(\partial D) \) satisfying (12), and which additionally has the property that each \( f_a \) is smoothly conjugate to the rigid rotation \( R_{2\pi a}|_\partial D \). More precisely, suppose that we find a smooth map \( g \in C^\infty(\mathbb{R} \times \partial D, \partial D) \), so that for each \( a \in \mathbb{R} \) the map \( g_a := g(a, \cdot) \) is an element of \( \text{Diff}^\infty_+(\partial D) \), and with the property that the path \( f_a := g_a R_{2\pi a} g_a^{-1} \) satisfies (12). Then we may modify \( H \in C^\infty(\mathbb{R} \times Z, \mathbb{R}) \) near \([-1, 1] \times \partial Z \) so that for each \( a \in \mathbb{R} \) the time-one map of the time-dependent Hamiltonian \( H_a := H(a, \cdot) \) equals \( f_a \) on the boundary of the disk. For each \( a \in \mathbb{R} \) let

\[
\phi_a : \mathbb{R} \times \partial Z \to \partial Z
\]

\[
(t, z) \mapsto \phi_a^t(z)
\]

denote the 1-parameter family of maps generated by \( X_{H_a,t} \) on \( \partial D \). So in particular \( \phi_a^0 = f_a \) for all \( a \in \mathbb{R} \). Now for each \( z \in \partial D \),

\[
S_z := \left\{ (a, t, g_a(\phi_a^t(z))) \in \mathbb{R} \times \mathbb{R} / Z \times \partial D \mid a \in \mathbb{R}, \ t \in \mathbb{R} \right\}
\]

is an immersed surface in \( \mathbb{R} \times \partial Z \), and the union

\[
S := \bigcup_{z \in \partial D} S_z
\]

is a foliation of \( \mathbb{R} \times \partial Z \). As \( \alpha \) is irrational each \( S_z \) is dense in \( \mathbb{R} \times \partial Z \). However, the relation

\[
f_a := g_a R_{2\pi a} g_a^{-1}
\]

for all \( a \in \mathbb{R} \) enables us to find a \( C^\infty \)-smooth almost complex structure \( J' \) on \( \mathbb{R} \times Z \), prescribed at points on \( \mathbb{R} \times \partial Z \) so that each \( S_z \) has \( J' \)-invariant tangent bundle. Indeed, differentiating the expression \( (a, t, g_a(\phi_a^t(z))) \) in \( a \) gives a vector field \( V_1 \) say, while differentiating it in \( t \) results in a vector field \( V_2 \). Both are non-vanishing and transverse as we will see, so we can set \( J' V_1 = V_2 \). That \( V_1 \) is indeed a well defined vector field uses (13). Moreover, one finds that:

\[
V_2 = \partial_t + X_{H_0^a}^z,
\]

while

\[
V_1 = \partial_a + V_3,
\]

for some \( V_3 \) on \( \mathbb{R} \times \partial Z \) that has no \( \partial_a \) component, and tends to zero in \( C^0 \) as \( \| \partial_a f_a \|_{C^0} \) tends to zero. We can arrange that \( \| \partial_a f_a \|_{C^0} \) is as small as we wish by “slowing everything down”, that is, replacing the interval \([-1, 1] \times Z \) by \([-N, N] \times Z \) for sufficiently large \( N > 0 \). Then, from these expressions for \( V_1, V_2 \) we see that \( J' \) extends to an almost complex structure \( J'' \) on \( \mathbb{R} \times Z \) with the following properties if \( \| \partial_a f_a \|_{C^0} \) is sufficiently small:

1. \( J'' \) coincides with the almost complex structure \( J \) outside of a small neighborhood of \([-N + 1, N - 1] \times \partial Z \).
2. Each surface \( L_c := \{ c \} \times \partial Z \) is totally real with respect to \( J'' \).
3. \( J'' \) is tamed by the symplectic form \( \Omega \) on \((-N, N) \times Z \).
And finally of course $S$ is a $J^\nu$-holomorphic filling of the boundary $\mathbb{R} \times \partial Z$.

The only remaining question is when the relation (43) can be arranged for all $a \in \mathbb{R}$. But this holds if and only if the circle maps $\varphi|_{\partial D}$ and $R_{2\pi a}|_{\partial D}$ are conjugate by an orientation preserving $C^\infty$-smooth diffeomorphism. Necessity is obvious, let us show sufficiency. Suppose that there exists $g \in \text{Diff}^\infty_+(\partial D)$ such that $\varphi|_{\partial D} = gR_{2\pi a}|_{\partial D}g^{-1}$. Since $g$ has degree +1 it is smoothly isotopic to the identity and we may find a smooth isotopy $g_a \in \text{Diff}^\infty_+(\partial D)$ over $a \in \mathbb{R}$, satisfying $g_a = g$ for all $a \leq -1$. Thus the smooth path $f_a \in \text{Diff}^\infty_+(\partial D)$ over $a \in \mathbb{R}$ defined by

$$f_a = \begin{cases} R_{2\pi a} & \text{if } a \geq 1 \\ \varphi|_{\partial D} & \text{if } a \leq -1 \end{cases}$$

and therefore has the properties we require.

**Appendix A.**

A.1. **Proof of proposition 2.** In this appendix we prove the following statement, which implies proposition 2. The idea of the proof was explained to me by Patrice LeCalvez.

We write $R_\theta : D \to D$ to denote the rigid rotation $z \mapsto e^{i\theta} z$ through angle $\theta \in \mathbb{R}$.

**Proposition 33.** Consider a sequence $\varphi_k \in \text{Homeo}_+(D)$ converging in the $C^0$-topology to $\varphi \in \text{Homeo}_+(D)$, where all maps fix $0 \in D$. Under the following additional assumptions it follows that $\varphi$ has no periodic points in $D \setminus \{0\}$.

1. For each $k \in \mathbb{N}$, there exists $g_k \in \text{Homeo}_+(D)$, fixing the origin, such that $\varphi_k = g_k^{-1}R_{2\pi \theta_k}g_k$, some $\theta_k \in \mathbb{R}$.
2. $\theta_k \to \theta$ as $k \to \infty$, where $\theta$ is irrational.

To prove this we need to recall the notions of positively and negatively returning disks due to Franks [13]. Let $\tilde{D} := D \cup \partial D$. Denote by $\tilde{A} := (0,1) \times \mathbb{R}$ the open annulus and its universal covering via the covering map

$$\pi : (0,1) \times \mathbb{R} \to \tilde{D} \setminus \{0\}$$

$$(x,y) \mapsto xe^{2\pi iy}.$$ 

Let $T : \tilde{A} \to \tilde{A}$ be the deck transformation $T(x,y) = (x, y+1)$. By an open disk $U \subset \tilde{A}$ is meant an open subset homeomorphic to $\tilde{D}$ with the subspace topology.

**Definition 34.** Let $f : A \to A$ be a homeomorphism homotopic to the identity, and $\tilde{f} : \tilde{A} \to \tilde{A}$ a lift via $\pi$. Consider an open disk $U \subset \tilde{A}$ for which

$$\tilde{f}(U) \cap U = \emptyset.$$
If there exist integers \( n > 0, k \neq 0 \), such that
\[
\tilde{f}^n(U) \cap T^kU \neq \emptyset
\]
then \( U \) is called a *positively*, resp. *negatively*, returning disk for \( \tilde{f} \) if \( k > 0 \), resp. \( k < 0 \).

**Remark 12.** Consider a disk \( U \subset \tilde{A} \) satisfying (44) and (45). If moreover the closure of the disk satisfies (44), that is, \( \tilde{f}(\bar{U}) \cap \bar{U} = \emptyset \), then for any sufficiently \( C^0 \)-small perturbation of \( \tilde{f} \), \( U \) will satisfy (44) and (45) for the perturbed map also.

The key result for us is the following strong generalization of the Poincaré-Birkhoff fixed point theorem, theorem 2.1 in [13].

**Theorem 35** (Franks). Let \( f : A \to A \) be a homeomorphism of the open annulus homotopic to the identity, and for which every point is non-wandering. If there exists a lift \( \tilde{f} : \tilde{A} \to \tilde{A} \) having a positively returning disk which is a lift of a disk in \( A \), and a negatively returning disk that is a lift of a disk in \( A \), then \( f \) has a fixed point.

Recall that a point \( x \in A \) is non-wandering for \( f \) if for every open neighborhood \( U \) of \( x \) there exists \( n > 0 \) such that \( f^n(U) \cap U \neq \emptyset \). It is easy to prove that every point is non-wandering for a homeomorphism \( f \) if \( f \) preserves a finite measure that is positive on open sets.

We now use theorem 35 to prove the proposition.

**Proof of proposition 33.** Arguing indirectly, suppose that \( \varphi \) has a periodic point \( z_0 \in D \setminus \{0\} \). Then we will show that some iterate of \( \varphi \), let us call it \( \varphi^n \), has a lift via the covering map
\[
\pi : (0, 1] \times \mathbb{R} \to D \setminus \{0\}
\]
\[
(x, y) \mapsto xe^{2\pi iy}
\]
to a map of the half closed infinite strip \( \tilde{\varphi}^n : (0, 1] \times \mathbb{R} \to (0, 1] \times \mathbb{R} \), having disks \( U_-, U_+ \subset (0, 1) \times \mathbb{R} \) which satisfy the following:

1. \( U_+ \) is a positively returning disk for \( \tilde{\varphi}^n \). \( U_- \) is a negatively returning disk for \( \tilde{\varphi}^n \).
2. The closures satisfy \( \tilde{f}(\bar{U}_+) \cap \bar{U}_+ = \emptyset \) and \( \tilde{f}(\bar{U}_-) \cap \bar{U}_- = \emptyset \).
3. \( U_+ \) and \( U_- \) are lifts of disks in \( \tilde{D} \setminus \{0\} \).

Suppose for a moment that we have established this. Then since \( \varphi_k : D \to D \) converges uniformly in the \( C^0 \)-topology to \( \varphi \), there exists a sequence of lifts
\[
\tilde{\varphi}_k : (0, 1] \times \mathbb{R} \to (0, 1] \times \mathbb{R}
\]
of \( \varphi_k \) which converges uniformly in the \( C^0 \)-topology to \( \tilde{\varphi} \). Hence by remark 12 there exists \( K \in \mathbb{N} \) such that for all \( k \geq K \), \( U_+ \) is a positively returning disk for \( \tilde{\varphi}_k^n \), and \( U_- \) is a negatively returning disk for \( \tilde{\varphi}_k^n \).
Recall that $U_+$ and $U_-$ lie in the interior $(0, 1) \times \mathbb{R}$, and both are lifts of disks in $\hat{D}\setminus\{0\}$. Therefore we may apply Franks’ theorem, theorem 35, to each map of the open annulus $\varphi^n_k = (\varphi_k)^n : \hat{D}\setminus\{0\} \to \hat{D}\setminus\{0\}$ for $k \geq K$. Indeed, being conjugate to a rigid rotation, every point is non-wandering for $\varphi^n_k$. We conclude that for all $k \geq K$, $\varphi^n_k$ has a fixed point in $D\setminus\{0\}$. But $\varphi^n_k$ is conjugate to the rigid rotation $R_{2\pi n \theta_k}$. So $n\theta_k \in \mathbb{Z}$ for all $k \geq K$, which contradicts the convergence of the sequence $\theta_k$ to an irrational number.

With this contradiction we will be done, and so it remains to establish that some iterate of $\varphi$ has a lift for which we can find disks $U_+, U_- \subset (0, 1) \times \mathbb{R}$ satisfying conditions (1), (2), and (3) listed above.

We are assuming that $\varphi$ has a periodic point besides $0 \in D$. From this it is not hard to see that for some $n > 0$ sufficiently large there exists a lift $\tilde{\varphi}^n : (0, 1] \times \mathbb{R} \to (0, 1] \times \mathbb{R}$ of $\varphi^n$ such that one of the following two possibilities occurs: (1) there is a fixed point $z \in \hat{D}\setminus\{0\}$ of $\varphi^n$ with a lift $(x, y) \in (0, 1] \times \mathbb{R}$, such that $\tilde{\varphi}^n(x, y) = (x, y - 1)$, and for all $y \in \mathbb{R}$, $\tilde{\varphi}^n(1, y) = (1, y')$ where $y' > y + 1$. Or (2), there is a fixed point $z \in \hat{D}\setminus\{0\}$ of $\varphi^n$ with a lift $(x, y) \in (0, 1] \times \mathbb{R}$, such that $\tilde{\varphi}^n(x, y) = (x, y + 1)$, and for all $y \in \mathbb{R}$, $\tilde{\varphi}^n(1, y) = (1, y')$ where $y' < y - 1$.

Let us consider the first situation, as the second is dealt with similarly. Any sufficiently small disk neighborhood $U_- \subset (0, 1] \times \mathbb{R}$ of $(x, y)$, satisfies $\tilde{\varphi}^n(U_-) \cap U_- = \emptyset$ and is a lift of a disk in $\hat{D}\setminus\{0\}$. Moreover $\tilde{\varphi}^n(U_-) \cap \hat{A}^{-1}(U_-)$ is non-empty as it contains the point $(x, y - 1)$. Thus $U_-$ is a negatively returning disk for $\tilde{\varphi}^n$ satisfying conditions (1), (2), and (3) above.

It remains to find a suitable positively returning disk for $\tilde{\varphi}^n$. Recall that for all $y \in \mathbb{R}$,

$$\tilde{\varphi}^n(1, y) = (1, y')$$

where $y' > y + 1$. As $\partial D$ is a compact invariant set for $\varphi^n$, there exists a point $z_1 \in \partial D$ that is non-wandering for $\varphi^n$ (e.g. take any point in the $\omega$-limit set of an orbit). Let $(1, y) \in (0, 1] \times \mathbb{R}$ be a lift of $z_1$. Then for every sufficiently small closed neighborhood $I \subset \partial \hat{A} : = \{1\} \times \mathbb{R}$ of $(1, y)$ there exists $m > 0$ and $k > 0$ such that

$$\tilde{\varphi}^n(I) \cap I = \emptyset$$

$$\tilde{\varphi}^{nm}(I) \cap \hat{T}^k(I) \neq \emptyset.$$

Taking $I$ sufficiently small we may also assume that it is a lift of an interval in $\partial D$, that is, a closed, connected, non-empty, simply connected set.

Consider the open neighborhoods of $I$ in $\hat{A}$ of the form

$$V : = \{ z \in (0, 1] \times \mathbb{R} \mid d(z, I) < \epsilon \}$$

for $\epsilon > 0$. Clearly,

$$\tilde{\varphi}^{nm}(V) \cap \hat{T}^k(V) \neq \emptyset$$
from (47) and is open in $[0, 1] \times \mathbb{R}$. And for all $\epsilon > 0$ sufficiently small we have

\[(49) \quad \tilde{\varphi}^n(\bar{V}) \cap \bar{V} = \emptyset\]

from (46). Set $U_+ := V \cap \tilde{A} = V \cap ((0, 1) \times \mathbb{R})$. From (48) we get

\[\tilde{\varphi}^{nm}(U_+) \cap T^k(U_+) \neq \emptyset.\]

From (49)

\[\tilde{\varphi}^n(U_+) \cap \bar{U}_+ = \emptyset\]

as $\bar{U}_+ = \bar{V}$. Moreover, for $\epsilon > 0$ sufficiently small $U_+$ is a disk in $\tilde{A}$ and is a lift of a disk in $D \setminus \{0\}$. Thus $U_+$ is a positively returning disk for $\tilde{\varphi}^n$ satisfying the required conditions (1), (2), and (3). $\blacksquare$

A.2. Nondegeneracy. Let $\varphi : D \rightarrow D$ be an irrational pseudo-rotation with rotation number $[\alpha] \in \mathbb{R}/\mathbb{Z}$. In this appendix we prove that the unique periodic point is non-degenerate for every iterate. More precisely:

**Lemma 36.** The linearization $D\varphi(0)$ has eigenvalues $\{e^{2\pi i \alpha}, e^{-2\pi i \alpha}\}$.

We begin with:

**Lemma 37.** Let $\pi : (0, 1] \times \partial D \rightarrow D \setminus \{0\}$ be the $C^\infty$-diffeomorphism $\pi(t, x) := \sqrt{tx}$. Then the diffeomorphism $\pi^{-1} \circ \varphi \circ \pi$ on the half open annulus $(0, 1] \times \partial D$ preserves the area form $dt \wedge d\theta$ and has an extension to a homeomorphism on the closed annulus $\tilde{\varphi} : [0, 1] \times \partial D \rightarrow [0, 1] \times \partial D$ given by

\[\tilde{\varphi}(0, x) = \left(0, \frac{D\varphi(0)[x]}{|D\varphi(0)[x]|}\right).\]

**Proof.** First, $\pi^*(dx \wedge dy) = \pi^*(rdr \wedge d\theta) = \sqrt{t}d(\sqrt{t}) \wedge d(\theta) = (1/2)dt \wedge d\theta$.

So $\tilde{\varphi}$ preserves the area form $dt \wedge d\theta$.

It remains to justify that the extension is continuous. Define $\Phi \in C^\infty(D, gl(\mathbb{R}^2))$ by

\[\Phi(z)[v] = \int_0^1 D\varphi(tz)[v]dt\]

for all $z \in D$ and $v \in \mathbb{R}^2$. As $\varphi(0) = 0$, by the fundamental theorem of calculus

\[(50) \quad \varphi(z) = \Phi(z)[z]\]

for all $z \in D$. Hence the continuous map $[0, 1] \times \partial D \rightarrow \mathbb{R}^2$ given by

\[(t, x) \mapsto \Phi(\sqrt{tx})[x]\]

is knowhere vanishing; on $t > 0$ because of relation (50) and that $\varphi$ maps only the origin to the origin; at $t = 0$ because $\Phi(0) = D\varphi(0)$ has no kernel.
A calculation shows that for all \( t > 0 \) and \( x \in \partial D \),
\[
\hat{\varphi}(t, x) = \left( |\varphi(\sqrt{tx})|^2, \frac{\varphi(\sqrt{tx})}{|\varphi(\sqrt{tx})|} \right).
\]
Using (50) we have, for \( t > 0 \) and \( x \in \partial D \),
\[
\frac{\varphi(\sqrt{tx})}{|\varphi(\sqrt{tx})|} = \frac{\Phi(\sqrt{tx})[x]}{|\Phi(\sqrt{tx})[x]|}.
\]
As we observed, the denominator in the right hand side is well defined, continuous, and nowhere vanishing, even at \( t = 0 \). Thus the right hand side of (52) defines a continuous extension of the left hand side, and at \( t = 0 \) takes the value
\[
\frac{D\varphi(0)[x]}{|D\varphi(0)[x]|}.
\]
Thus the right hand side of (51) extends continuously to \( t = 0 \), and the extended map satisfies
\[
\hat{\varphi}(0, x) = \left( 0, \frac{D\varphi(0)[x]}{|D\varphi(0)[x]|} \right).
\]
This proves lemma 37. ■

A real linear map \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) defines a circle map by
\[
f_T : \partial D \to \partial D
\]
\[
v \mapsto \frac{T v}{|Tv|}.
\]
Let us record some elementary properties of this correspondence:

1. If \( T \) is an orientation preserving isomorphism then \( f_T \) is an orientation preserving homeomorphism.
2. If \( T_1 \) and \( T_2 \) are two linear maps then \( f_{T_1 \circ T_2} = f_{T_1} \circ f_{T_2} \), and moreover \( f_{id} = id_{S^1} \).
3. \( T \) has a positive real eigenvalue if and only if \( f_T \) has a fixed point if and only if \( \text{rot}(f_T) = 0 \). Therefore \( T \) has a negative real eigenvalue implies \( \text{rot}(f_T) \in \{ 0, 1/2 \} \).

Proof of lemma 36. By lemma 37 we may conjugate the map \( \varphi : D \setminus \{0\} \to D \setminus \{0\} \) via a smooth orientation preserving change of coordinates \( \pi : (0, 1) \times \partial D \to D \setminus \{0\} \) to obtain an area preserving map on the half open annulus that extends to a homeomorphism on the closed annulus
\[
\hat{\varphi} : [0, 1] \times \partial D \to [0, 1] \times \partial D.
\]
Moreover, on the “new” boundary \( \{0\} \times \partial D \) the extended map is the circle map induced by the differential of \( \varphi \) at the removed fixed point. That is,
\[
\hat{\varphi}(0, x) = \left( 0, \frac{D\varphi(0)[x]}{|D\varphi(0)[x]|} \right)
\]
for all $x \in \partial D$. The annulus map $\hat{\varphi}$ preserves Lebesgue measure and is homotopic to the identity and so the Poincaré-Birkhoff fixed point theorem applies. That is, since $\varphi$ has no periodic points on $D \setminus \{0\}$ the map $\hat{\varphi}$ has no periodic points in $(0, 1] \times \partial D$, so the restriction of $\hat{\varphi}$ to its two boundary components give circle homeomorphisms with equal rotation numbers. In terms of $\varphi$ this is

$$\text{rot}(f_{D\varphi}(0)) = \text{rot}(\varphi|_{\partial D})$$

where $f_{D\varphi}(0)$ is the circle map determined by the linear map $T = D\varphi(0) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as described in (53) above. As $\varphi|_{\partial D}$ has no periodic points, its rotation number $[\alpha] := \text{rot}(f_{D\varphi}(0)) \in \mathbb{R}/\mathbb{Z}$ is irrational. Therefore, $\text{rot}(f_{D\varphi}(0))$ is irrational and so $D\varphi(0)$ cannot have any real eigenvalues from the discussion immediately following (53) above. Hence, being symplectic, the eigenvalues of $D\varphi(0)$ must be of the form

$$\sigma(D\varphi(0)) = \{e^{2\pi i \theta}, e^{-2\pi i \theta}\}$$

for some $\theta \notin (\mathbb{Z} \cup 1/2\mathbb{Z})$. From the real canonical form, $D\varphi(0)$ is therefore similar to the rotation through $2\pi \theta$ and also to the rotation through $-2\pi \theta$. Since these two rotations are themselves conjugate to each other via an orientation reversing linear map, we conclude that there always exists an orientation preserving real linear map $P : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$P^{-1}D\varphi(0)P = R_{2\pi \theta} \quad \text{or} \quad P^{-1}D\varphi(0)P = R_{-2\pi \theta}.$$

Therefore, $[\alpha] = \text{rot}(f_{D\varphi(0)}) = \text{rot}(f_{R_{\pm 2\pi \theta}}) = \pm \theta$. Meaning that $[\alpha] = [\theta]$ or $[\alpha] = -[\theta]$. So by (54) the eigenvalues of $D\varphi(0)$ are $\{e^{2\pi i \alpha}, e^{-2\pi i \alpha}\}$. ■

A.3. Intersection theory. The intersection number $F \cdot G$ between pairs of closed oriented surfaces $F$ and $G$ in a closed four manifold is of course a homological invariant. For pseudoholomorphic curves it is especially useful since its vanishing implies the two curves are either equal or disjoint due to the positivity phenomenon in [32]. Moreover, via the so called adjunction formula, the embeddedness properties of a surface are related to its self-intersection number which is homotopy invariant.

Siefring showed that a homotopy invariant intersection number can also be associated to pseudoholomorphic curves with punctures asymptotic to nondegenerate periodic orbits [35], that retains many of the properties that hold for closed curves.

In this appendix we state without proof the properties of an intersection number that we used in section 7 for curves with a boundary component. We also state a corresponding adjunction formula. A more detailed explanation will appear in [5].

We use the notation from section 7.1. Fix an homology class $A \in H_1(\partial Z, \mathbb{Z})$. Let $C^\infty(\gamma_+, \partial W, A)$ denote the set of maps $\tilde{u} \in C^\infty(\mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}, W)$ equipped with the $C^\infty_{\text{loc}} \cap C^0$ topology which satisfy:

- $\tilde{u}(0, \cdot) \in L_c$ for some $c \in \mathbb{R}$ (not fixed), and represents $A$. 


• \( \tilde{u} \) is asymptotically cylindrical in the sense of [35], with asymptotic orbit \( \gamma_+ \).
• \( \tilde{u} \) takes interior points to \( W \setminus \partial W \).
• \( \tilde{u} \) meets \( \partial W \) transversely along the loop \( \tilde{u}(0, \cdot) \).

Remark 13. Recall that \( \gamma_+ \) lies in the positive end of \( (W, \tilde{J}) \), and that it is nondegenerate and simply covered.

The homotopy invariant intersection index of Siefring can be extended to these curves with boundary. This allows us to associate to any pair of curves \( \tilde{u}, \tilde{v} \in C^\infty(\gamma_+, \partial W, A) \) a half integer \( \tilde{u} \cdot \tilde{v} \in \frac{1}{2} \mathbb{Z} \) with the following properties:

1. Symmetry: \( \tilde{u} \cdot \tilde{v} = \tilde{v} \cdot \tilde{u} \).
2. Homotopy invariance: For a continuous path \( \tilde{u}_\tau \) in \( C^\infty(\gamma_+, \partial W, A) \), \( \tilde{u}_\tau \cdot \tilde{v} \) is independent of \( \tau \).
3. Positivity: Suppose \( \tilde{u} \) and \( \tilde{v} \) are also \( \tilde{J} \)-holomorphic, embedded on their boundary circles, and do not have identical images. Then, \( \tilde{u} \cdot \tilde{v} \geq 0 \), and equality implies that the two curves have disjoint images.

Remark 14. This is more delicate for pairs of curves \( \tilde{u} \in C^\infty(\gamma_+, \partial W, A) \) and \( \tilde{v} \in C^\infty(\gamma_+, \partial W, B) \) for which \( A \neq B \). We will not consider these cases.

Following [37, 38], which is for curves without boundary, we define the normal Chern number of a \( \tilde{J} \)-holomorphic map \( \tilde{u} \in C^\infty(\gamma_+, \partial W, A) \), to be the integer

\[
(55) \quad c_N(\tilde{u}) = \text{ind}(\tilde{u}) - 1 - p(\gamma_+).
\]

Here \( p(\gamma_+) \) denotes the parity of the Conley-Zehnder index of \( \gamma_+ \), and \( \text{ind}(\tilde{u}) \) is the Fredholm index of the linearized Cauchy-Riemann operator at \( \tilde{u} \) viewed as a solution of the free boundary problem.

In our simple situation expression (55) seems hardly to warrant a name. But it allows us to draw parallels with the general situation for curves without boundary. The terminology arises, see [37], because if \( \tilde{u} \) is immersed then \( c_N(\tilde{u}) \) has an interpretation as a relative first Chern number of a normal bundle over \( \tilde{u} \). In the case at hand this can be described as follows.

Let \( \zeta \subset T(\partial W) \) denote the unique \( \tilde{J} \)-invariant 2-plane distribution in the boundary of \( W \). Each 2-torus \( L \subset \partial W \) is totally real and therefore transverse to \( \zeta \). To the non-degenerate periodic orbit \( \gamma_+ \) one can associate an operator, the so-called asymptotic operator [19], with discrete real spectrum. Each eigenspace of this operator yields a homotopy class of sections of \( \gamma_+^*TD \rightarrow \mathbb{R}/\mathbb{Z} \). Let \( \Phi_+ \) denote the class associated to the maximal negative eigenvalue of this operator. Let \( \Phi_0 \) denote the homotopy class of sections of \( \tilde{u}(0, \cdot)^*\zeta \rightarrow \mathbb{R}/\mathbb{Z} \) that corresponds to the orientable line bundle \( l := \zeta \cap TL_\zeta \). Let \( N_{\tilde{u}} \rightarrow \mathbb{R}_+ \times \mathbb{R}/\mathbb{Z} \) be a \( \tilde{J} \)-invariant normal bundle to \( \tilde{u} \) in \( W \) which, outside of a compact set, is equal to the tangent planes to the disks, and which over the boundary points of \( \tilde{u} \) coincides with \( \zeta \). Then (55) can be interpreted...
as the first Chern number of the complex line bundle \((N_{\tilde{u}}, \hat{J}) \to \mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}\) relative to the homotopy classes \(\Phi_+\) and \(\Phi_0\).

With this interpretation of the normal Chern number, the adjunction formula in our situation for curves with embedded boundary takes the following form.

**Theorem 38.** Suppose \(\tilde{u} \in C^\infty(\gamma_+, \partial W, A)\) is \(\hat{J}\)-holomorphic and embedded on its boundary circle. Then

\[
\tilde{u} \cdot \tilde{u} = 2\delta(\tilde{u}) + c_N(\tilde{u}) \in \mathbb{Z}
\]

where \(\delta(\tilde{u}) \geq 0\) is an integer that vanishes if and only if \(\tilde{u}\) is embedded.

**Remark 15.** \(\delta(\tilde{u})\) is the count of double and singular points of \(\tilde{u}\) as defined in Siefring \[35\] and up to a factor of 2 is the same as in Micallef-White \[32\]. (There is no contribution to \(\delta(\tilde{u})\) from the periodic orbit because it is simply covered.) Note that there are no additional contributions to consider from singular or double points on the boundary as we assume \(\tilde{u}\) is embedded on the boundary. There are no interior-boundary double points because \(\tilde{u} \in C^\infty(\gamma_+, \partial W, A)\).

**Remark 16.** For general punctured pseudoholomorphic curves the adjunction formula is due to Siefring \[35\]. The expression above in \(56\) is however closer to (A.6) in \[38\]. The additional terms there have vanished here because the asymptotic periodic orbit \(\gamma_+\) is simply covered.

**Remark 17.** The adjunction formula usually requires the curve to be somewhere injective. This is automatically the case here.

The utility of this formula is most apparent when \(c_N(\tilde{u}) \geq 0\). Then the vanishing of the left hand side guarantees that \(\tilde{u}\) is embedded. This is convenient as \(\tilde{u} \cdot \tilde{u}\) is a homotopy invariant.

**Proposition 39.** Suppose that \(\tilde{u} \in C^\infty(\gamma_+, \partial W, A)\) is \(\hat{J}\)-holomorphic, embedded on its boundary circle, that \(\text{ind}(\tilde{u}) = 2\) as a solution to the free boundary problem, and that \(\tilde{u} \cdot \tilde{u} = 0\). Then

1. \(\tilde{u}\) is embedded.
2. \(\tilde{u}\) is Fredholm regular.
3. An open neighborhood of the image of \(\tilde{u}\) in \(W\) is foliated by a smooth family of embedded \(\hat{J}\)-holomorphic curves in \(C^\infty(\gamma_+, \partial W, A)\).

By Fredholm regular we mean that the linearized Cauchy-Riemann operator at \(\tilde{u}\), viewed as a solution of the free boundary problem, is surjective.

**Proof.** It follows from \(55\) that \(c_N(\tilde{u}) \geq 0\). Since \(\tilde{u}\) is embedded on the boundary we can apply the adjunction formula. Therefore \(\tilde{u} \cdot \tilde{u} = 0\) implies that \(\tilde{u}\) is embedded and that \(c_N(\tilde{u}) = 0\).

Automatic transversality arguments in \(36, 39\) then show that \(\tilde{u}\) is Fredholm regular. More precisely this is because \(\tilde{u}\) is immersed and \(c_N(\tilde{u}) < \text{ind}(\tilde{u})\).
Since moreover \( \text{ind}(\tilde{u}) = 2 \), \( \tilde{u} \) is embedded and Fredholm regular, meets the boundary transversely in an embedded totally real submanifold, and has interior disjoint from \( \partial W \), an implicit function theorem in \([36, 39]\), generalizing that in \([23]\), produces a local 2-dimensional family of \( J \)-holomorphic curves in \( C^\infty(\gamma_+, \partial W, A) \). Since also \( c_N(\tilde{u}) = 0 \) these curves foliate an open neighborhood of the image of \( \tilde{u} \).

\[ \square \]

References


Barney Bramham, Institute for Advanced Study, Princeton, NJ 08540
E-mail address: bramham@ias.edu