LECTURE II: δ-RINGS

Fix a prime p. In this lecture, we discuss some aspects of the theory of δ-rings. This theory provides a good language to talk about rings with a lift of Frobenius modulo p. Some of the material discussed below can be found in [1, 2, 3].

1. Definition and examples

To motivate the definition of a δ-ring, note that if A is a commutative ring equipped with a map $\phi : A \to A$ that is a lift for Frobenius on A/p, then for each element $f \in A$, we have an equation of the form

$$\phi(f) = f^p + p\delta$$

in A. If A is p-torsion free, then $\delta = \delta(f)$ is uniquely determined by the preceding formula, so we can regard $\delta(-)$ as an endomorphism of the set A. Moreover, the fact that $\phi(-)$ is a ring homomorphism can be encoded in terms of the behaviour of $\delta(-)$ under addition and multiplication. If A is not necessarily p-torsionfree, it is better record $\delta(-)$ instead of $\phi(-)$ as $\delta(-)$ records *why* $\phi(-)$ is a lift of Frobenius. This motivates the following:

Definition 1.1 (Joyal)**.** A *δ-ring* is a pair (A, δ) where A is a commutative ring $\delta : A \to A$ is a map of sets with $\delta(0) = \delta(1) = 0$, and satisfying the following two identities

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$

and

$$\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x + y)^p}{p} = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}.$$

There is an evident category of δ-rings. If the δ-structure on a ring A is clear from context, we often suppress it from the notation and simply call A a δ-ring. In the literature, a δ-structure is often called a *p-derivation*.

Before giving examples, let us give the obligatory lemma justifying the previous discussion.

Lemma 1.2. *Let A be a commutative ring.*

(1) If $\delta : A \to A$ provides a δ-structure on A, then the map $\phi : A \to A$ given by $\phi(f) = f^p + p\delta(f)$ defines an endomorphism of A that is a lift of the Frobenius on A/p.

(2) When A is p-torsionfree, the construction in (1) gives a bijective correspondence between δ-structures on A and Frobenius lifts on A.

From now on, given a δ-ring A, we usually write $\phi : A \to A$ for the associated Frobenius map. Note that the multiplicative identity for δ can be written asymmetrically as

$$\delta(xy) = \phi(x)\delta(y) + y^p\delta(x). \tag{1}$$

This form is often convenient in computations.

Proof. It is elementary to prove (1), i.e., to check ϕ is a ring homomorphism. Let us give the proof for additivity.

$$\phi(f + g) = (f + g)^p + p\delta(f + g) = (f + g)^p + p\delta(f) + p\delta(g) + f^p + g^p - (f + g)^p = \phi(f) + \phi(g).$$

For (2), as A is p-torsionfree, the formula $\phi(f) = f^p + p\delta$ uniquely defines $\delta = \delta(f)$ given ϕ, so it suffices to show that the function $\delta(-)$ defined this way satisfies the relevant identities, which is easy to do by hand. □

The following lemma is very useful in computations.

Lemma 1.3. *Let A be a δ-ring. Then $\phi : A \to A$ is a δ-map, i.e., $\phi(\delta(x)) = \delta(\phi(x))$ for any $x \in A$.*

Proof. We give the proof in the p-torsionfree case; the general case can be reduced to this one by Lemma 2.5 below. When A is p-torsionfree, we have $\delta(x) = \frac{\phi(x)-x^p}{p}$, so writing out $\delta(\phi(x))$ and using that ϕ is a ring homomorphism gives the required identity. □

Example 1.4. As a p-torsionfree ring A with a lift ϕ of Frobenius is a δ-ring, we obtain many easy examples such as:

(1) The ring $\mathbf{Z}$ with ϕ being the identity map. Explicitly, we have $\delta(n) = \frac{n-n^p}{p}$. In fact, it is not difficult to see that this is the initial object in the category of δ-rings: the identities on δ in a δ-ring A force the map $\mathbf{Z} \to A$ to be compatible with δ.

(2) The ring $\mathbf{Z}[x]$ with ϕ determined by $\phi(x) = x^p + pg(x)$ for any $g(x) \in \mathbf{Z}[x]$.

(3) For any perfect field k of characteristic p, the ring $W(k)$ of Witt vectors of k with ϕ being the standard lift of Frobenius. In this case, note that there is a unique lift of Frobenius, so $W(k)$ admits only one δ-structure.

(4) If A is a $\mathbf{Z}[1/p]$-algebra, then any endomorphism ϕ of A provides a δ structure on A as the condition that ϕ lift Frobenius on A/p is vacuously true.

It is slightly non-trivial to give examples of δ-rings with p-torsion, but they do exist. We shall later give a systematic source of examples via the Witt vector construction. For now, we simply mention one example:

(5) There is a unique δ-structure on the ring $\mathbf{Z}[x]/(x^p, px)$ such that $\delta(x) = 0$.

The next lemma shows that there are no p-power torsion δ-rings and its proof justifies the terminology "p-derivation" for the δ operator: it lowers p-adic order of vanishing by 1.

Lemma 1.5. *There is no nonzero δ-ring A such that $p^n = 0$ in A for some $n \geq 1$.*

Proof. Assume such a δ-ring A exists. Then A is a $\mathbf{Z}_{(p)}$-algebra. We shall check the following:

($*$) For any $u \in \mathbf{Z}_{(p)}^*$ and $m \geq 1$, we have $\delta(p^m u) = p^{m-1} v$ for some $v \in \mathbf{Z}_{(p)}^*$.

This implies the lemma by induction: if $p^m = 0$ in A, then $\delta^m(p^m)$, which is a unit in A by ($*$), is also $\delta^m(0) = 0$, whence $A = 0$.

To prove ($*$), we first observe that ($*$) holds true when $u = 1$ simply because ϕ is identity on $\mathbf{Z}$:

$$\delta(p^m) = \frac{\phi(p^m) - p^{mp}}{p} = \frac{p^m - p^{mp}}{p} = p^{m-1}(1 - p^{mp-m}),$$

which has the required form. For the general form of ($*$), using (1) and the fact that ϕ must be the identity on powers of p, we get

$$\delta(p^m u) = p^m \delta(u) + u^p \delta(p^m) = p^m \delta(u) + p^{m-1} u^p w,$$

where w is a unit in $\mathbf{Z}_{(p)}^*$ by the previous case. Simplifying, this gives

$$\delta(p^m u) = p^{m-1}(u^p w + p\delta(u)).$$

So we must show that $v = u^p w + p\delta(u)$ is in $\mathbf{Z}_{(p)}^*$. But this is clear: u^p and w are units by construction, while $p\delta(u)$ lies in the Jacobson radical. □

Remark 1.6. Specifying a δ-structure on a ring A is the same as specifying the structure of a p-typical λ-ring. We do not elaborate on this here, but it motivates the following terminology: an element $x \in A$ has *rank* 1 if $\delta(x) = 0$. In this case, we have $\phi(x) = x^p$.

2. The category of δ-rings

We want to explain some basic constructions with δ-rings. For this, it is useful to have an alternate perspective on the δ-ring structure.

Construction 2.1 (The truncated Witt vector functor)**.** For any ring A, the ring $W_2(A)$ of p-typical length 2 Witt vectors is defined as follows: we have $W_2(A) = A \times A$ as sets, and addition and multiplication are defined via

$$(x,y) + (z,w) := (x+z, y+w+\frac{x^p + z^p - (x+z)^p}{p}) \quad \text{and} \quad (x,y)\cdot(z,w) = (xz, x^p w + z^p y + pyw).$$

Ignoring the second component gives a ring homomorphism $\epsilon : W_2(A) \to A$. It is immediate from the definitions that specifying a δ-structure on A is the same as specifying a ring map $w : A \to W_2(A)$ such that $\epsilon \circ w = \mathrm{id}$: the correspondence attaches the map $w(x) = (x, \delta(x))$ to a δ-structure $\delta : A \to A$ on A.

Remark 2.2. If A is a p-torsionfree ring, then $W_2(A)$ can also be defined as the fibre product of the canonical map $A \xrightarrow{can} A/p$ with the map $A \xrightarrow{can} A/p \xrightarrow{\phi} A/p$, where ϕ is the Frobenius. Explicitly, this identification is given by sending $(x,y) \in W_2(A)$ to the pair $(x^p + py, x) \in A \times A$, noting that $x^p + py$ and $\phi(x)$ agree in A/p. We may thus view $\mathrm{Spec}(W_2(A))$ as obtained by glueing two copies of $\mathrm{Spec}(A)$ using the Frobenius on $\mathrm{Spec}(A/p) \subset \mathrm{Spec}(A)$. Note that it is evident from this interpretation that specifying a map $A \to W_2(A)$ splitting the projection down to A is the same as specifying a Frobenius lift on A.

Lemma 2.3 (Limits and colimits)**.** *The category of δ-rings has all limits and colimits, and these commute with the forgetful functor to commutative rings.*

Proof. Fix a diagram $\{A_i\}$ of δ-rings. It is easy to see that there is a unique δ-structure on $\lim_i A_i$ compatible with the δ-structures on each A_i via the projection. For colimits, we use the description via the truncated Witt vectors. Given a diagram $\{A_i\}$ as before, the maps $A_i \to W_2(A_i)$ from Construction 2.1 encoding the δ-structure are compatible in i. Taking colimits, we get a map $\mathrm{colim}\, A_i \to \mathrm{colim}_i\, W_2(A_i)$. Composing with the natural map $\mathrm{colim}_i\, W_2(A_i) \to W_2(\mathrm{colim}_i\, A)$ coming from functoriality of $W_2(-)$, we get a map $\mathrm{colim}_i\, A_i \to W_2(\mathrm{colim}_i\, A_i)$. It is easy to see that composing this map with the projection $W_2(\mathrm{colim}_i\, A_i) \to \mathrm{colim}_i\, A_i$ gives the identity map, so $\mathrm{colim}_i\, A_i$ acquires a δ-structure. One checks that this construction provides the desired colimit. ☐

Remark 2.4. Combining Lemma 2.3 with the adjoint functor theorem, we learn that the forgetful functor from δ-rings to commutative rings has both a left adjoint and a right adjoint. The left adjoint provides a notion of a "free δ-ring" $\mathbf{Z}\{S\}$ on a set S: apply the left adjoint to the polynomial ring $\mathbf{Z}[\{x_s\}_{s\in S}]$. The right adjoint is given by the Witt vector functor [3] (but we won't be using this fact in any essential way). In particular, for any commutative ring A, the ring $W(A)$ of Witt vectors of A is naturally a δ-ring; this provides many examples of δ-rings containing p-torsion elements as $W(A)$ for any non-reduced ring A of characteristic p contains p-torsion.

Lemma 2.5 (Free δ-rings)**.** *The free δ-ring $\mathbf{Z}\{x\}$ on a variable x is the polynomial ring $\mathbf{Z}[x_0, x_1, x_2, ...]$ with $x = x_0$ and $\delta(x_i) = x_{i+1}$. In particular, for any set S, the δ-ring $\mathbf{Z}\{S\}$ is p-torsionfree.*

Proof. Consider the ring $A := \mathbf{Z}[x_0, x_1, x_2, ...]$. This ring admits a Frobenius lift ϕ given by $\phi(x_i) = x_i^p + px_{i+1}$. As A is p-torsionfree, this Frobenius lift has a unique associated δ-structure determined by $\delta(x_i) = x_{i+1}$. This shows that the object in the statement of the lemma is well-defined. To identify A with the free δ-ring on $x = x_0$, we check the universal property. Fix a δ-ring S with some $f \in S$. Then there is a unique map $A \to S$ of commutative rings determined by $x_i \mapsto \delta^i(f)$. Using the identities describing the behaviour of δ under multiplication and addition on both A and S, it follows that $A \to S$ is a map of δ-rings. This map carries x to f by construction, and is clearly the unique map with this property. $\square$

Remark 2.6. Using the existence of pushouts and free δ-rings, one can construct more δ-rings using "generators and relations". For example, there exists a free δ-ring $\mathbf{Z}\{x,y\}/(x^2+y^3+xy)_\delta$ on two variables x and y satisfying the equation $x^2+y^3+xy=0$. It can be constructed as a pushout

$$\begin{array}{ccc} \mathbf{Z}\{z\} & \xrightarrow{z \mapsto x^2+y^3+xy} & \mathbf{Z}\{x,y\} \\ \downarrow{\scriptstyle z\mapsto 0} & & \downarrow \\ \mathbf{Z} & \longrightarrow & \mathbf{Z}\{x,y\}/(x^2+y^3+xy)_\delta, \end{array}$$

where we recall that pushouts in δ-rings are computed on underlying rings. As free δ-rings are large, it can be rather tricky to analyse these pushouts. For example, is the pushout above the 0 ring modulo p?

Let us next record the stability of δ-structures under some natural ring-theoretic operations.

Lemma 2.7 (Localizations of δ-rings)**.** *Fix a δ-ring A and $S \subset A$ a multiplicative subset with $\phi(S) \subset S$. There is a unique δ-structure on the localization $S^{-1}A$ compatible with the one on A.*

Proof. Assume first that A is p-torsionfree. Then $S^{-1}A$ is also p-torsionfree. Since $\phi(S) \subset S$, the map $\phi : A \to A$ extends uniquely to a map $\psi : S^{-1}A \to S^{-1}A$. As ϕ lifts Frobenius on A/p, ψ must lift Frobenius on $S^{-1}A/p$, so it follows that ψ is associated to a δ-structure on $S^{-1}A$. The uniqueness of this structure and compatibility with the one on A is clear.

In general, one chooses a surjection $F \to A$ where F is free δ-ring. The preimage $T \subset F$ of $S \subset A$ is a multiplicative subset such that $\phi(T) \subset T$. The preceding paragraph then gives a unique δ-structure on $T^{-1}F$ compatible with the one on F. Base changing along $F \to A$ then gives the desired δ-structure on $S^{-1}A \simeq T^{-1}A$ as pushouts of δ-rings are computed on underlying rings. $\square$

Exercise 2.8. Let A be a δ-ring.

(1) Assume that $x \in A$ admits a p^n-th root. Then $\delta(x) \in p^n A$. Deduce that if A is p-adically separated, then any element x that admits p^n-th roots for all $n \geq 1$ must have rank 1. (Hint: reduce to the free case and translate to a question about $\phi(x)$).)

(2) (Completions) Fix a finitely generated ideal $I \subset A$ that contains p. Then the I-adic completion of A admits a unique δ-structure.

It follows that $\mathbf{Z}_p$ with the δ-structure prescribed by requiring $\phi = \mathrm{id}$ is the initial object in the category of p-adically complete δ-rings.

Lemma 2.9 (Étale extensions of δ-rings)**.** *Fix a map $A \to B$ of p-adically complete and p-torsionfree rings. Assume A is equipped with a δ-structure and that $A \to B$ is étale modulo p. Then B has a unique δ-structure compatible with the one on A.*

Using the étale localization property of the Witt vectors (which is not too difficult in the p-adically nilpotent or complete cases) and the interpretation of δ-structures in Construction 2.1, one can drop the p-torsionfreeness hypothesis in the above lemma (Rezk).

Proof. As both A and B are p-torsionfree, it suffices to show that B admits a unique Frobenius lift compatible with the one on A. By p-adic completeness, it suffices to do this modulo p^n for all

n. For $n = 1$, this is simply the well-known statement that the pushout of Frobenius on A is the Frobenius on B via the relative Frobenius $B^{(1)} := B \otimes_{A,F} A \to B$ for $A \to B$. For larger n, one argues using the topological invariance of the étale site. ☐

Lemma 2.10 (Quotients of δ-rings)**.** *Fix a δ-ring A. Let $I \subset A$ be an ideal such that $\delta(I) \subset I$. Then A/I admits a unique δ-structure compatible with the one on A.*

Proof. It suffices to show that for $x \in A$ and $\epsilon \in I$, we have $\delta(x+\epsilon) \equiv \delta(x) \mod I$. This follows from the additivity formula for δ. ☐

3. Perfect δ-rings

The following class of δ-rings will be important for relating δ-rings to perfectoid rings.

Definition 3.1. A δ-ring A is called *perfect* if the Frobenius $\phi : A \to A$ is an isomorphism.

Our goal is to classify such rings as follows.

Proposition 3.2 (Perfect δ-rings = perfect rings)**.** *The following categories are equivalent:*

(1) The category of perfect δ-rings that are p-adically complete.

(2) The category of p-adically complete and p-torsionfree rings that are perfect modulo p.

(3) The category of perfect rings of characteristic p.

The functor from (1) to (2) is the forgetful functor; the functor from (2) to (3) is $A \mapsto A/p$; the functor from (3) to (1) is $A \mapsto W(A)$.

In other words, every p-adically complete perfect δ-ring has the form $W(R)$ (with its natural Frobenius) for a perfect $\mathbf{F}_p$-algebra R. In fact, the proof below does not use the definition of the Witt vector functor $W(-)$, and provides an alternative way to think about it on perfect rings. One of the two main ingredients in the proof of Proposition 3.2 is the following.

Lemma 3.3. *Let A be a δ-ring and let $x \in A$ with $px = 0$. Then $\phi(x) = 0$. In particular, if ϕ is injective, then A is p-torsionfree.*

Proof. We trivially have $\phi(x) = 0$ in $A[1/p]$, so we may assume that A is a $\mathbf{Z}_{(p)}$-algebra. Applying δ to $px = 0$ shows that $p^p\delta(x) + \phi(x)\delta(p) = 0$. As $\delta(p)$ is a unit in $\mathbf{Z}_{(p)}$, it suffices to show that $p^p\delta(x) = 0$. But we have

$$p^p\delta(x) = p^{p-1} \cdot p\delta(x) = p^{p-1} \cdot (\phi(x) - x^p) = p^{p-2}\big(\phi(px) - (px)x^{p-1}\big) = 0,$$

where the last equality follows as $px = 0$. ☐

Remark 3.4. One can show that specifying a δ-structure on a ring A is the same as specifying a *derived* Frobenius lift on A, i.e., if $\overline{A} := A \otimes^L_{\mathbf{Z}} \mathbf{Z}/p$ denotes the derived mod p reduction on A, then specifying a δ-structure on A is equivalent to giving an endomorphism $\phi : A \to A$ together with a homotopy between the composite $A \xrightarrow{\phi} A \xrightarrow{can} \overline{A}$ and the composite $A \xrightarrow{can} \overline{A} \xrightarrow{\text{Frob}} \overline{A}$. To see this, one must show that $W_2(A)$ can be described as the fibre product of $A \xrightarrow{can} \overline{A} \xleftarrow{\text{Frobocan}} A$. This holds true (Remark 2.2) when A is p-torsionfree, and the general case follows by left Kan extensions. (Details omitted)

Using this interpretation, Lemma 3.3 has a slightly more conceptual interpretation: as the p-torsion of A identifies with $\pi_1(\overline{A})$, the lemma follows from the fact that Frobenius acts trivially on the higher homotopy groups of a simplicial commutative ring.

The other ingredient is the following well-known lemma.

Lemma 3.5. *Let A be a perfect $\mathbf{F}_p$-algebra. Then the cotangent complex $L_{A/\mathbf{F}_p}$ vanishes. Consequently, the following categories are equivalent:*

(1) Perfect $\mathbf{F}_p$-algebras.

(2) For fixed $n \geq 1$, the category of flat $\mathbf{Z}/p^n$-algebras $\widetilde{A}$ with $\widetilde{A}/p$ being perfect.

(3) p-adically complete and p-torsionfree $\mathbf{Z}_p$-algebras $\widetilde{A}$ with $\widetilde{A}/p$ perfect.

There are obvious functors in the direction (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1). To go from (1) to (3), one may explicitly use the Witt vector functor $A \mapsto W(A)$, though we shall not need its precise structure.

Proof. The Frobenius endomorphism of any $\mathbf{F}_p$-algebra A induces the 0 map on its cotangent complex: this is clear for polynomial $\mathbf{F}_p$-algebras, and thus follows in general as Frobenius is functorial (and because there is a functorial simplicial resolution of any $\mathbf{F}_p$-algebra by polynomial algebras). On the other hand, if A is perfect, then Frobenius is an isomorphism, so it must act as an isomorphism on the cotangent complex as well. Combining these two observations shows that $L_{A/\mathbf{F}_p} \simeq 0$. The equivalence of (1) and (2) follows from standard relations between the cotangent complex and deformation theory. The equivalence of (2) and (3) follows as the category of p-adically complete and p-torsionfree $\mathbf{Z}_p$-algebras can be described as the inverse limit of the categories of flat $\mathbf{Z}/p^n$-algebras. ☐

Proof of Proposition 3.2. Lemma 3.3 ensures that the forgetful functor goes from (1) to (2). The equivalence of (2) and (3) comes from Lemma 3.5. To get from (3) to (1), fix a perfect ring A, and let $\widetilde{A}$ denote the corresponding object in (2), so $\widetilde{A}$ is a p-adically complete and p-torsionfree ring. As $A \mapsto \widetilde{A}$ is a functor, the Frobenius on A lifts to a unique automorphism of $\widetilde{A}$, so $\widetilde{A}$ comes equipped with a Frobenius lift. As $\widetilde{A}$ is p-torsionfree, this defines the δ-structure on $\widetilde{A}$, giving an object in (1). Using the uniqueness of lifts, it is easy to check that these constructions provide mutually inverse equivalences. ☐

Using Proposition 3.2, we can give a conceptual construction of the Teichmuller map $R \to W(R)$ and the Teichmuller expansion of an element $f \in W(R)$ for R perfect.

Construction 3.6 (The Teichmuller expansion)**.** Let R be a perfect $\mathbf{F}_p$-algebra, and let $W(R)$ be its ring of Witt vectors. Using the characterization of the latter as the unique p-adically complete and p-torsionfree ring lifting R, let us describe a "normal form" for elements of $W(R)$.

First, we show that the projection $W(R) \to R$ has a unique multiplicative section $R \to W(R)$ denoted $x \mapsto [x]$. It is enough to define such a section for $W(R)/p^n \to R$. For this, given $x \in R$, choose $y \in W(R)/p^n$ lifting $x^{1/p^n} \in R$. Using the elementary observation that if $a = b \mod p^k$ then $a^p = b^p \mod p^{k+1}$ (in any commutative ring), it follows that $y^{p^n} \in W(R)/p^n$ is well-defined (i.e., independent of choices) and lifts $x \in R$. We set $[x] = y$; the multiplicativity of $x \mapsto [x]$ is immediate from the construction. For uniqueness, consider two multiplicative lifts $[\cdot], [\cdot]' : R \to W(R)/p^n$. Then for any $a \in W(R)$, we can write $[a] = [a]' + pb$ for some $b \in W(R)/p^n$. Raising to the p^n-th power and using multiplicativity then shows that $[a^{p^n}] = [a^{p^n}]'$ in $W(R)/p^n$. As the p-power map on R is bijective, it follows that $[x] = [x]'$ for all $x \in R$, as wanted.

Now given any $f \in W(R)$, if we write $\overline{f} \in R$ for its image, then $f = [\overline{f}] \mod p$, so we can write $f = [\overline{f}] + pf_1$ for a unique $f_1 \in W(R)$ (where uniqueness of f_1 is due to p-torsionfreeness of $W(R)$). Applying the same reasoning to f_1 and continuing, we find that f admits a unique p-adic expansion $\sum_{i=0}^{\infty}[a_i]p^i$, called the *Teichmuller* expansion of f.

Exercise 3.7. Let R be a perfect $\mathbf{F}_p$-algebra. Show that $f \in W(R)$ has rank 1 exactly when $f = [a]$ for some $a \in R$. (Hint: first show that the Frobenius lift $\phi : W(R) \to W(R)$ is simply given by $\sum_{i=0}^{\infty}[a_i]p^i \mapsto \sum_{i=0}^{\infty}[a_i^p]p^i$.)

References

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