TOPICS IN ALGEBRAIC GEOMETRY I: ÉTALE COHOMOLOGY (MATH 731)

BHARGAV BHATT

Description. Let X be an algebraic variety over the field of rational numbers. Can the Betti numbers $\dim H^i(X(\mathbf{C}), \mathbf{Q})$ be defined purely algebraically? Étale cohomology provides an affirmative answer to this question by realising the related vector spaces $H^i(X(\mathbf{C}), \mathbf{Q}_\ell) \simeq H^i(X(\mathbf{C}), \mathbf{Q}) \otimes \mathbf{Q}_\ell$ purely algebraically for any prime number ℓ. It follows that the spaces $H^i(X(\mathbf{C}), \mathbf{Q}_\ell)$ come equipped with an action of the group $G = \mathrm{Aut}(\mathbf{C})$ of automorphisms of the field $\mathbf{C}$. This interaction between the topology of $X(\mathbf{C})$ and the representation theory of G leads to a fascinating story that has been at the heart of numerous fundamental advances in twentieth century mathematics. The goal of the present class is to develop enough étale cohomology theory to appreciate some of these applications, most notably to arithmetic algebraic geometry.

Plan. The plan for this course is to take a scenic route to the proof of the Weil conjectures. Here's a rough outline of the major topics that will be covered:

- Basics of flat, étale, smooth, and unramified ring homomorphisms.
- Grothendieck topologies.
- Some basic constructible sheaf theory.
- Calculations of the étale cohomology groups of curves .
- The proper and smooth base change theorems, and applications.
- Poincaré duality and, more generally, Verdier duality.
- The Artin-Grothendieck comparison theorem for varieties over $\mathbf{C}$.
- The Grothendieck-Lefschetz trace formula for varieties over finite fields, and applications.
- Deligne's proof of the Weil conjectures from [Del74] (or, time permitting, [Del80]).

Background. I will probably assume that students attending have prior familiarity with algebraic geometry, say at the level of chapters 2 and 3 of Hartshorne's book [Har77] (or equivalent, and including the section on homological algebra in chapter 3), or are willing to work hard enough.

Office hours. I will hold office hours every Wednesday from 3pm to 4pm in my office (4835 EH).

Homework. There will be problem sets posted on the course website
`http://www-personal.umich.edu/~bhattb/mat731fall2011/`

References. My favourite reference is [Del77], which I will follow (in spirit, at least) in this course; this book is available online from the Springer website. Lots of other references can be found at
`http://www-personal.umich.edu/~bhattb/mat731fall2011/refs.html`

REFERENCES

[Del74] Pierre Deligne. La conjecture de Weil. I. *Inst. Hautes Études Sci. Publ. Math.*, (43):273–307, 1974.

[Del77] Pierre Deligne. *Cohomologie étale*. Springer-Verlag, Berlin, 1977. Séminaire de Géométrie Algébrique du Bois-Marie SGA $4\frac{1}{2}$, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier, Lecture Notes in Mathematics, Vol. 569.

[Del80] Pierre Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.