Problem set 8

1. Let $f : X \to S$ be a separated morphism of finite type between noetherian schemes, and fix $\ell \in \mathbf{Z}$. Show that there are only finitely many possibilities for the Betti numbers $\dim H^i(X_s, \mathbf{Z}/\ell)$ as s varies through all possible geometric points of f, and $i \in \mathbf{Z}$. Can you show the same without fixing ℓ?

2. The goal of this exercise is to carry out an algebraic analogue of a part of the topological proof of Poincare duality sketched in class: we will compute the compactly supported étale cohomology of a "small ball" around a point on a curve. Let C be a smooth curve over an algebraically closed field k, and let ℓ be an odd prime invertible on k. Fix a point $x \in C(k)$, and let $C_x = \lim U$ denote the strict henselization of C at x, where the limit takes place over all diagrams of the form $x \in U \to C$ with $U \to C$ étale; we call such diagrams etale neighbourhoods of x.

   (a) Show that $\lim_U H^0_c(U, \mu_\ell) = 0$ where the limit takes place as above.

   (b) Let U be a smooth affine curve over k with rational point $x \in U(k)$. By using Riemann-Roch, show that, after possibly replacing U with a Zariski open $U' \subset U$ containing x, there exist finite étale covers $V \to U$ of degree ℓ which "totally ramify" at ∞, i.e, if $\overline{V} \to \overline{U}$ denotes the induced map on compactifications, then $f^{-1}(z)$ is set-theoretically supported at a point, for all $z \in \overline{U} - U$. The formal local structure of such a map at z is given by $k[\![s]\!] \to k[\![t]\!]$ via $s \mapsto t^\ell$.

   (c) Let $f : \mathrm{Spec}(k[\![t]\!]) \to \mathrm{Spec}(k[\![s]\!])$ be the map defined by $s \mapsto t^\ell$. Show that the trace map $f_*\mu_\ell \to \mu_\ell$ (induced by the norm map $f_*\mathbf{G}_m \to \mathbf{G}_m$) is constant.

   (d) Using the excision exact sequence for $U \subset \overline{U}$, show that $\lim_U H^1_c(U, \mu_\ell) \simeq 0$, where the limit takes place as above.

   (e) For an étale neighbourhood U of x with projective compactification $\overline{U}$, show that $H^2_c(U, \mu_\ell) \simeq H^2_c(\overline{U}, \mu_\ell) \simeq \mathrm{Pic}(\overline{U})/\ell$. Given a map $f : V \to U$ of étale neighbourhoods of x, show that the previous isomorphism identifies the pushforward $f_! : H^2_c(V, \mu_\ell) \to H^2_c(U, \mu_\ell)$ with the norm map $\mathrm{Pic}(\overline{V})/\ell \to \mathrm{Pic}(\overline{U})/\ell$ (induced by applying H^1 to the norm map $f_*\mathbf{G}_m \to \mathbf{G}_m$). Using this, conclude that $f_!$ is bijective (what is $f_!(\mathcal{O}_{\overline{V}}(x))$?).

   (f) Show that $\lim_U H^2_c(U, \mu_\ell) \simeq \mathbf{Z}/\ell$, where the limit takes place as above over étale neighbourhoods of x.

3. Fix two separably closed fields K and L, possibly of different characteristics. Fix an integer N invertible on both K and L. Show that there is a canonical isomorphism $R\Gamma(\mathbf{P}^n_K, \mathbf{Z}/N) \simeq R\Gamma(\mathbf{P}^n_L, \mathbf{Z}/N)$ that commutes with cup products. Using the case $K = \mathbf{C}$, show that the standard $\mathrm{GL}_n(K)$-action on $\mathbf{P}^n_K$ induces the trivial action on $H^k(\mathbf{P}^n_K, \mathbf{Z}/N)$ for any k.

4. The goal of this exercise is to use the Lefschetz trace formula to show that a genus 0 curve over a finite field is always isomorphic to $\mathbf{P}^1$. Let k be a finite field of characteristic p with size $q = p^f$, and let C be a smooth projective geometrically connected curve of genus 0 over k. Fix an auxilliary odd integer $n > q + 1$.

(a) Show that "degree" induces an isomorphism $\mathrm{Pic}(C_{\overline{k}}) \simeq \mathbf{Z}$.

(b) Show that C admits a separable degree 2 map $\pi : C \to \mathbf{P}^1_k$.

(c) Show that π induces the multiplication by 2 map $\pi^* : \mathrm{Pic}(\mathbf{P}^1_{\overline{k}}) \to \mathrm{Pic}(C_{\overline{k}})$ once both sides are identified with $\mathbf{Z}$ via the degree map.

(d) Show that π induces an isomorphism $\pi^* : H^2(\mathbf{P}^1_{\overline{k}}, \mathbf{Z}/n) \to H^2(C_{\overline{k}}, \mathbf{Z}/n)$. Observe that this isomorphism is equivariant for the geometric Frobenius on either side.

(e) Using the Lefschetz trace formula, show that $\#C(k) = q + 1 \mod n$. Since $n > q + 1$, conclude that C has a rational point $x \in C(k)$.

(f) Show that $C \simeq \mathbf{P}^1$.

(g) (*) For any n, show that twisted forms of $\mathbf{P}^n$ over k are all trivial, i.e., isomorphic to $\mathbf{P}^n$ over k.

5. Let X be a proper variety over a separably closed field k. Let $\sigma \in \mathrm{Aut}(X)$ be an automorphism of order ℓ^n, where ℓ is a prime number invertible on k, and $n \in \mathbf{N}$. Assume that the fixed points of σ are isolated.

   (a) Show that the automorphism $\sigma^* : H^k(X, \mathbf{Z}/\ell) \to H^k(X, \mathbf{Z}/\ell)$ always has an eigenvector with eigenvalue 1.

   (b) Show that $\#X(k)^\sigma = \chi(X, \mathbf{Z}/\ell) \mod \ell$.

6. Let X be a smooth proper variety over an algebraically closed based field k. Let $\phi : X \to X$ be an endomorphism of X with finitely many fixed points. Let $p \in X(k)$ be a fixed point of ϕ. Show that if $d\phi_p : T_p(X) \to T_p(X)$ does not have an eigenvalue equal to 1, then $\langle \Gamma_\phi, \Delta \rangle_P = 1$.

7. The goal of this exercise is give a sense of how big $\pi_1(\mathbf{A}^1_k)$ can be, in positive characteristic. We work with $k = \overline{\mathbf{F}_p}$ for this exercise.

   (a) Let $U \subset V$ be a open immersion of connected noetherian schemes with V normal and excellent. Show that $\pi_1(U) \to \pi_1(V)$ is surjective; find an example to show that the normality is necessary.

   (b) Let X be a smooth projective geometrically connected curve over k. Show that there exists a generically étale map $f : X \to \mathbf{P}^1_k$.

   (c) Let $U \subset \mathbf{A}^1_k$ be a non-empty open subset. Using that $\mathbf{A}^1(k) - U(k)$ lies in a finite subgroup of $\mathbf{A}^1(k)$, show that there exists a finite étale map $f : V \to \mathbf{A}^1_k$ for a suitable open $V \subset U$.

   (d) For any smooth projective curve C over k, show that there exists a finite flat map $f : C \to \mathbf{P}^1_k$ ramified only at ∞.

   (e) For any smooth projective curve C over k, show that $\pi_1(\mathbf{A}^1_k)$ contains a subgroup that maps onto $\pi_1(C)$.

8. The goal of this exercise is to give the definition of ℓ-adic cohomology, and a sample theorem. Let $\mathcal{C}$ be a Grothendieck topology, and let $\mathrm{Ab}(\mathcal{C})^{\mathbf{N}}$ denote the category of projective systems $(\mathcal{F}_e, \phi_e)$ where $\mathcal{F}_e \in \mathrm{Ab}(\mathcal{C})$ and $\phi_e : \mathcal{F}_e \to \mathcal{F}_{e-1}$ is a map; we often suppress the maps ϕ_e from the notation.

   (a) Show that $\mathrm{Ab}(\mathcal{C})^{\mathbf{N}}$ is an abelian category with enough injectives. I suggest trying to realise $\mathrm{Ab}(\mathcal{C})^{\mathbf{N}}$ as $\mathrm{Ab}(\mathcal{C})$-valued sheaves on a suitable site.

   (b) Show that taking limits in the category of sheaves defines a left exact functor $\lim : \mathrm{Ab}(\mathcal{C})^{\mathbf{N}} \to \mathrm{Ab}(\mathcal{C})$ with a left adjoint $\mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}(\mathcal{C})^{\mathbf{N}}$ given by associating to $\mathcal{F} \in \mathrm{Ab}(\mathcal{C})$ the constant projective system with value $\mathcal{F}$.

(c) Show that the functor $\lim$ preserves injectives. The resulting derived functor $D^+(\mathrm{Ab}(\mathcal{C})^{\mathbf{N}}) \to D^+(\mathrm{Ab}(\mathcal{C}))$ is denoted $R\lim$, and its cohomologies are denoted $R^i \lim$.

(d) When $\mathcal{C} = *$, show that $R^i \lim = 0$ for $i > 1$, and that $R^1 \lim_e A_e \simeq \lim^1_e A_e$ vanishes when $\{A_e\}$ satisfies the Mittag-Leffler condition. In particular, if each A_e is finite, this $\lim^1$ vanishes.

(e) Fix a system $\{\mathcal{F}_e\} \in \mathrm{Ab}(\mathcal{C})^{\mathbf{N}}$. Assume that the presheaves $U \mapsto \lim_e H^i(U, \mathcal{F}_e)$ and $U \mapsto \lim^1_e H^{i-1}(U, \mathcal{F}_e)$ sheafify to 0 for $i > 0$. Show that $R^i \lim \mathcal{F}_e \simeq 0$ for $i > 0$.

(f) Fix $k \in \mathbf{N}$. Show that the functor $\mathrm{pr}_k : \mathrm{Ab}(\mathcal{C})^{\mathbf{N}} \to \mathrm{Ab}(\mathcal{C})$ defined by $\{\mathcal{F}_e\} \mapsto \mathcal{F}_k$ is a left exact functor that preserves injectives.

(g) Show that there is a left exact functor $\mathrm{Ab}(\mathcal{C})^{\mathbf{N}} \to \mathrm{Ab}^{\mathbf{N}}$ defined by taking global sections $\Gamma(*, -)$ in each "slot." We denote this functor by $\{\Gamma(*, -)\}$. Show that this functor preserves injectives.

(h) Show that there is a commutative diagram of left exact functors

$$\begin{array}{ccc} \mathrm{Ab}(\mathcal{C})^{\mathbf{N}} & \xrightarrow{\lim} & \mathrm{Ab}(\mathcal{C}) \\ \big\downarrow{\scriptstyle \{\Gamma(*,-)\}} & & \big\downarrow{\scriptstyle \Gamma(*,-)} \\ \mathrm{Ab}^{\mathbf{N}} & \xrightarrow{\lim} & \mathrm{Ab} \end{array}$$

Conclude that there exists an isomorphism of functors $D^+(\mathrm{Ab}(\mathcal{C})^{\mathbf{N}}) \to D^+(\mathrm{Ab})$

$$R\Gamma(*, R\lim \mathcal{F}_e) \simeq R\lim R\{\Gamma(*, \mathcal{F}_e)\}.$$

Show also that the "projection to the k-th slot," i.e., $R\mathrm{pr}_{k,*}$, of $R\{\Gamma(*, \mathcal{F}_e)\}$ is quasi-isomorphic to $R\Gamma(*, \mathcal{F}_k)$.

(i) For the purposes of this exercise, the ℓ-adic cohomology of $\mathcal{C}$ is defined as $R\Gamma(*, \mathbf{Z}_\ell) := R\lim_e R\Gamma(*, \mathbf{Z}/\ell^e)$, and $H^i(*, \mathbf{Z}_\ell) := H^i(R\Gamma(*, \mathbf{Z}_\ell))$. Show that there exists a short exact sequence

$$1 \to \lim{}^1 H^{i-1}(*, \mathbf{Z}/\ell^e) \to H^i(*, \mathbf{Z}_\ell) \to \lim_e H^i(*, \mathbf{Z}/\ell^e) \to 1,$$

where the group $\lim^1$ on the left is the classical $\lim^1$ defined by Milnor. Show that if $H^i(*, \mathbf{Z}/\ell^e)$ is finite for all i and e, then one has an isomorphism

$$H^i(*, \mathbf{Z}_\ell) \simeq \lim_e H^i(*, \mathbf{Z}/\ell^e).$$

(j) Let X be a locally contractible topological space homotopy equivalent to a finite CW complex, and let $\mathcal{C}$ be the Grothendieck topology associated to X. Show that the presheaves $U \mapsto \lim_e H^i(U, \mathbf{Z}/\ell^e)$ and $U \mapsto \lim^1_e H^{i-1}(U, \mathbf{Z}/\ell^e)$ sheafifyto 0 for $i > 0$, and that the projective system $H^i(X, \mathbf{Z}/\ell^e)$ has a vanishing $\lim^1$ for all i. Conclude that there are isomorphisms

$$R\Gamma(X, \mathbf{Z}_\ell) \simeq R\Gamma(X, \lim_e \mathbf{Z}/\ell^e) \quad \text{and} \quad H^i(X, \mathbf{Z}_\ell) \simeq \lim_e H^i(X, \mathbf{Z}/\ell^e).$$

(k) Give an example of a scheme X such that the presheaf $U \mapsto \lim_e H^i(U, \mathbf{Z}/\ell^e)$ does *not* sheafify to 0 on $X_{\text{ét}}$. I suggest working with spectra of finite fields.

(l) (*) Give an example of a scheme X such that the projective system $\{H^i(X, \mathbf{Z}/\ell^e)\}$ has a non-vanishing $\lim^1$. I suggest working with spectra of number fields.

(m) For a scheme X, we apply the above definitions to $\mathcal{C} = X_{\text{ét}}$ to define $H^i(X, \mathbf{Z}_\ell)$ as $H^i(R\Gamma(*, \mathbf{Z}_\ell))$. Now let X be a proper finite type scheme over an algebraically closed field k. Show that $H^i(X, \mathbf{Z}_\ell) = \lim_n H^i(X, \mathbf{Z}/\ell^n)$ for all i, and that $H^i(X, \mathbf{Z}_\ell) = 0$ for $i > 2\dim(X)$.

Remark 0.1. The theory sketched above is worked out in Jannsen's paper "Continuous étale cohomology," and I suggest looking at it for further information.