Problem set 7

1. Let S be a scheme, and let $\mathcal{F}$ be a sheaf on $S_{\text{ét}}$.

   (a) If s and t are geometric points with s specialising t, construct natural specialisation maps $\mathcal{F}_s \to \mathcal{F}_t$. In the case that $S = \operatorname{Spec}(A)$ is affine, and $\mathcal{F} = \widetilde{M}$ is a quasi-coherent $\mathcal{O}_S$-module, this construction should recover the usual localisation map $M \otimes_A A_s^{sh} \to M \otimes_A A_t^{sh}$.

   (b) Assume $\mathcal{F}$ is constructible with finite stalks, and S is noetherian. Then show that $\mathcal{F}$ is locally constant if and only if all the specialisation maps $\mathcal{F}_s \to \mathcal{F}_t$ as above are isomorphisms.

2. Let U and V be two open subsets of a scheme X such that $X = U \cup V$. Let $j_U : U \to X$, $j_V : V \to X$ and $j_{U\cap V} : U \cap V \to X$ denote the inclusions.

   (a) For any abelian sheaf $\mathcal{F} \in \operatorname{Ab}(X_{\text{ét}})$, show that there is a short exact sequence

   $$1 \to j_{U\cap V,!}\big(\mathcal{F}|_{U\cap V}\big) \to j_{U,!}\big(\mathcal{F}|_U\big) \oplus j_{V,!}\big(\mathcal{F}|_V\big) \to \mathcal{F} \to 1$$

   (b) Assume X is a separated k-scheme for some separably closed field k. For any constructible sheaf $\mathcal{F} \in \operatorname{Ab}(X_{\text{ét}})$, show that there is a long exact sequence

   $$\ldots H_c^i(U \cap V, \mathcal{F}|_{U\cap V}) \to H_c^i(U, \mathcal{F}|_U) \oplus H_c^i(V, \mathcal{F}|_V) \to H_c^i(X, \mathcal{F}) \to H_c^{i+1}(U \cap V, \mathcal{F}|_{U\cap V}) \ldots$$

   (c) (*) Formulate and prove a generalisation of the previous conclusion to open covers of X with more than two elements.

   (d) Assume X is a variety of dimension n over a separably closed field k. Using vanishing results, show that the association $U \mapsto H_c^{2n}(U, \mathcal{F})$ defines a *cosheaf* on X_{Zar}. Can you guess what this cosheaf is when $\mathcal{F} = \mathbf{Z}/n$ and X is smooth?

3. Let $S = \operatorname{Spec}(A)$ be the spectrum of a henselian local scheme, and $\widehat{S} = \operatorname{Spec}(\widehat{A})$ be the completion. Let $f : X \to S$ be a proper morphism, and let $\widehat{f} : \widehat{X} = X \times_S \widehat{S} \to \widehat{S}$ be the base change. Assume that the map $A \to \widehat{A}$ can be written as a filtered colimit of *smooth* A-algebras. Show that base change defines an equivalence between categories of finite étale covers of X and $\widehat{X}$.

   Remark 0.1. The condition that $A \to \widehat{A}$ is a filtered colimit of smooth A-algebras is often satisfied in practice: by a theorem of Popescu, this is true whenever A is excellent. Earlier work of Artin already handled the case where A is finite type over $\mathbf{Z}$. When combined with the previous exercise and some standard limit arguments, this result can be used to show: if $X \to \operatorname{Spec}(A)$ is a finitely presented proper morphism over any strictly henselian local scheme, then $H^1_{\text{ét}}(X, \underline{A}) \to H^1_{\text{ét}}(X_0, \underline{A})$ is surjective for any finite abelian group A, where X_0 denotes the special fibre. This fact was used implicitly in class during the proof of the proper base change theorem.

4. Let V be a finite dimensional vector space over an algebraically closed field k of characteristic p. Let $\phi : V \to V$ be a p-linear map, i.e., ϕ is additive and satisfies $\phi(r \cdot v) = r^p \cdot \phi(v)$. Show that the map $v \mapsto \phi(v) - v$ is surjective.

5. The goal of this exercise is to fill one gap left in the proof of the proper base change theorem presented in class: we will now tackle p-torsion in characteristic p.

   (a) Let X be an $\mathbf{F}_p$-scheme such that $H^i(X, \mathcal{O}_X) = 0$ for $i > d$. Show that $H^i_{\text{ét}}(X, \mathbf{Z}/p) = 0$ for $i > d + 1$.

   (b) Let X be a quasi-projective k-scheme of dimension d over an algebraically closed field k of characteristic p. Show that $H^d(X, \mathcal{O}_X)$ is a finite dimensional k-vector space. Using the previous exercise on p-linear algebra, conclude that $H^i_{\text{ét}}(X, \mathbf{Z}/p) = 0$ for $i > d$.

   (c) Let $f : X \to S$ be a proper morphism with $S = \mathrm{Spec}(A)$ the spectrum of a complete noetherian local ring with algebraically closed residue field, and let X_0 denote the special fibre. Assume that $\dim(X_0) \leq 1$. Show that $H^i_{\text{ét}}(X, \mathbf{Z}/p) \to H^i_{\text{ét}}(X_0, \mathbf{Z}/p)$ is surjective for all i.

   (d) Following the proof given in class, show that the proper base change theorem is valid for any torsion abelian sheaf.

6. The goal of this exercise is to show that $R^i f_!$ cannot be defined as a derived functor of $f_!$. All schemes in this exercise are assumed to be separated, and relative to a fixed algebraically closed field k.

   (a) Let X be a scheme, and let $\mathcal{F}$ be a sheaf on X. For any $f \in \mathcal{F}(X)$, show that the support of f is a closed subset of X. In particular, if X is proper over k, the support of f is also proper over k.

   (b) Let U be an affine k-scheme. Show that for any abelian sheaf $\mathcal{F}$ on U, the group $H^0_c(U, \mathcal{F})$ is identified with the set of $f \in \mathcal{F}(U)$ such that the support of f is a finite set of closed points.

   (c) Conclude that there is an equality of functors on $\mathrm{Ab}(X_{\text{ét}})$:

   $$H^0_c(U, -) \simeq \oplus_{x \in U^0} H^0_x(U, -),$$

   where U^0 denotes the set of closed points of U, $H^0_x(U, -)$ is the functor of "sections with support at x", i.e., $H^0_x(U, \mathcal{F}) = \mathrm{Hom}(i_* \mathbf{Z}, \mathcal{F})$, and $i : x \hookrightarrow X$ is the usual map.

   (d) (*) Show that if U is a smooth affine curve with a closed point $x \in X$, then $H^2_x(U, \mathbf{Z}/n)$ is isomorphic to $\mathbf{Z}/n$.

   (e) Conclude that the derived functors of $\mathcal{F} \mapsto H^0_c(U, \mathcal{F})$ do not coincide with the functors $H^i_c(U, \mathcal{F})$ for U a smooth affine curve.

   Remark 0.2. The essential problem encountered above was the paucity of proper subvarieties inside an affine variety. If we pass to the analytic realm, then one can show that there are "enough" closed compact subsets in an affine variety over $\mathbf{C}$. Consequently, the derived functors of "sections with proper support" produce the correct theory in the analytic world.

7. By contemplating $H^1(C, \mathbf{Z})$ for both nodal and smooth curves C, give an example to show that the conclusion of the proper base change theorem fails with non-torsion coefficients.

8. Give an example of the failure of Abhyankar's lemma when dealing with covers of degree p, i.e., construct a smooth curve $f : X \to S$ over $S = \mathrm{Spec}(V)$ the spectrum of a complete dvr with residue characteristic p, and a finite étale degree p cover $Y_\eta \to X_\eta$ of the generic fibre of f that does not extend to a finite étale cover of X after any finite base change on S. I suggest thinking about a family of genus 1 curves in characteristic p with ordinary generic fibre, and supersingular special fibre.

9. Give an example showing the necessity of the normality assumption on the source for the formulation of Zariski-Nagata purity given in class, i.e., construct a finite injective map $f : R \to S$ with R a regular local domain of dimension 2 such that f is unramified outside the closed point of $\mathrm{Spec}(R)$, but ramifies over the closed point.