Problem set 6

Notation 0.1. All sheaves are étale sheaves unless otherwise specified. Some exercises below need additional background not yet covered in class: Exercise 12 uses formal geometry, Exercises 13 and 14 use proper base change, and Exercise 15 uses complex analytic spaces. However, please try them!

1. Let X be an affine $\mathbf{F}_p$-scheme. Show that $H^i_{\text{ét}}(X, \mathbf{Z}/p) = 0$ for $i > 1$.
2. Let X be an $\mathbf{F}_p$-scheme. Show that $H^i_{\text{fppf}}(X, \mathbf{Z}/p) = H^i_{\text{ét}}(X, \mathbf{Z}/p)$ for all i.
3. Give an example of an affine $\mathbf{F}_p$-scheme X such that $H^2_{\text{ét}}(X, \mu_p) = 0$, but $H^2_{\text{fppf}}(X, \mu_p) \neq 0$.
4. Let $X/\mathbf{F}_p$ be a projective variety. Show that $\chi(X, \mathbf{Z}/p) := \sum_i (-1)^i \dim_{\mathbf{F}_p}(H^i_{\text{ét}}(X, \mathbf{Z}/p))$ is 0.
5. Let $j : U \to X$ be an étale morphism, and let $f : Y \to X$ be any morphism. Show that $f'^* \circ j_! \simeq j'_! \circ f^*$ where $f' : Y \times_X U \to U$ and $j' : Y \times_X U \to Y$ are the two base changes.
6. (Brauer groups of regular schemes inject into that of the fraction fields) Let X be a regular noetherian scheme. Let $j : \eta \to X$ denote the inclusion of the generic point.
   (a) Show that there is an exact sequence
   $$0 \to \mathbf{G}_m \to j_* \mathbf{G}_m \to \mathrm{Div}_X \to 0,$$
   where $\mathrm{Div}_X = \oplus_{x \in X^1} i_{x,*} \mathbf{Z}$, where X^1 is the set of codimension one points of X, and $i_x : \mathrm{Spec}(\kappa(x)) \to X$ is the natural map.
   (b) Show that $H^2(X, \mathbf{G}_m) \to H^2(X, j_* \mathbf{G}_m)$ is injective.
   (c) Using Hilbert's theorem 90, show that $R^1 j_* \mathbf{G}_m = 0$.
   (d) Using the Leray spectral sequence, show that $H^2(X, \mathbf{G}_m) \to H^2(\eta, \mathbf{G}_m)$ is injective.
7. Let C be a smooth projective geometrically connected curve of arithmetic genus g over $k = \overline{k}$, and let $U \subset C$ be the open complement of a set $Z = \{z_1, \ldots, z_r\}$ of r distinct points on C. Fix an integer n invertible on k. We chased constructible sheaves in class to see that $H^1(U, \mu_n) \simeq (\mathbf{Z}/n)^{2g+r-1}$. Now we will see it directly; this is just the same wine in a very slightly different packaging!
   (a) Show that there is an natural identification of $H^1(U, \mu_n)$ with the π_0 of the category $B\mu_n(U)$ of pairs $(\mathcal{L}, \phi)$ where $\mathcal{L} \in \mathrm{Pic}(U)$, and $\phi : \mathcal{L}^{\otimes n} \simeq \mathcal{O}_U$ is an isomorphism.
   (b) Let $\mathcal{C}$ denote the category of triples $(\mathcal{M}, D, \psi)$ where $\mathcal{M} \in \mathrm{Pic}^0(X)$, D is a degree 0 divisor on C supported on Z, and $\psi : \mathcal{M}^{\otimes n} \to \mathcal{O}_C(D)$ is an isomorphism. Show that there is an essentially surjective functor $\mathcal{C} \to B\mu_n(U)$ given by $(\mathcal{M}, D, \psi) \mapsto (\mathcal{M}|_U, \psi|_U)$.

(c) Let $\mathcal{D} \subset \mathcal{C}$ be the full subcategory of triples $(\mathcal{O}_C(D), nD, \mathrm{can})$ where $D \subset C$ is a divisor of degree 0 supported on Z. Show that there is a natural action of $\mathcal{D}$ on $\mathcal{C}$ given by tensoring; you can do this at the level of isomorphism classes if you want.

(d) Show that $\pi_0(B\mu_n(U)) = \pi_0(\mathcal{C})/\pi_0(\mathcal{D})$. By associating to a triple $(\mathcal{M}, D, \psi)$ as above the valuation of D at each z_i, show that there is a map $\pi_0(B(\mu_n(U)) \to \oplus_{i=1}^r \mathbf{Z}/n$.

(e) By studying the kernel and image of the map $\pi_0(B\mu_n(U)) \to \oplus_{i=1}^r \mathbf{Z}/n$, show that there is a short exact sequence

$$1 \to \pi_0(B\mu_n(X)) \to \pi_0(B\mu_n(U)) \to \oplus_{i=1}^r \mathbf{Z}/n \stackrel{\mathrm{sum}}{\to} \mathbf{Z}/n \to 1$$

(f) Using our computation of $\pi_0(B\mu_n(X))$ from class, show that $H^1(U, \mu_n) \simeq (\mathbf{Z}/n)^{2g+r-1}$.

8. (Mayer-Vietoris) Let X be a scheme, and let $U, V \subset X$ be open subsets such that $U \cup V = X$.

(a) For any sheaf $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$, show that there is a short exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V).$$

(b) Using the fact that $h_{U\cap V} \to h_U$ and $h_{U \cap V} \to h_V$ are monomorphisms, show that if $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$ is *injective*, then the short exact sequence above is exact on the right as well.

(c) By contemplating injective resolutions, show that for any sheaf $\mathcal{F}$, there is a long exact sequence of cohomology groups

$$\dots H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \oplus H^i(V, \mathcal{F}) \to H^i(U \cap V, \mathcal{F}) \to H^{i+1}(X, \mathcal{F}) \to \dots.$$

(d) Categorify this to show that $\mathrm{Ab}(X_{\text{ét}}) \simeq \mathrm{Ab}(U_{\text{ét}}) \times_{\mathrm{Ab}((U\cap V)_{\text{ét}})} \mathrm{Ab}(V_{\text{ét}})$ (the category of triples $(\mathcal{F}, \mathcal{G}, \phi)$ with $\mathcal{F} \in \mathrm{Ab}(U_{\text{ét}})$, $\mathcal{G} \in \mathrm{Ab}(V_{\text{ét}})$, and $\phi : \mathcal{F}|_{U \cap V} \simeq \mathcal{G}|_{U\cap V}$ an isomorphism).

(e) Show that $U \mapsto H^1(U_{\text{ét}}, \mathbf{Z}/n)$ defines a Zariski sheaf on a normal noetherian scheme X. Is normality necessary?

(f) Show that $U \mapsto H^2(U_{\text{ét}}, \mathbf{G}_m)$ defines a Zariski sheaf on a regular noetherian scheme X. Is regularity necessary?

9. (Refined Mayer-Vietoris) Let X be a scheme. Let $j : U \to X$ be an open immersion, and let $f : V \to X$ be an étale morphism such that $f(V) \cup U = X$, and f is an isomorphism over $X - U$. The cartesian square defined by f and j is often called a "elementary" Nisnevich square. The reason for this terminology is that j, f define a Nisnevich cover, and, moreover, such Nisnevich covers "generate" the Nisnevich topology in a suitable sense. We will see below that a Mayer-Vietoris sequence also holds in étale cohomology for such elementary squares.

(a) For any sheaf $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$, show that there is a short exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \times_X V) \oplus \mathcal{F}(V \times_X V).$$

(b) If $\Delta : V \to V \times_X V$ is the diagonal, then show that Δ is an isomorphism onto a connected component, so $V \times_X V = \Delta(V) \sqcup W$. Show also that the composite map $W \to V \times_X V \to X$ has image contained entirely in U.

(c) Writing $V \times_X V = \Delta(V) \sqcup W$ and using the previous exercise, show that for any sheaf $\mathcal{F} \in \mathrm{Ab}(X_{\text{ét}})$, there is a short exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \times_X V).$$

(d) Show that the sequence above is right exact when $\mathcal{F}$ is injective. Deduce as before that there is a long exact sequence of cohomology groups

$$\dots H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \oplus H^i(V, \mathcal{F}) \to H^i(U \times_X V, \mathcal{F}) \to H^{i+1}(X, \mathcal{F}) \to \dots.$$

(e) Categorify this to show that $\mathrm{Ab}(X_{\text{ét}}) \simeq \mathrm{Ab}(U_{\text{ét}}) \times_{\mathrm{Ab}((U \times_X V)_{\text{ét}})} \mathrm{Ab}(V_{\text{ét}})$.

(f) Give an example to show that the hypothesis that f is an isomorphism over $X - U$ is necessary for the conclusion on long exact sequences above.

(g) (*) Show that $U \mapsto H^1(U_{\text{ét}}, \mathbf{Z}/n)$ defines a Nisnevich sheaf on a normal noetherian scheme X.

(h) (*) Show that $U \mapsto H^2(U_{\text{ét}}, \mathbf{G}_m)$ defines a Nisnevich sheaf on a regular noetherian scheme X.

10. Show that on a noetherian scheme, all torsion sheaves are filtered colimits of constructible ones.

11. The goal of this exercise is to construct sheaves by glueing along stratifications; this will be used to give a very explicit description of the category of sheaves constructible along a *fixed* stratification in terms of local systems on the strata. Let X be a scheme. Let $i : Z \to X$ be a closed immersion with open complement $j : U \to X$. Let $\mathcal{C}$ be the category of triples (F_Z, F_U, ϕ) where $F_Z \in \mathrm{Shv}(Z_{\text{ét}})$, $F_U \in \mathrm{Shv}(U_{\text{ét}})$, and $\phi : F_Z \to i^* j_* F_U$ is a map; the morphisms are defined in the evident manner.

(a) Show that there is a natural functor $\mathrm{can} : \mathrm{Shv}(X_{\text{ét}}) \to \mathcal{C}$ that lifts the restriction functor $\mathrm{Shv}(X_{\text{ét}}) \to \mathrm{Shv}(Z_{\text{ét}}) \times \mathrm{Shv}(U_{\text{ét}})$.

(b) Show that there is a natural functor $\mathrm{Glue} : \mathcal{C} \to \mathrm{Shv}(X_{\text{ét}})$ defined by

$$\mathrm{Glue}(F_Z, F_U, \phi) = i_* F_Z \times_{i_* i^* j_* F_U} j_* F_U.$$

(c) By contemplating behaviour on stalks, show that can and Glue define a mutually inverse equivalence of categories. What do locally constant sheaves in $\mathrm{Shv}(X_{\text{ét}})$ correspond to?

(d) Show that the constant sheaf $\underline{\mathbf{Z}/n}$ is not isomorphic to $j_! \underline{\mathbf{Z}/n} \oplus i_* \underline{\mathbf{Z}/n}$ in general.

(e) Let $X = \mathrm{Spec}(R)$ be a strictly henselian discrete valuation ring with fraction field K. Show that there is an equivalence of categories between $\mathrm{Ab}(X)$ and triples (A, M, ψ) where A is an abelian group, M is G_K-module (where G_K is the absolute Galois group of K), and $A \to M^{G_K}$ is a map of abelian groups.

Remark 0.2. Note that there are absolutely no conditions on the homomorphism $A \to M^{G_K}$. This shows that the proper base change theorem alone is woefully inadequate for describing the relation between the cohomology of the generic and special fibres of a proper family; *smooth* base change will give a better handle on this situation.

(f) (*) Let $X = \mathbf{A}^2_{\mathbf{C}}$. Explicitly describe the full subcategory of $\mathrm{Ab}(X_{\text{ét}})$ spanned by constructible sheaves $\mathcal{F}$ that are locally constant sheaves of $\mathbf{Z}/p$-vector spaces along the two-step stratification $\{\mathbf{A}^2 - \mathbf{A}^1 \times \{0\}, \mathbf{A}^1 \times \{0\}\}$. Can you describe the analogous category for the three-step stratification $\{\mathbf{A}^2 - \mathbf{A}^1 \times \{0\}, (\mathbf{A}^1 - \{0\}) \times \{0\}, \{(0,0)\}\}$? Do you see a general pattern here?

12. The goal of this exercise is to discuss some further consequences of formal GAGA beyond those discussed in class. Let $S = \mathrm{Spec}(A)$ be the spectrum of a complete noetherian local ring $(A, \mathfrak{m})$, and let $\widehat{S} = \mathrm{Spf}(A)$. Fix a proper morphism $f : X \to S$, and let $\widehat{f} : \widehat{X} \to \widehat{S}$ be the formal completion of X along the special fibre $X_0 = f^{-1}(\mathrm{Spec}(A/\mathfrak{m})) \subset X$ (and the same convention for non-proper schemes).

    (a) Let $\widehat{Z} \to \widehat{X}$ be a finite morphism of formal $\widehat{S}$-schemes. Show that there exists a unique finite morphism $Z \to X$ that recovers $\widehat{Z} \to \widehat{X}$ on formal completion.

    (b) Let $Z \to S$ be any separated morphism of finite type. Let $\widehat{X} \to \widehat{Z}$ be a morphism of formal $\widehat{S}$-schemes. Show that there is a unique morphism $X \to Z$ that recovers $\widehat{X} \to \widehat{Z}$ on formal completion.

    (c) Give an example of two S-schemes Z_1 and Z_2, and a map $\widehat{Z_1} \to \widehat{Z_2}$ of formal $\widehat{S}$-schemes that does not come from a map $Z_1 \to Z_2$ of S-schemes.

    (d) Let $\widehat{g} : \widehat{Z} \to \widehat{S}$ be a proper morphism of formal schemes. Assume that $\dim(Z_0) = 1$. By lifting suitable line bundles, show that $\widehat{g}$ is algebraisable, i.e., that there exists a proper morphism $g : Z \to S$ which recovers $\widehat{g}$ on formal completion.

    (e) (*) Can you give an example of a morphism of relative dimension ≥ 2 where the conclusion of the previous exercise fails?

    (f) Let $g_0 : Z_0 \to \mathrm{Spec}(A/\mathfrak{m})$ be a smooth projective curve. Using a previous exercise on deformation theory, show that there exists a smooth projective morphism $g : Z \to S$ that recovers g_0 as its special fibre. Deduce that every smooth projective curve over a positive characteristic field "lifts" to characteristic 0.

    (g) (*) Can you show the previous claim for a *nodal* curve?

13. Let K/k be an extension of algebraically closed fields. Given a projective variety X/k and a constructible sheaf $\mathcal{F}$ on X, show that the base change map $H^i(X, \mathcal{F}) \to H^i(X_K, \mathcal{F}_K)$ is an isomorphism for all i (using proper base change).

14. Let $S = \mathrm{Spec}(R)$ be the spectrum of a complete dvr with an algebraically closed residue field k. Fix an algebraic closure $\overline{K}$ of the fraction field K of R. Let $s : \mathrm{Spec}(k) \to S$ and $\overline{\eta} : \mathrm{Spec}(\overline{K}) \to S$ be the natural maps. Let $g : C \to S$ be a smooth projective morphism whose fibres are geometrically connected. Consider the sheaf $\mathcal{F} = R^1 g_*(\mathbf{Z}/n)$. We will understand how this sheaf interpolates the cohomology of the generic and special fibres of g.

    (a) Using the proper base change theorem, show that $\mathcal{F}(S) = \mathcal{F}_s = H^1(C_s, \mathbf{Z}/n)$.

    (b) Using the proper base change theorem, show that $\mathcal{F}_{\overline{\eta}} = H^1(C_{\overline{\eta}}, \mathbf{Z}/n)$.

    (c) Show that there is a natural "cospecialisation" map $c : H^1(C_s, \mathbf{Z}/n) \to H^1(C_{\overline{\eta}}, \mathbf{Z}/n)$. Moreover, show that c is invariant under the natural action of $\mathrm{Gal}(\overline{K}/K)$ on the target.

    (d) The cospecialisation map relates isomorphism classes of $\mathbf{Z}/n$-étale covers of C_s to isomorphism classes of $\mathbf{Z}/n$-étale covers of $C_{\overline{\eta}}$. Categorify this as follows: show that there is a functor $\widetilde{c}$ from the category of finite étale covers of C_s to $\mathrm{Gal}(\overline{K}/K)$-equivariant finite étale covers of $C_{\overline{\eta}}$.

    (e) (*) Show that c is always injective, even if n is not invertible on S. I suggest first showing that the restriction functor from finite étale covers of C to that of C_η is fully faithful, then varying the dvr S, i.e., replacing it by a finite extension, and then taking a limit to get fraction field $\overline{K}$.

(f) Give an example to show that the injectivity of c fails if g is not assumed to be proper.

(g) Now assume that g is a relative curve, and that n is invertible on S. Show that the map c is an isomorphism by counting sizes. What does this say about the functor $\widetilde{c}$? Can you see it directly? What does this say about the $\mathrm{Gal}(\overline{K}/K)$-representation $H^1(C_{\overline{\eta}}, \mathbf{Z}/n)$?

Remark 0.3. Later in the class we will see that the assumption on the relative dimension is not necessary, and nor is the assumption that we work in cohomological degree 1. The general conclusion is that $\mathrm{Gal}(\overline{K}/K)$ acts trivially on $H^i(X_{\overline{\eta}}, \mathbf{Z}/n)$ for all proper smooth morphisms $X \to S$ and integers n invertible on S; such representations are called *unramified*.

15. Let X be a smooth $\mathbf{C}$-variety. Let X_{an} denote the analytic étale site of X, i.e., the objects are maps $U \to X$ of complex analytic spaces that are local isomorphisms on U. Let $\mathcal{O}_{X_{\mathrm{an}}}$ denote the structure sheaf on this site. We will prove compatibility of algebraic and analytic first Chern class maps.

(a) Show that there is a natural morphism $X_{\mathrm{an}} \to X(\mathbf{C})_{top}$ of sites; here $X(\mathbf{C})_{top}$ is the site defined by the topological space $X(\mathbf{C})$. Show this map induces isomorphisms $H^i(X(\mathbf{C}), A) \simeq H^i(X_{\mathrm{an}}, \underline{A})$ for any i and any abelian group A.

(b) Show that analytification defines a morphism of ringed sites $f : (X_{\mathrm{an}}, \mathcal{O}_{X_{\mathrm{an}}}) \to (X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}})$.

(c) Show that analytification induces a map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathrm{an}}) \simeq H^1(X_{\mathrm{an}}, \mathcal{O}^*_{X_{\mathrm{an}}})$.

(d) Fix an integer n. Show that $f^*\mu_n = \mu_n$. Deduce that there is a natural map $H^*(X_{\text{ét}}, \mu_n) \to H^*(X_{\mathrm{an}}, \mu_n)$.

(e) Using Serre's GAGA theorem, show that $H^1(X_{\text{ét}}, \mu_n) \simeq H^1(X_{\mathrm{an}}, \mu_n)$ if X is projective. We will see later that this statement is true without projectivity, and in any degree.

(f) Show that the usual formulas define an "exponential" sequence on X_{an}

$$0 \to \mathbf{Z} \to \mathcal{O}_{X_{\mathrm{an}}} \to \mathcal{O}^*_{X_{\mathrm{an}}} \to 0.$$

(g) Show that the pushout of the exponential sequence along $\mathbf{Z} \to \mu_n$ (given by $1 \mapsto e^{\frac{2\pi i}{n}}$) is isomorphic to the analytic "Kummer" sequence

$$0 \to \mu_n \to \mathcal{O}^*_{X_{\mathrm{an}}} \to \mathcal{O}^*_{X_{\mathrm{an}}} \to 0.$$

Deduce that the modulo n first Chern class map $c_1 : \mathrm{Pic}(X_{\mathrm{an}}) \to H^2(X_{\mathrm{an}}, \mathbf{Z}) \to H^2(X_{\mathrm{an}}, \mu_n)$ is given by the coboundary map for the analytic Kummer sequence.

(h) Show that there is a map of short exact sequences from f^* applied to the Kummer sequence on $X_{\text{ét}}$ to the Kummer sequence on X_{an}.

(i) Deduce that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(X_{\mathrm{an}}) \\ \downarrow & & \downarrow \\ H^2(X, \mu_n) & \longrightarrow & H^2(X_{\mathrm{an}}, \mu_n). \end{array}$$

Here the horizontal maps are the analytification ones, the left vertical map comes from the Kummer sequence, and the right vertical map is the classical first Chern class map.