Problem set 5

Notation 0.1. When talking about coherent sheaves, we use f^{-1} to denote the set-theoretic inverse image functor associated a map $f : X \to Y$ of schemes.

1. The goal of this exercise is prove a theorem of Nagata showing that henselizations of noetherian local rings are also noetherian. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring.

   (a) Show that $R/\mathfrak{m}^n \simeq R^h/\mathfrak{m}^n$, and hence $\widehat{R} \simeq \widehat{R^h}$.

   (b) Using the criterion formulated in terms of the base-change behaviour of finitely generated ideals, show that $R^h \to \widehat{R^h}$ is faithfully flat.

   (c) Show that R^h is noetherian.

   (d) (*) Show that R^{sh} is noetherian (or read EGA III, Chapter 0, Lemma 10.3.1.3).

2. (*) Let $(R, \mathfrak{m}, k)$ be a henselian local ring. Show that reduction modulo $\mathfrak{m}$ defines an equivalence of categories between finite étale covers of R, $\widehat{R}$, and k.

3. (Topological description of henselian rings) Let $(R, \mathfrak{m}, k)$ be a henselian ring.

   (a) Given a finite separable extension $i : k \to k'$, show that there is a unique henselian ring $(R', \mathfrak{m}', k')$ finite étale over $(R, \mathfrak{m}, k)$ such that $R \to R'$ reduces to i modulo $\mathfrak{m}$.

   (b) Show that the association $k' \mapsto R'$ (from the previous exercise) extends to a sheaf on $\mathrm{Spec}(k)_{\text{ét}}$.

   (c) Given a pointed scheme (X, x) with $i : \mathrm{Spec}(\kappa(x)) \to X$ denoting the residue field, show that $\Gamma(\mathrm{Spec}(\kappa(x))_{\text{ét}}, i^{-1}\mathcal{O}_{X_{\text{ét}}}) \simeq \mathcal{O}_{X,x}^h$.

4. Let $(R, \mathfrak{m}, k)$ be a local normal domain with fraction field K. Assume that K is separably closed. Please do not assume R is noetherian for this exercise.

   (a) Let $A \to B$ be a faithfully flat ring map of domains. If there is a factorisation $A \to B \to \mathrm{Frac}(A)$ with $B \to \mathrm{Frac}(A)$ injective, then show that $A = B$.

   (b) Show that any $R \to S$ étale can be written as $S = \prod_{i=1}^k S_i$ with each $R \to S_i$ an open immersion, and at least one $R \to S_i$ an isomorphism.

   (c) Show that R is strictly henselian.

5. (The Nisnevich topology) Let X be a noetherian scheme, and fix $n \in \mathbf{Z}$. Define the small *Nisnevich site* X_{Nis} as follows: objects are étale maps $U \to X$, and a family $\{U_i \to U\}$ is a Nisnevich covering if it is an étale covering with the additional property that any field valued point $\mathrm{Spec}(k) \to U$ lifts to some U_i. We use the subscript $_{\mathrm{Nis}}$ to denote cohomology in the Nisnevich topology.

(a) Show that descent and cohomology for quasi-coherent sheaves work as expected, i.e., quasi-coherent $\mathcal{O}_X$-modules define Nisnevich sheaves, and the Nisnevich cohomology of the resulting sheaves agrees with Zariski cohomology.

(b) Given a point $x \in X$, let $i : \mathrm{Spec}(\kappa(x)) \to X$ denote the corresponding map. Show that $\Gamma(\mathrm{Spec}(k)_{\mathrm{Nis}}, i^{-1}\mathcal{O}_{X_{\mathrm{Nis}}}) \simeq \mathcal{O}^h_{X,x}$.

(c) Show that exactness of a sequence of Nisnevich sheaves can be detected by pullbacks along all maps of the form $i : \mathrm{Spec}(\kappa(x)) \to X$, where $x \in X$ is a point.

(d) Let X be a normal connected scheme. Show that $H^1_{\mathrm{Nis}}(X, \mathbf{Z}/n) = 0$. This shows that Nisnevich cohomology does not agree with étale cohomology in general.

(e) Let X be a normal connected scheme. By analysing the Leray spectral sequence for $X_{\text{ét}} \to X_{\mathrm{Nis}}$, show that the presheaf $U \mapsto H^1_{\text{ét}}(U, \mathbf{Z}/n)$ defines a *sheaf* for the Nisnevich topology.

(f) Let $X \subset \mathbf{P}^2$ be an irreducible nodal cubic. Show that $H^1_{\mathrm{Nis}}(X, \mathbf{Z}/2) \simeq \mathbf{Z}/2$ while $H^1_{\mathrm{Zar}}(X, \mathbf{Z}/2) \simeq 0$. This shows that Nisnevich cohomology does not agree with Zariski cohomology in general.

6. (Finite morphisms, following Morel-Voevodsky) We will see that finite morphisms are acyclic for the Nisnevich topology (like étale), but not for the Zariski topology.

   (a) Let $f : X \to Y$ be a morphism of schemes. Show that there is a morphism of sites $f : X_{\mathrm{Nis}} \to Y_{\mathrm{Nis}}$. Moreover, if f is finite, show that f_* is exact.

   (b) Let $f : X \to Y$ be a morphism of schemes, with Y the spectrum of a local ring. Show that the functor $f : X_{\mathrm{Zar}} \to Y_{\mathrm{Zar}}$ induces an exact functor f_* if and only if $H^i(X_{\mathrm{Zar}}, F) = 0$ for all $F \in \mathrm{Ab}(X_{\mathrm{Zar}})$ and $i > 0$.

   Let X be the spectrum of the semilocal ring of $\mathbf{A}^2$ at two fixed points x_0 and x_1. Choose two irreducible curves $C_1, C_2 \subset \mathbf{A}^2$ such that $C_1 \cap C_2 = \{x_0, x_1\}$. Let $U \xrightarrow{j_1} V \xrightarrow{j_2} X$ be the sequence of open immersions defined by $V = X - \{x_0, x_1\}$, and $U = X - (C_1 \cup C_2)$; let $j = j_2 \circ j_1$.

   (c) Using the Mayer-Vietoris sequence for the open cover $V = (V - (C_1 \cap V)) \cup (V - (C_2 \cap V))$, show that $H^1(V_{\mathrm{Zar}}, j_{1,!}\mathbf{Z}))$ is non-zero.

   (d) Using the Mayer-Vietoris sequence for the open cover $X = (X - \{x_0\}) \cup (X - \{x_1\})$, show that $H^2(X_{\mathrm{Zar}}, j_!(\mathbf{Z}))$ is non-zero.

   (e) Show that there is a finite morphism $X \to \mathrm{Spec}(R)$ for a local ring R. Conclude that finite morphisms need not be acyclic for the Zariski topology.

7. (Transfers) The goal of this exercise is to discuss the existence of norm maps in étale cohomology with coefficients in $\mathbf{G}_m$.

   (a) Let $f : R \to S$ be a finite locally free map of algebras. Given $s \in S$, show that multiplication by s action of s on S gives a well-defined characteristic polynomial $\phi_s \in R[t]$ of degree $\deg(f)$. In particular, show that there is a well-defined element $\mathrm{Nm}(s) \in R$ such that $\mathrm{Nm}(f(r)) = r^{\deg(f)}$.

   (b) Let $f : Y \to X$ be a finite locally free map of schemes with X connected. Prove that there is a natural "norm" map $\mathrm{Nm}_f : f_*\mathbf{G}_m \to \mathbf{G}_m$ in $\mathrm{Ab}(\mathrm{Sch}_{/X,\mathrm{fppf}})$ such that the composite $\mathbf{G}_m \xrightarrow{f^*} f_*f^*\mathbf{G}_m \xrightarrow{\mathrm{Nm}_f} \mathbf{G}_m$ is $x \mapsto x^{\deg(f)}$.

(c) Let $f : Y \to X$ be a finite locally free map of schemes with X connected. Show that there is a natural map $H^i(\mathrm{Nm}_f) : H^i_{\mathrm{fppf}}(Y, \mathbf{G}_m) \to H^i_{\mathrm{fppf}}(X, \mathbf{G}_m)$ such that the composite

$$H^i_{\mathrm{fppf}}(X, \mathbf{G}_m) \stackrel{f^*}{\to} H^i_{\mathrm{fppf}}(Y, \mathbf{G}_m) \stackrel{H^i(\mathrm{Nm}_f)}{\to} H^i_{\mathrm{fppf}}(X, \mathbf{G}_m)$$

is multiplication by $\deg(f)$, and similarly for étale cohomology.

(d) Show that for $X = \mathrm{Spec}(R)$ with R henselian local, the groups $H^i_{\text{ét}}(k, \mathbf{G}_m)$ are torsion for $i > 0$. Can you give an example of a henselian ring R for which the torsion orders of $H^2_{\text{ét}}(\mathrm{Spec}(R), \mathbf{G}_m)$ can be arbitrarily large?

8. (More transfers) The goal of this exercise is to show norm maps exist for constant coefficients as well. As a corollary, we will see that étale cohomology with $\mathbf{Q}$-coefficients is typically uninteresting. Fix a scheme X. Let A abusively denote the constant sheaf on $X_{\text{ét}}$ associated to an abelian group A.

   (a) Let $f : Y \to X$ be a finite étale map. Show that there is a norm map $\mathrm{Nm}_f : f_* f^* A \to A$ in $\mathrm{Shv}(X_{\text{ét}})$ such that the composite $A \stackrel{f^*}{\to} f_* f^* A \stackrel{\mathrm{Nm}_f}{\to} A$ is multiplication by $\deg(f)$.

   (b) Assume that X is a henselian local scheme. Show that $H^i_{\text{ét}}(X, A)$ is torsion for $i > 0$.

   (c) Assume that X is a henselian local scheme. Show that $H^i_{\text{ét}}(X, \mathbf{Q}) = 0$ for $i > 0$. Deduce that $H^1_{\text{ét}}(X, \mathbf{Z}) = 0$ for X henselian.

   (d) By contemplating singular curves, give an example of a local ring R such that $H^1_{\text{ét}}(\mathrm{Spec}(R), \mathbf{Z})$ is non-zero.