Problem set 4

Notation 0.1. A family $\{f_i : X_i \to X\}$ of maps is called an *fppf* cover if each f_i is locally finitely presented and flat, and $\sqcup_i X_i \to X$ is surjective. An fppf cover is a *Zariski cover* if each $X_i \to X$ is an open immersion.

1. The goal of this exercise is to give a criterion for testing when a presheaf is an fppf sheaf; the formulation below is meant to formalise the intuition that all maps are affine Zariski locally on the source and target. Fix a base scheme S, and let $F \in \mathrm{PShv}(\mathrm{Sch}_{/S})$. Show that F is an fppf sheaf if and only if the following two conditions are satisfied:

   (a) For $T \in \mathrm{Sch}_{/S}$, and any Zariski cover $\{T_i \hookrightarrow T\}$, the sequence

   $$F(T) \to \prod_i F(T_i) \rightrightarrows \prod_{i \neq j} F(T_i \cap T_j).$$

   is exact.

   (b) For any fppf map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine S-schemes, the sequence

   $$F(A) \to F(B) \rightrightarrows F(B \otimes_A B)$$

   is exact.

   Use this criterion to show that $h_X \in \mathrm{PShv}(\mathrm{Sch}_{/S})$ is an fppf sheaf for any $X \in \mathrm{Sch}_{/S}$.

2. Fix a base scheme S. Let $u : \mathrm{Aff}_{/S} \subset \mathrm{Sch}_{/S}$ be the full subcategory spanned by maps $X \to S$ with X affine; equip $\mathrm{Aff}_{/S}$ with the fppf topology, i.e., a family of maps in $\mathrm{Aff}_{/S}$ is an fppf cover if it is so after applying u. Show that there is a morphism of sites $\mathrm{Sch}_{/S,\mathrm{fppf}} \to \mathrm{Aff}_{/S,\mathrm{fppf}}$ that induces an equivalence $\mathrm{Shv}(\mathrm{Sch}_{/S,\mathrm{fppf}}) \simeq \mathrm{Shv}(\mathrm{Aff}_{/S,\mathrm{fppf}})$.

3. (Descent for quasi-affine maps) Let $f : T \to S$ be a faithfully flat and finitely presented map of schemes. Let $F \in \mathrm{Shv}(\mathrm{Sch}_{/S,\mathrm{fppf}})$ be such that f^*F is representable by quasi-affine morphism $X \to T$. Show that f is representable by a quasi-affine morphism.

4. The goal of this exercise is to show that cohomology groups can be computed in either the big or the small topologies without affecting the answer. Fix a base scheme X. Let $\mathcal{C} = \mathrm{Sch}_{/X,\text{ét}}$ and $\mathcal{D} = X_{\text{ét}}$ denote the big and small sites of X respectively.

   (a) Show that there is a morphism of sites $f : \mathcal{C} \to \mathcal{D}$.

   (b) Show that $f_* : \mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}(\mathcal{D})$ is exact.

   (c) Show that f_* preserves injectives and resolutions, and hence injective resolutions.

   (d) Conclude that for any $F \in \mathrm{Sch}_{/X,\text{ét}}$, there is a natural isomorphism $H^*(X_{\text{ét}}, f_*F) \simeq H^*(X, F)$, where we view the right (resp. left) hand side as cohomology in $\mathcal{C}$ (resp. $\mathcal{D}$).

5. (Cech description of H^1) Let $\mathcal{C}$ be a Grothendieck topology, and let $G \in \mathrm{Ab}(\mathcal{C})$.

   (a) Show that for any $U \in \mathcal{C}$, and $\alpha \in H^k(U, G)$ with $k > 0$, there exists a cover $\{f_i : U_i \to U\}$ such that $f_i^* \alpha = 0$. (Use injective resolutions)

   (b) Show that there is a natural isomorphism $\check{H}^1(U, G) \simeq H^1(U, G)$ for any $U \in \mathcal{C}$. (Use the spectral sequence)

6. Let $\mathcal{C}$ be a Grothendieck topology, and let $G \in \mathrm{Shv}(\mathcal{C})$ be a sheaf of (not necessarily abelian) groups. Show that the formula defining the first Cech cohomology group $\check{H}^1(U, G)$ still makes sense, but gives a pointed set, rather than a group. When $\mathcal{C}$ is the category of H-sets for some group H, this is nicely discussed in the Appendix to Chapter VII in Serre's *Local fields* book.

7. (Torsors) Let $\mathcal{C}$ be a Grothendieck topology with a final object $*$, and let $G \in \mathrm{Shv}(\mathcal{C})$ be a sheaf of groups. A *G-torsor* is a sheaf $F \in \mathrm{Shv}(\mathcal{C})$ together with a simply transitive action of G such that $F(U_i) \neq \emptyset$ for some cover $\{U_i \to *\}$; here simply transitive means that for any $T \in \mathcal{C}$ and $g \in F(T)$, the orbit map $G(T) \to F(T)$ is an isomorphism.

   (a) Let H be a sheaf of sets with a G-action. Show that H is a G-torsor if and only if the map $G \times H \to H \times H$ defined by $(g, h) \mapsto (gh, h)$ is an isomorphism, and H has sections over some cover of $*$.

   (b) Given a morphism $f : \mathcal{D} \to \mathcal{C}$ of sites (with final objects), show that G-torsors pullback to f^*G-torsors in the obvious sense.

   (c) Show that there is a bijection between $\check{H}^1(*, G)$ and isomorphism classes of G-torsors, and that this bijection respects pullbacks along morphisms of sites.

8. Let $\mathcal{C} = \mathrm{Sch}_{/k,\mathrm{fppf}}$, and let $G \to \mathrm{Spec}(k)$ be a group scheme, i.e., a k-scheme such that $\mathrm{Hom}_k(-, G)$ is valued in groups. Examples include $G = \mathbf{G}_a, \mathbf{G}_m, \mu_n, \mathrm{GL}_n$ and abelian varieties.

   (a) Show that if G is affine, then any G-torsor is representable by an affine scheme, i.e., if a sheaf F is a G-torsor, then $F = \mathrm{Hom}_k(-, X)$ for some affine k-scheme X with a G-action.

   (b) Show the converse to the above statement, i.e., show that if a G-torsor F is representable by an affine k-scheme X, then G is itself affine.

   (c) Compute $H^1_{\mathrm{fppf}}(\mathrm{Spec}(\mathbf{F}_q), \mathbf{Z}/n)$ and $H^1_{\mathrm{fppf}}(\mathbf{P}^1_{\mathbf{C}}, \mathbf{Z}/n)$ for any n.

9. (Galois descent) Let T be a scheme with a G-action. Assume that there is a finite group G acting on T. Recall that a G-equivariant sheaf on T is the data of a sheaf $E \in \mathrm{Shv}(T_{\text{ét}})$ together with isomorphisms $\sigma_g : g^*E \simeq E$ for each $g \in G$ such that $\sigma_{gh} = \sigma_h \circ h^*(\sigma_g)$. The category of all such sheaves is denoted $\mathrm{Shv}(T_{\text{ét}})^G$. Let $f : T \to S$ be a G-invariant morphism.

   (a) Show that pullback defines a functor $\overline{f}^* : \mathrm{Shv}(S_{\text{ét}}) \to \mathrm{Shv}(T_{\text{ét}})^G$.

   (b) Assume T is an étale G-torsor over S via f. Show that $\overline{f}^*$ is an equivalence.

   (c) What does the previous item say for f being the map $\mathrm{Spec}(\mathbf{C}) \to \mathrm{Spec}(\mathbf{R})$ with $G = \mathbf{Z}/2$?

10. (Special case of Grothendieck's theorem) Let $G \to S$ be a smooth affine group scheme, and let $F \in \mathrm{Shv}(\mathrm{Sch}_{/S,\mathrm{fppf}})$ be a G-torsor. Show that there exists an *étale* cover $\{S_i \to S\}$ such that $F(S_i) \neq \emptyset$. (The point here is that we only assume existence of sections over an fppf cover, and deduce sections over an étale cover)

11. (Twisted forms) Let $\mathcal{C}$ be a Grothendieck topology with a final object $*$, and let $F \in \mathrm{Shv}(\mathcal{C})$. A *twisted form* of F is a sheaf $G \in \mathrm{Shv}(\mathcal{C})$ which is isomorphic to F over some cover of $*$.

    (a) Construct a bijection between $\check{H}^1(*, \mathrm{Aut}(F))$ and isomorphism classes of twisted forms of F.

    (b) The identity map $\check{H}^1(*, \mathrm{Aut}(F)) \to \check{H}^1(*, \mathrm{Aut}(F))$ can be viewed as a procedure for associating $\mathrm{Aut}(F)$-torsors to twisted forms of F. What is this procedure explicitly?

    (c) Assume that $F \in \mathrm{Shv}(\mathcal{C})$ is a sheaf of groups. Then the left action F on itself defines a group homomorphism $F \to \mathrm{Aut}(F)$, and so a map $H^1(*, F) \to H^1(*, \mathrm{Aut}(F))$. What is this map in terms of torsors and twisted forms?

    (d) Now assume $F \in \mathrm{Mod}(R)$ for some sheaf of rings $R \in \mathrm{Shv}(\mathcal{C})$. Construct a bijection between $\check{H}^1(*, \mathrm{Aut}_R(F))$ and twisted forms of F as an R-module, i.e., R-modules M which are locally isomorphic to F as R-modules.

12. (Non-abelian Hilbert 90) Let X be a scheme, and let $\mathcal{C} = \mathrm{Sch}_{/X,\mathrm{fppf}}$.

    (a) Show that the functor $U \mapsto \mathrm{GL}_n(\Gamma(U, \mathcal{O}_U))$ is representable by a group scheme. In particular, it defines a sheaf of groups on $\mathcal{C}$. We call the representing scheme and the sheaf GL_n.

    (b) Show that there is a natural bijection between $\check{H}^1_{\mathrm{fppf}}(U, \mathrm{GL}_n)$, $\check{H}^1_{\text{ét}}(U, \mathrm{GL}_n)$, $\check{H}^1_{\mathrm{Zar}}(U, \mathrm{GL}_n)$, and the isomorphism classes of rank n vector bundles on U (in the classical sense); here the subscript is telling us the site to compute Cech cohomology in.

    (c) What is the GL_n-torsor associated to a rank n vector bundle (in any of the above topologies)?

13. Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites, and let $F \in \mathrm{Ab}(\mathcal{C})$. Show that $R^i f_* \mathcal{F}$ is the sheafification of the presheaf $U \mapsto H^i(u_f(U), F)$ where $u_f : \mathcal{D} \to \mathcal{C}$ is the continuous functor underlying f.

14. Fix a field k. Consider the presheaf $F \in \mathrm{PShv}(\mathrm{Sch}_{/k})$ defined by $F(U) = \Gamma(U, \Omega^1_{U/k})$.

    (a) Show that F is an étale sheaf.

    (b) Give an example to show that F is not an fppf sheaf.

    (c) (*) What happens in the smooth topology?

15. The goal of this exercise is to discuss certain strange phenomena that occur on small sites. Let k be a field of characteristic p. Let X_0 be a reduced k-scheme, and let $X = X_0 \times_k \mathrm{Spec}(k[\epsilon])$. Let $i : X_0 \to X$ be the closed immersion defined by $\epsilon \mapsto 0$.

    (a) Show that i^* induces an equivalence $\mathrm{Shv}(X_{\text{ét}}) \simeq \mathrm{Shv}(X_{0,\text{ét}})$. (Use the equivalence between $X_{0,\text{ét}}$ and $X_{\text{ét}}$ from a previous problem set)

    (b) Show that $H^i(X_{0,\text{ét}}, \mu_p) = 0$ for all values of i.

    (c) Show that $i^*(\mu_p) = \mathbf{G}_a$. (This is not a contradiction to a previous exercise because μ_p is *not* representable on the small site $X_{\text{ét}}$)

    (d) Give an example to show that $H^i(X_{\text{ét}}, \mu_p)$ is not always 0.

    (e) Check for yourself that the preceding phenomenon does not occur in characteristic 0: the group scheme μ_p *is* representable on the small site in that case.