Problem set 3

Notation 0.1. For an object U in a category $\mathcal{C}$, we let $\mathcal{C}_{/U}$ denote the category of objects lying over U, and let $h_U \in \mathrm{PShv}(\mathcal{C})$ denote the representable functor $\mathrm{Hom}(-, U)$. For a Grothendieck topology $\mathcal{C}$, the sheafification functor $\mathrm{PShv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C})$ is denoted $F \mapsto F^{\#}$. For a morphism $f : \mathcal{C} \to \mathcal{D}$ of sites, we let $u_f : \mathcal{D} \to \mathcal{C}$ denote the underlying continuous functor, and set $f_* = u^s : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ and $f^* = u_s : \mathrm{Shv}(\mathcal{D}) \to \mathrm{Shv}(\mathcal{C})$ denote the induced adjoint pair. For a scheme X, we use $\mathrm{Sm}_{/X}$ to denote the category of *smooth* X-schemes. A subscript is used to indicate the topology when appropriate; for example, $\mathrm{Sm}_{/X,\text{ét}}$ denotes the category of all smooth X-schemes with the topology defined by étale covers. We will ignore set-theoretic problems.

1. Let $\mathcal{C}$ be a category, and let $F \in \mathrm{PShv}(\mathcal{C})$.
   (a) Show that $\mathrm{Hom}_{\mathrm{PShv}(\mathcal{C})}(h_V, \mathcal{F}) = F(V)$ for any $V \in \mathcal{C}$.
   (b) Show that there is a tautological isomorphism
   $$\operatorname*{colim}_{h_V \to F} h_V \simeq F$$
   in the category $\mathrm{PShv}(\mathcal{C})$. Here the colimit takes place over all diagrams $h_V \to F$ with V variable, and the map is the obvious one.
   (c) Assume now that $\mathcal{C}$ is a Grothendieck topology, and F is a sheaf. Show that we have an isomorphism
   $$\operatorname*{colim}_{h_V \to F} h_V^{\#} \simeq F$$
   in the category $\mathrm{Shv}(\mathcal{C})$. This property is often useful in testing the behaviour of left adjoint functors out of $\mathrm{Shv}(\mathcal{C})$.

2. Let $\mathcal{C}$ be a Grothendieck topology. Show the natural map $\mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ defined by $U \mapsto h_U^{\#}$ preserves finite limits.

3. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor between Grothendieck topologies. Show that the induced functor $u_s : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ satisfies
   $$u_s(h_V^{\#}) = h_{u(V)}^{\#}$$
   for any object $V \in \mathcal{C}$, i.e., representable sheaves pullback to representable ones.

4. The goal of this exercise is to give an example of a procedure for comparing topologies on a fixed category. Let $\mathcal{C}$ be a category, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote two Grothendieck topologies on $\mathcal{C}$, i.e., the underlying category for both $\mathcal{C}_1$ and $\mathcal{C}_2$ is simply $\mathcal{C}$, but the topologies could be different.

(a) Assume that $\mathcal{C}_2$ is finer than $\mathcal{C}_1$, i.e., if $\{U_i \to U\}$ is a covering in $\mathcal{C}_1$, then it is also a covering in $\mathcal{C}_2$. Show that there is a morphism of sites $f : \mathcal{C}_2 \to \mathcal{C}_1$, and that the associated functor $f_* : \mathrm{Shv}(\mathcal{C}_2) \to \mathrm{Shv}(\mathcal{C}_1)$ is fully faithful.

(b) Assume that $\mathcal{C}_2$ is finer than $\mathcal{C}_1$ as above, and also that every covering in $\mathcal{C}_2$ can be refined by one in $\mathcal{C}_1$, i.e., given $\{U_i \to U\}$ a covering in $\mathcal{C}_2$, there exist maps $V_i \to U_i$ such that the induced family $\{V_i \to U\}$ is a covering in $\mathcal{C}_1$. Then show that the pushforward functor $f_* : \mathrm{Shv}(\mathcal{C}_2) \to \mathrm{Shv}(\mathcal{C}_1)$ is an equivalence, i.e., every presheaf that satisfies the sheaf axiom for $\mathcal{C}_1$ also satisfies it for $\mathcal{C}_2$.

5. Let X be a scheme. Then show that there is a natural equivalence $\mathrm{Shv}(\mathrm{Sch}_{/X,\text{ét}}) \simeq \mathrm{Shv}(\mathrm{Sch}_{/X,\mathrm{sm}})$.

6. The goal of this exercise is to explain the construction $j_!$ for a suitable morphism of sites. This will be used in the next exercise to define $j_!$ for an étale morphism j of schemes. Let $\mathcal{C}$ be a Grothendieck topology. Assume $\mathcal{C}$ has finite limits, and let $U \in \mathcal{C}$. We will denote objects of $\mathcal{C}_{/U}$ as maps $V \to U$.

   (a) Show that there is a morphism of sites $j : \mathcal{C}_{/U} \to \mathcal{C}$ with u_j defined by $V \mapsto (V \times U \to U)$.

   (b) For any $F \in \mathrm{Shv}(\mathcal{C})$, show that $j^*(F)(V \to U) = F(V)$. In particular, observe that j^* commutes with all small limits. Conclude that j^* must have a left adjoint.

   (c) Show that the forgetful functor is a continuous functor $u : \mathcal{C}_{/U} \to \mathcal{C}$. Let $j_! = u_s : \mathrm{Shv}(\mathcal{C}_{/U}) \to \mathrm{Shv}(\mathcal{C})$ denote the induced functor. Show that
$$j_!(h^{\#}_{V\to U}) = h^{\#}_V.$$

   (d) Show that $j_!$ is a left adjoint to j^*.

   (e) Assume that $U \to *$ is a monomorphism, where $*$ is a final object in $\mathcal{C}$. Show that there is a natural isomorphism $j^* j_! F \simeq F$ for any $F \in \mathrm{Shv}(\mathcal{C}_{/U})$. Deduce a natural map $j_! F \to j_* F$.

7. Let X be a scheme, and let $j : U \to X$ be an étale morphism.

   (a) Show that there is an equivalence $X_{\text{ét},/U} \simeq U_{\text{ét}}$ of Grothendieck topologies. Conclude that $\mathrm{Shv}(X_{\text{ét},/U}) \simeq \mathrm{Shv}(U_{\text{ét}})$.

   (b) Show that $X_{\text{ét}}$ has finite limits, i.e., check that products and equalizers exist.

   (c) Let $j_! : \mathrm{Shv}(U_{\text{ét}}) \to \mathrm{Shv}(X_{\text{ét}})$ be the functor defined using the previous exercise. Write down an explicit formula for $j_!$.

   (d) Copy the arguments above to show that there exists a functor $j_!^{\mathrm{Ab}} : \mathrm{Ab}(U_{\text{ét}}) \to \mathrm{Ab}(X_{\text{ét}})$, left adjoint to $j^* : \mathrm{Ab}(X_{\text{ét}}) \to \mathrm{Ab}(U_{\text{ét}})$.

   (e) For any $U \in X_{\text{ét}}$, let $\underline{\mathbf{Z}} \in \mathrm{Shv}(U_{\text{ét}})$ denote the sheafification of the constant presheaf with value $\mathbf{Z}$. Show that the collection $S = \{j_!^{\mathrm{Ab}}(\underline{\mathbf{Z}})\}$, indexed by $U \in X_{\text{ét}}$, forms a collection of generators for $\mathrm{Ab}(X_{\text{ét}})$, i.e., any $F \in \mathrm{Ab}(X_{\text{ét}})$ admits an epimorphism from a direct sum of objects in S. Note that since we require étale morphisms to be locally finitely presented, the category $X_{\text{ét}}$ is equivalent to a small category.

   (f) Using Grothendieck's theorem on abelian categories, show that $\mathrm{Ab}(X_{\text{ét}})$ has enough injectives.

   (g) For any ring Λ, show analogous arguments show that $\mathrm{Mod}(X_{\text{ét}}, \underline{\Lambda})$ is an abelian category with enough injectives, where $\underline{\Lambda} \in \mathrm{Shv}(X_{\text{ét}})$ denotes the constant sheaf on Λ.

8. Let X be a topological space. Let $j : U \to X$ be an open subset, with complement $i : Z \to X$. Fix $F \in \mathrm{Ab}(X)$.

   (a) Show that $j^* j_!^{\mathrm{Ab}} F \simeq F$, and $i^* j_!^{\mathrm{Ab}} F = 0$, i.e., $j_!^{\mathrm{Ab}}$ is as an "extension by zero" functor.

   (b) Show that there is an exact sequence
$$0 \to j_!^{\mathrm{Ab}} j^* F \to F \to i_* i^* F \to 0.$$
   This sequence gives us a method to decompose a sheaf on X in terms of those on U and Z. This decomposition will be very useful when we prove theorems about cohomology by doing induction on dimension.

9. The goal of this exercise is to lead you through an example of a continuous functor between naturally occuring sites that does not define a morphism, i.e., the associated pullback is not left exact. This example is due to Behrend, and can be found in Olsson's paper "Sheaves on algebraic stacks."

   Notation 0.2. For a scheme X, let $\mathrm{Sm}_{/X,\mathrm{Zar}}$ denote the category $\mathrm{Sm}_{/X}$ equipped with the Zariski topology. For a morphism $f : X \to Y$ of schemes, temporarily use f^{-1} to denote the left adjoint functor $u_{f,s} : \mathrm{Shv}(\mathrm{Sm}_{/Y,\mathrm{Zar}}) \to \mathrm{Shv}(\mathrm{Sm}_{/X,\mathrm{Zar}})$ induced by the continuous functor $u_f : \mathrm{Sm}_{/Y,\mathrm{Zar}} \to \mathrm{Sm}_{/X,\mathrm{Zar}}$ defined by taking fibre products (check that there is such a continuous functor!).

   Let $X = \mathrm{Spec}(k)$, $Y = \mathbf{A}^1_k$, and $i : X \to Y$ be the map defined by 0.

   (a) Let $\mathcal{O}_Y \in \mathrm{Shv}(\mathrm{Sm}_{/Y,\mathrm{Zar}})$ denote the structure sheaf, i.e., the functor $(U \to Y) \to \mathcal{O}(U)$. Show that $\mathcal{O}_Y$ is a representable sheaf, i.e., there is an object $G \in \mathrm{Sm}_{/Y}$ such that $h_G = h_G^{\#} = \mathcal{O}_Y$.

   (b) Show that there is a natural morphism $a : \mathcal{O}_Y \to \mathcal{O}_Y$ of sheaves defined by multiplication by t, where t is the co-ordinate on $\mathbf{A}^1_k = Y$.

   (c) Show that a has a trivial kernel.

   (d) Show that i^{-1} preserves representable objects.

   (e) Show that $i^{-1}(a)$ does not have a trivial kernel. Conclude that i^{-1} is not left exact, and thus the continuous functor u_i does not define a morphism of sites $\mathrm{Sm}_{/X,\text{ét}} \to \mathrm{Sm}_{/Y,\text{ét}}$.

   (f) Show that the category $\mathrm{Sm}_{/Y,\mathrm{Zar}}$ has fibre products, but no equalizers. In particular, it does not have finite limits.

10. Let $u : \mathcal{C}_2 \to \mathcal{C}_1$ be a continuous functor. Assume that $\mathcal{C}_2$ has all finite limits. Then show that u induces a morphism $\mathcal{C}_1 \to \mathcal{C}_2$ of sites.

11. Let X be a scheme. The goal of this exercise is to show that the containment $\mathrm{Shv}(\mathrm{Sch}_{/X,\text{ét}}) \subset \mathrm{Shv}(\mathrm{Sch}_{/X,\mathrm{Zar}})$ is strict. Fix an $n \in \mathbf{Z}$ invertible on X. Consider the presheaf $F \in \mathrm{PShv}(\mathrm{Sch}_{/X})$ defined by
$$F(U) = \mathcal{O}(U)^*/(\mathcal{O}(U)^*)^n.$$

    (a) Show that the sheafification of F for the étale topology is 0.

    (b) By computing values on a suitable test scheme, show that the Zariski sheafification of F is not zero.

(c) (*) Identify the Zariski sheafification of F with the Zariski sheafification of

$$U \mapsto \{(\mathcal{L}, \alpha) \mid \mathcal{L} \in \mathrm{Pic}(U), \alpha : \mathcal{L}^{\otimes n} \simeq \mathcal{O}_U\}/\cong .$$

Later we will identify the right hand side with $H^1(U_{\text{ét}}, \mu_n)$. This illustrates a typical phenomenon: the presheaf defined by cohomology groups in a given topology does not usually sheafify to 0 in a coarser topology, even though it always sheafifies to 0 in the original topology.