Problem set 2

1. Give an example of a ring map $A \to B$ which satisfies the infinitesimal lifting criterion for being étale, but is not flat itself.

   Hint: Consider the ring $k[\mathbf{Q}_{\geq 0}]$ for some characteristic p field k and build a map.

2. Give an example of a flat ring map $A \to B$ such that $\Omega^1_{B/A} = 0$, but $A \to B$ does not satisfy the infinitesimal lifting criterion for being étale. Note that such a map cannot be finitely presented.

   Hint: try playing with quotients of perfect rings in characteristic p.

3. Let $f : X \to S$ be a locally finitely presented unramified morphism. Show that any section of f is an open immersion. If f is additionally assumed to be separated, then show that a section has to be an isomorphism onto a connected component.

   Hint: Consider the map $X \to X \times_S X$ defined by $(s \circ f, \mathrm{id})$ where $s : S \to X$ is a section of f, and use that $\delta : X \to X \times_S X$ is an open immersion if f is unramified (and also closed if f is separated).

4. A morphism $f : X \to S$ is called a *finite étale cover* if f is finite, surjective, and étale. Classify all finite étale covers of:

   (a) $\mathrm{Spec}(\mathbf{R})$ where $\mathbf{R}$ is the field of real numbers.

   (b) $\mathbf{P}^1_{\mathbf{C}}$ and $\mathbf{P}^1_{\overline{\mathbf{F}_p}}$.

   (c) $\mathbf{A}^1_{\mathbf{C}} = \mathrm{Spec}(\mathbf{C}[t])$.

   (d) $\mathbf{G}_{m,\mathbf{C}} = \mathrm{Spec}(\mathbf{C}[t, t^{-1}])$.

   (e) $\mathrm{Spec}(\mathcal{O}_{C,x})$ where C is a smooth projective curve over $\mathbf{C}$, and $x \in C$ is a closed point. Answer in terms of the projective geometry of C.

   (f) An artinian local $\mathbf{C}$-algebra.

   (g) A complete local $\mathbf{C}$-algebra.

   (h) An elliptic curve E over $\mathbf{C}$.

   (i) The nodal cubic in $\mathbf{P}^2_{\mathbf{C}}$.

   (j) The cuspidal cubic in $\mathbf{P}^2_{\mathbf{C}}$. (The next exercise may be useful here.)

   (k) $\mathrm{Spec}(\mathbf{Z})$.

   (l) $\mathrm{Spec}(\mathbf{Z}_{(p)})$, the local scheme of $\mathrm{Spec}(\mathbf{Z})$ at a prime p. Answer in terms of number fields.

   (m) $\mathrm{Spec}(\mathbf{Z}_p)$, the completion of $\mathrm{Spec}(\mathbf{Z})$ at a prime p.

   (n) (*) $\mathbf{A}^2_{\mathbf{C}} - \{0\}$

5. Let S be a noetherian, integral, normal, *excellent* scheme; you may take S to be a variety if you wish. Let $U \subset S$ be a dense open subset. Show that the restriction functor $X \mapsto X_U := X \times_S U$ is fully faithful on the category of finite étale covers of S. Give an example to show that the normality condition cannot be dropped.

   Hint: For the first part, "normalise." For the example, see list above.

6. Let k be a field, and let X be a k-scheme of finite type. Show that X is smooth over k of dimension d if and only if for every affine $U \subset X$, the ring $\mathcal{O}(U)$ is geometrically regular of dimension d, i.e., for any field extension L/k, the base change $\mathcal{O}(U) \otimes_k L$ is a regular ring of dimension d.

   For the forward direction, use the fact that, locally around x, X is étale over affine space. For the backwards direction, reduce to the case that k is algebrically closed. Now show that $x \mapsto \dim \Omega^1_{X/k} \otimes_k \kappa(x)$ is a constant function by comparing $\dim \Omega^1_{X/k} \otimes \kappa(x)$ with $\dim \mathfrak{m}_x/\mathfrak{m}_x^2$ and using regularity. Conclude by Nakayama that $\Omega^1_{X/k}$ is locally free of rank d. Now find appropriate maps to affine space.

7. (Local structure for étale morphisms) Let $f : X \to S$ be a locally finitely presented morphism which is étale at $x \in X$. Then show that, Zariski locally on S around $f(x)$, there exists a presentation $\mathcal{O}_{X,x} \simeq (\mathcal{O}_S[t]/(f))_{\mathfrak{p}}$, with f monic, and $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_S[t]/(f))$ some prime where f' is invertible.

   Solution sketch: using Zariski's main theorem, realise f as $X \subset \overline{X} \to S$ with $\overline{X} \to S$ finite, and $X \subset \overline{X}$ an open immersion. May work locally on S, so we may assume S is local. Now $\kappa(x)/\kappa(s)$ has a single generator $\overline{g}$ by the primitive element theorem. Lift this generator to an element $g \in \mathcal{O}_{\overline{X}}$ with the property that its image in all other residue fields lying above s is 0; we can do this by the Chinese remainder theorem. This generator defines a map $X \subset \overline{X} \to \mathbf{A}^1_S$. Choose some monic polynomial $f \in \mathcal{O}_S[t]$ that $g \in \overline{X}$ satisfies. Then the map factors as $X \subset \overline{X} \xrightarrow{i} \operatorname{Spec}(\mathcal{O}_S[t]/(f))$. Now observe that by the choice of g, this last map has the property that $i^{-1}(i(x)) = x$ (because the element t is invertible in $\kappa(i(x))$, and hence also for any point in the preimage). Hence, by Nakayama applied to $\mathcal{O}_S[t]/(f)$, conclude that the map $\mathcal{O}_S[t]/(f) \to \mathcal{O}_{\overline{X}}$ is surjective around $i(x)$. Since g a primitive element at x, there is a factorisation $(\mathcal{O}_S[t]/(f))_{f'} \to \mathcal{O}_{X,x}$. This map is étale since both the source and target are so. Hence, the composite $\operatorname{Spec}(\mathcal{O}_{X,x}) \to \operatorname{Spec}((\mathcal{O}_S[t]/(f))_{f'}) \subset \operatorname{Spec}(\mathcal{O}_S[t]/(f(t))$ is an immersion at $x \in X$ that is also étale. Show that this forces the map to be an open immersion.

8. Let $f : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local homomorphism of noetherian local rings. Assume that f is essentially finitely presented and étale. Then show

   (a) A is regular if and only if B is so.
   (b) A is Cohen-Macaulay if and only if B is so.
   (c) A is Gorenstein if and only if B is so.
   (d) A is an integral domain if B is so; the converse fails (example?).

   Hint: use the characterisation of étale ring maps as flat and unramified ones, and homological criteria.

9. Let S be a scheme, and $S_0 \subset S$ be a closed subscheme defined by a quasi-coherent ideal of square zero. The goal of this exercise is to show that the functor $X \mapsto X_0 := X \times_S S_0$ induces an equivalence between categories of étale schemes over S and S_0.

   (a) Show using the infinitesimal lifting criterion that if X and Y are S-schemes with X étale, then $\operatorname{Hom}_S(Y, X) \simeq \operatorname{Hom}_{S_0}(Y_0, X_0)$. Note that Y is not assumed to be affine.
   (b) Show that if X_0 is a standard étale S_0-scheme (i.e., one of the form $\operatorname{Spec}((\mathcal{O}_{S_0}[t]/(f))_{f'})$ with f monic), then X_0 comes from an étale S-scheme X. Moreover, show that X is determined uniquely up to unique isomorphism once X_0 is specified.

(c) Using the uniqueness in the previous step, show by glueing that all étale S_0-schemes arise from those over S by base change.

10. The goal of this exercise is to introduce some ideas of deformation theory, putting the preceding exercise in some perspective. Let $S_0 \subset S$ be a closed immersion of affine schemes defined a quasi-coherent ideal $\mathcal{I}$ of square zero on a scheme S. Let $f_0 : X_0 \to S_0$ be a smooth finitely presented morphism. A *deformation* of f_0 to S is defined to be a pair (f, i) where $f : X \to S$ is a flat morphism, and $i : X_0 \to X \times_S S_0$ is an S_0-isomorphism; we sometimes abusively refer to f as the deformation. Let $\mathrm{Def}(f_0, S)$ denote the category of all deformations of f_0 to S, and let $\pi_0(\mathrm{Def}(f_0, S))$ denote the set of isomorphism classes.

    (a) Show that if f_0 arises via base change from a flat morphism $f : X \to S$, then f has to be smooth.

    (b) Show that $\mathrm{Def}(f_0, S)$ is a groupoid, i.e., any morphism in $\mathrm{Def}(f_0, S)$ is an automorphism.

    (c) If f_0 admits a deformation $f : X \to S$, then show that the kernel of $\mathrm{Aut}_S(X) \to \mathrm{Aut}_{S_0}(X_0)$ is identified naturally with $\mathrm{Hom}_{X_0}(\Omega^1_{X_0/S_0}, f_0^*\mathcal{I})$. Note that this is simply the automorphism group in the category $\mathrm{Def}(f_0, S)$ at the point defined by f, and turns out to be independent of f.

    (d) If f_0 is affine, then show that $\mathrm{Def}(f_0, S)$ is non-empty and connected, i.e., there exists a deformation $f : X \to S$, and any two deformations are isomorphic.

    Hint: To show non-emptiness, embed in affine space, and lift a set of well-chosen equations. To show uniqueness (up to non-unique isomorphism), use the infinitesimal lifting property to lift the identity map to an isomorphism between any two deformations.

    (e) Show that $\pi_0(\mathrm{Def}(f_0, S))$ is a torsor for $\mathrm{Ext}^1_{X_0}(\Omega^1_{X_0/S_0}, f_0^*\mathcal{I})$, i.e., the latter group acts simply transitively on the former set, provided this set is non-empty.

    Hint: Given two deformations f_1 and f_2, pick an affine chart for X_0, write down isomorphisms between f_1 and f_2 over those charts using the previous claim, and see what you get.

    (f) (*) Give an example to show that $\mathrm{Def}(f_0, S)$ could be empty.

    (g) Show that there exists a canonical element $\mathrm{ob} \in \mathrm{Ext}^2_{X_0}(\Omega^1_{X_0/S_0}, f_0^*\mathcal{I})$ which is zero if and only if $\mathrm{Def}(f_0, S)$ is non-empty.

    Hint: Pick an affine chart for X_0, deformations of pieces of the chart to S, and isomorphisms over the overlaps. What is the obstruction to glueing?

    (h) Observe what happens when f_0 is étale.

11. The goal of this exercise is to introduce some ideas that become useful when thinking about the cotangent complex. Let $A \to B$ be a ring homomorphism. Let $\mathcal{C}$ be the category of A-algebras equipped with a map to B. For a B-module M, let $B \oplus M$ denote the square-zero extension of B by M, i.e., we define multiplication by

$$(b_1, m_1) \cdot (b_2, m_2) = (b_1 b_2, b_1 m_2 + b_2 m_1).$$

    (a) Show that the association $M \mapsto B \oplus M$ defines a functor $\mathcal{F} : \mathrm{Mod}(B) \to \mathcal{C}$. Observe that the two maps $0 \to M$ and $M \to 0$ define maps $B \simeq \mathcal{F}(0) \to \mathcal{F}(M)$ and $\mathcal{F}(M) \to \mathcal{F}(0) \simeq B$ for any B-module M.

    (b) Show that for any $R \in \mathcal{C}$, there is a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(R, \mathcal{F}(M)) \simeq \mathrm{Der}_A(R, M)$. Conclude that if $A \to B$ satisfies the infinitesimal lifting criterion for smoothness, then $\Omega^1_{B/A}$ is a projective B-module. Note the absence of finiteness conditions.

(c) (*) Observe that $\mathcal{F}(M)$ is an abelian group object of $\mathcal{C}$ (i.e., that $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{F}(M))$ is naturally valued in abelian groups) thanks to the formula above in terms of derivations. Show that every abelian group object in $\mathcal{C}$ is (uniquely, up to unique isomorphism) of this form, i.e., that $\mathcal{F}$ factors as an equivalence $\mathrm{Mod}(B) \simeq \mathrm{Ab}(\mathcal{C})$ followed by the forgetful functor $\mathrm{Ab}(\mathcal{C}) \to \mathcal{C}$.

(d) Reinterpret the preceding results to show that the forgetful functor $\mathrm{Ab}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint "abelianisation" functor $(-)^{\mathrm{Ab}} : \mathcal{C} \to \mathrm{Ab}(\mathcal{C})$ defined by $R \mapsto R^{\mathrm{Ab}} := \mathcal{F}(\Omega^1_{R/A} \otimes_R B)$. Note that you could also have predicted the existence of $(-)^{\mathrm{Ab}}$, and hence that of theory of Kahler differentials, by simply observing that $\mathcal{F}$ preserves limits, and applying the adjoint functor theorem.