Problem set 1

Review

Recall that for a variety X over a finite field $k = \mathbf{F}_q$, the zeta function is defined as

$$Z(X,t) := \exp(\sum_{r=1}^{\infty} \#X(\mathbf{F}_{q^r}) \cdot \frac{t^r}{r}) \in \mathbf{Q}[\![t]\!]$$

This function satisfies

$$t \cdot \frac{d}{dt} \log(Z(X,t)) = \sum_{r=1}^{\infty} \#X(\mathbf{F}_{q^r}) \cdot t^r.$$

Two useful identities in the game are the following:

- If $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$, and $a_r = \sum_{i=1}^{r} \alpha_i^r$, then

$$\sum_{i=1}^{n} \frac{\alpha_i \cdot t}{1 - \alpha_i \cdot t} = \sum_{r=1}^{\infty} a_r \cdot t^r.$$

- The power series expansion for log:

$$\log(1-x) = -1 \cdot \sum_{r=1}^{\infty} \frac{x^r}{r}.$$

Problems

The hints (in gray) are provided for your benefit, but please try doing the problem yourself first!

1. Do Exercise 5.5 in Appendix C of Hartshorne.
2. Let k be a finite field of characteristic different from 2. Let $X = V(x^2 + y^2 + z^2) \subset \mathbf{P}_k^2$ be the smooth conic. Compute the zeta function of X.

   Hint: Show first that X is a smooth genus 0 curve. Next, show that once X has a k-rational point $x \in X(k)$, then projection from x defines an isomorphism $X \simeq \mathbf{P}^1$ (how many points can a line and a conic meet at?). Finally, show that X always has a k-rational point by contemplating whether or not -1 is a square. In the case -1 is not a square, what does the lack of solutions to $x^2 + y^2 = -z^2$ mean for the set of squares in k^*?.

3. Let k be a finite field. Compute the zeta function of the variety Flag_n parametrising complete flags in an n-dimensional vector space.

   Hint: Let $G = \mathrm{GL}_n$ be the general linear group over k, and let $B \subset G$ be the algebraic subgroup of upper triangular matrices. Compute $\#G(\mathbf{F}_q)$ and $\#B(\mathbf{F}_q)$ by hand. Then show that $\mathrm{Flag}_n(\mathbf{F}_q) \simeq G(\mathbf{F}_q)/B(\mathbf{F}_q)$, and use the preceding computations.

4. Let $k = \mathbf{F}_q$ be a finite field. Let X/k be a variety, and let X^0 be the set of closed points of the underlying scheme. Then show that

$$Z(X,t) = \prod_{x \in X^0} \frac{1}{1 - t^{\deg(\kappa(x))/k}}.$$

   Hint: First show that $\#X(\mathbf{F}_{q^r}) = \sum_{e|r} e \cdot \#\{x \in X^0 \mid \deg(\kappa(x)/\mathbf{F}_q) = e\}$, and then use that.

5. Let $k = \mathbf{F}_q$ be a finite field. Let X/k be a variety. Then show that

$$Z(X \times_{\mathbf{F}_q} \mathbf{F}_{q^e}, t^e) = \prod_{i=0}^{e-1} Z(X, \zeta^i \cdot t)$$

   where ζ is a primitive e-th root of 1.

6. Let $k = \mathbf{F}_q$ be a finite field, and let X be a smooth, projective, geometrically connected curve of genus g over k. The goal of the following series of exercises is to lead you through a proof of all the Weil conjectures for X.

   (a) Choose $e \in \mathbf{N}$ be such that $\deg : \mathrm{Pic}(X) \to \mathbf{Z}$ maps onto $e\mathbf{Z}$. Show $e|(2g-2)$, and that for any $n \in e\mathbf{Z}$, $\#\mathrm{Pic}^n(X) = \#\mathrm{Pic}^0(X)$.

   (b) (*) Show directly $Z(X,t)$ is rational of the form $\frac{b(t^e)}{(1-t^e)(1-qt^e)}$ with $b(t)$ a polynomial of degree $\leq 2g$.

   Hint: First show using one of the previous exercises that

$$Z(X,t) = \prod_{x \in X^0} \sum_{j=0}^{\infty} t^{j \cdot \deg(\kappa(x)/k)}.$$

   Next, using the correspondence between $\mathbf{N}$-linear combinations of points, effective divisors, and line bundles equipped with a section (up to scaling), reinterpret the above sum to be

$$Z(X,t) = \sum_{D, D \geq 0} t^{\deg(D)} = \sum_{\mathcal{L} \in \mathrm{Pic}(X)} \#\mathbf{P}(H^0(X,\mathcal{L})) \cdot t^{\deg(\mathcal{L})} = \sum_{\mathcal{L} \in \mathrm{Pic}(X)} \frac{q^{h^0(\mathcal{L})} - 1}{q - 1} \cdot t^{\deg(\mathcal{L})}.$$

   Note that there is no contribution from line bundles without a section, and hence none from $\mathcal{L}$ with $\deg(\mathcal{L}) < 0$. Now break up the sum on the right as $g_1 + g_2$, with g_1 recording contributions of terms corresponding to line bundles $\mathcal{L}$ with $\deg(\mathcal{L}) \leq 2g - 2$, and g_2 the rest. Using that $\mathrm{Pic}^0(X)$ is finite, show that g_1 is a polynomial. Using Riemann-Roch, compute g_2, and make conclusions.

   (c) Show that $Z(X,t)$ has a pole at $t = 1$.

   (d) Compare the poles at $t = 1$ of $Z(X,t)$ with $Z(X \times_{\mathbf{F}_q} \mathbf{F}_{q^e})$ to show that $e = 1$. Conclude that $Z(X,t) = \frac{b(t)}{(1-t)(1-qt)}$ with $\deg(b(t)) = 2g$.

   Note that this already shows that any genus 0 curve over k has to be isomorphic to $\mathbf{P}^1$ (since it has a degree 1 line bundle), and any genus 1 curve over k has to have a k-rational point (since $C \simeq \mathrm{Pic}^1(C)$ for such curves by the obvious map). Both these consequences generalise, as we will see later.

(e) Prove the functional equation

$$Z(X, \frac{1}{qt}) = q^{1-g} \cdot t^{2-2g} \cdot Z(X, t).$$

Hint: follow the method used to prove rationality above, and use Serre duality.

(f) Conclude using the functional equation that the set $\{\alpha_1, \ldots, \alpha_{2g}\}$ of roots of $b(t)$ (with multiplicity) is invariant under the operation $\alpha \mapsto \frac{q}{\alpha}$.

(g) Show that RH for X, i.e., the assertion that $|\alpha_i| = q^{\frac{1}{2}}$, is equivalent to the assertion that $a_r \leq 2g \cdot q^{\frac{r}{2}}$ where $a_r = \sum_{i=1}^{2g} \alpha_i^r$.

Hint: follow the proof shown in class for elliptic curves. Specifically, use the functional equation to show that it suffices to show $|\alpha_i| \leq q^{\frac{1}{2}}$, and then study the formula

$$\sum_{i=1}^{2g} \frac{\alpha_i \cdot t}{1 - \alpha_i \cdot t} = \sum_{r=1}^{\infty} a_r \cdot t^r,$$

especially its poles in the region $|t| < q^{\frac{1}{2}}$.

(h) Using the fixed point formula (which we know now for X), show that RH for X is equivalent to showing that $\#X(\mathbf{F}_{q^r}) = q^r + O(q^{\frac{r}{2}})$. Now do exercise 1.10 in chapter V of Hartshorne.

7. (This exercise requires some familiarity with abelian varieties). Let k be an arbitrary field. Let (A, e) be an abelian variety over k, and let X be a torsor for A, i.e., X is a proper smooth k-variety, and there is an A-action $\mathrm{act} : A \times X \to X$ such that for any k-scheme L and a point $x \in X(L)$, the induced "orbit" map $A_L \to X_L$ given by $a \mapsto a + x$ is an isomorphism. The goal of this exercise is to show that when $k = \mathbf{F}_q$ is a finite field, X always has a k-rational point, and thus $A \simeq X$.

(a) Show that the assertion is true for $X = \mathrm{Pic}^n(C)$ and $A = \mathrm{Pic}^0(C)$ for some smooth, projective and geometrically connected curve C.

Hint: use the previous exercise.

(b) Show that X is projective.

Hint: Use the finite surjective map $X_L \to X$ for a suitable field L and use that abelian varieties are projective.

(c) (*) Show that there is a natural map $\mathrm{sub} : X \times X \to A$ given by at the level points by $(x, y) \mapsto x - y$. Show also that for each integer $d \geq 0$, there is a natural map $\mathrm{Sym}^{d+1}(X) \times \mathrm{Sym}^d(X) \to X$ which is given, at the level points, by $(x_1, \ldots, x_{d+1}), (y_1, \ldots, y_d) \mapsto (\sum_{i=1}^d x_i - y_i) + x_{d+1}$.

Hint: Consider the "universal point" $L = X \xrightarrow{\mathrm{id}} X$ to get sub. Get the map $X^{d+1} \times X^d \to X$ using sub. Now show it factors through appropriate quotients.

(d) (*) Show that there exists a smooth, projective, geometrically connected curve C/k and a map $\mathrm{Pic}^n(C) \to X$ for some n. Conclude that X must have a k-rational point.

Hint: Choose a general smooth curve $C \subset X$ by intersecting general hyperplane sections for a sufficiently big projective embedding (one can always do this thanks to theorems of Poonen and Gabber). Consider the composite map $\mathrm{Sym}^{d+1}(C) \times \mathrm{Sym}^d(C) \to \mathrm{Sym}^{d+1}(X) \times \mathrm{Sym}^d(X) \to X$. Show that X admits no maps from $\mathbf{P}^1$. Conclude that for d sufficiently large, this map factors through a map $\mathrm{Pic}^{d+1}(C) \times \mathrm{Pic}^d(C) \to X$. Now use that $\mathrm{Pic}^n(C)$ always has a rational point for every n sufficiently large.