On the derived category of perverse sheaves

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Let $D = D^b(X, \mathbb{Q}_\ell)$ be the usual derived category of $\mathbb{Q}_\ell$-sheaves on a certain scheme X, and $M = M(X) \subset D$ be the category of perverse sheaves for middle perversity. Now consider the derived category $D^b(M)$ of an abelian category M; we have the natural exact functor $D^b(M) \to D$. The aim of this note is to show that this functor is an equivalence of categories. The same result result holds for M = the category of algebraic holonomic $\mathcal{D}$-modules and D = the derived category of complexes of $\mathcal{D}$-modules with holonomic cohomology.

One may look at this from two complementary points of view. First we see that Yoneda-type Ext's in M are computable by easy topological means (since they coincide with Ext's in D). Secondly, the niche D where M dwells, may be recovered from M (note, that a priori D is quite transcendental with respect to M); this may be of use in a future motivic sheaf theory.

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§1. Notation and statement of the main theorem

1.1. Fix a base field k; in what follows the base schemes will be separated of finite type over k. For a scheme X denote $D(X)^{(i)}$, $i = 1,\dots,5$, the following triangulated categories:

(i) $D(X)^{(1)} := D^b_c(X,R)$ = the derived category of complexes of étale R-sheaves having bounded constructible cohomology; here R is a finite ring of characteristic prime to char k.

(ii) $D(X)^{(2)} := D^b_c(X, \bar{\mathbb{Q}}_\ell)$. Here $\ell \neq$ char k and k is assumed to be algebraically closed; see e.g. [1] (2.2.18).

(iii) $D(X)^{(3)} := D^b_c(X(\mathbb{C}), R)$ = the derived category of complexes of R-sheaves on the classical topology of X, having algebraically constructible cohomology; here $k = \mathbb{C}$ and R is any field, see e.g. [1] (2.2.1.).

(iv) $D(X)^{(4)} = D^b_m(X, \bar{\mathbb{Q}}_\ell)$ = the derived category of mixed sheaves, see e.g. [1] (5.1.5.).

(v) $D(X)^{(5)} = D_{Hol}(X)$ = the derived category of complexes of $\mathcal{D}$-modules having bounded holonomic cohomology, see e.g. [3] §4 (here char k= 0).

Each of these triangulated categories $D(X)^{(i)}$ has a canonical filtered counterpart - the f-category $DF(X)^{(i)}$ over $D(X)^{(i)}$ (for f-categories see the appendix). In the cases i = 1,3 this is the derived category of complexes of sheaves with finite decreasing filtration such that each graded quotient belongs to $D(X)^{(i)}$; in the other cases $D(X)^{(i)}$ is the corresponding $\mathbb{Q}_\ell$ - or $\mathcal{D}$-module analogue.

There are various standard functors between $D(X)^{(i)}$ such as $\otimes$, Hom, the direct and inverse image functors $f_!$, f_*, $f^!$, f^* (more precisely, in case i = 1 $\otimes$ and Hom may take values in unbounded complexes); all these functors have a canonical f-lifting to $DF(X)^{(i)}$. We will consider $D(X)^{(i)}$ as t-categories with a t-structure defined by the middle perversity for i = 1,...,4 and with the obvious t-structure for i = 5. The hearts $M(X)^{(i)}$ of this t-structures are categories of constructible perverse sheaves in the cases i = 1,...,4 and the category of holonomic modules $M_{Hol}(X)$ in the case i = 5.

1.2. Assume that for any scheme X over k we are given a strictly full t-subcategory D(X) in $D(X)^{(i)}$ above (i = 1,...,5 is fixed) closed under $\otimes$, Hom and f_*, $f_!$, f^*, $f_!$ (i.e. for any morphism $f : X \to Y$ of schemes one should have $f_*, f_! : D(X) \to D(Y)$; $f^*, f_! : D(Y) \to D(X)$).

Examples. Clearly, we may take $D(X) \equiv D(X)^{(i)}$ or $D(X) \equiv 0$. In the case i = 1 this are the only possibilities. In the cases i = 2,3 we may take for D(X) the subcategories generated by quasiunipotent local systems (according to Kashiwara and Gabber, see [5]), or by local systems having geometric origin ([1] (6.2.4.)). In the case i = 5 we may take $D(X) = D_{RS}(X)$ = the subcategory generated by lisse holonomic modules having regular singularities at ∞ (see [3] §4), or, more generally, $D(X) = D_{RS\Lambda}(X)$ ([3] (4.8)). In what follows assume that D(X) is not identically zero.

Remark 1.2.1. a). In case $k = \mathbb{C}$ we have the canonical t-exact functor $DR : D(X)^{(5)} = D_{Hol}(X) \longrightarrow D(X)^{(3)} = D^b_c(X(\mathbb{C}), \mathbb{C})$, whose restriction on $D_{RS}(X) \subset D_{Hol}(X)$ is an equivalence of categories commuting with any standard functor (see [3] §5).

b) clearly D (spec k) contains all the Tate modules R(j) (cases i = 1,3,5), or $\bar{\mathbb{Q}}_\ell(j)$ (cases i = 2,4); hence the functors Φ_f, Ψ_f,

Ξ_f (see [2]) preserve $D(X)$.

1.3. Let $M(X) \subset D(X)$ be the heart of $D(X)$. Clearly, $D(X)$ coincides with the full subcategory in $D(X)^{(i)}$ of complexes having all t-cohomology in $M(X)$. Let $DF(X) \subset DF(X)^{(i)}$ be the full subcategory of objects having each a graded quotient in $D(X)$. Clearly, $DF(X)$ is an f-category over $D(X)$. It defines a canonical t-exact functor $real_X : D^b(M(X)) \to D(X)$ that induces the identity functor between hearts $M(X)$ (see appendix; in holonomic case real is obvious functor). Now we may formulate

Main theorem 1.3. This functor is an equivalence of categories.

Remarks. a) The corresponding statement for the category of sheaves lisse along a fixed stratification is usually false.

b) I don't know whether the analogous fact remains true for perverse sheaves of other perversities different from the middle one, say, for ordinary constructible $\mathbb{Q}_\ell$-sheaves. Also I am ignorant of the analytic cases, both constructible and holonomic.

1.4. Note that the main theorem just claims that Yoneda-type Ext's between the objects of $M(X)$ (i.e. Ext's computed in $D^b(M(X))$ coincide with usual Ext's computed in $D(X)$. Namely, the following simple general lemma holds (proof is similar to [1] (3.1.16)).

Lemma 1.4. Let $F : D_1 \to D_2$ be a t-exact functor between t-categories. D_i with hearts $C_i \subset D_i$. Assume that $F|_{C_1} : C_1 \to C_2$ is an equivalence of categories, and $D_2 = D_2{}^b$. Then the following statements are equivalent:

(i) F is an equivalence of categories;

(ii) For any $M,N \in \mathrm{Ob}\, C_1$ and $i > 0$ the map $\mathrm{Hom}^i_{D_1}(M,N) \longrightarrow \mathrm{Hom}^i_{D_2}(F(M), F(N))$ is an isomorphism;

(iii) Assume that $D_1 = D^b(C_1)$. For any $M,N \in \mathrm{Ob}\, C_1$, $i > 0$ and $x \in \mathrm{Hom}^i_{D_2}(F(M), F(N))$ there exists an injection $N \hookrightarrow N'$ in C_1 such that the image of x in $\mathrm{Hom}^i_{D_2}(F(M), F(N'))$ is zero. $\square$

Clearly 1.3. falls into this situation, so it suffices to prove for $F = real_X$ either 1.4. (ii) or 1.4 (iii). For $M,N \in \mathcal{M}(X)$ put $\mathrm{Ext}^i_{\mathcal{M}(X)}(M, N) := \mathrm{Hom}^i_{D^b(\mathcal{M}(X))}(M, N)$, $\mathrm{Ext}^i_{D(X)}(M,N) := \mathrm{Hom}^i_{D(X)}(M,N)$. So to prove 1.3. we have to show that these Ext's coincide.

§2. Proofs

The proof of the theorem 1.3 is divided into two steps: first we show that it is valid at the generic point of X (lemma 2.1.1.), and then by means of glueing (see [2]) we use this to reduce the problem to lower dimensions.

2.1. Let $\eta \in X$ be a generic point; $D(\eta) = 2\text{-}\varinjlim_{\eta\in U} D(U)$ the 2-limit of t-categories $D(U)$, U runs the Zariski open sets containing η. Clearly $D(\eta)$ is t-category with the heart $\mathcal{M}(\eta) = 2\text{-}\lim \mathcal{M}(U)$; we also have our t-exact functor $\mathrm{real}_\eta : D^b(\mathcal{M}(\eta)) = 2\text{-}\lim D^b(\mathcal{M}(U)) \to D(\eta)$.

Lemma 2.1.1. $\mathrm{real}_\eta : D^b(\mathcal{M}(\eta)) \to D(\eta)$ is an equivalence of categories.

Proof. First notice that in case $D(X) = D(X)^{(1)}$ (see 1.1.) the lemma is trivial since here $\mathcal{M}(\eta)$ = finite Galois R-modules, $D(\eta)$ = derived category of complexes of arbitrary Galois R-modules with finite bounded cohomology groups. Therefore we assume that we are in one of the situations (ii)-(v) of 1.1; in particular the coefficient ring is a field. Assume also that X is reduced; this changes nothing.

According to 1.4. (iii) it suffices to show the following. Let $U \subset X$ be a Zariski open set, $\eta \in U$, and M_U, N_U are in $\mathcal{M}(U)$. Then for some open set $V \subset U$, $\eta \in V$, there exists O_V in $\mathcal{M}(V)$ and an injection $N_V = N_U|_V \hookrightarrow O_V$ such that for any $i > 0$ the induced arrow $\mathrm{Ext}^i_{D(U)}(M_U, N_U) \to \mathrm{Ext}^i_{D(V)}(M_V, O_V)$ is zero.

The proof will be carried by induction in $\dim X$; clearly 2.1.1. holds for X of dimension zero, so assume that we have 2.1.1. for any Y of dimension less then X.

Shrinking U if necessary we may assume that M_U, N_U are lisse, that U is irreducible and there exists a smooth affine $\P : U \to Z$ with 1-dimensional fibers such that Z is regular and $L^q = R^q\P_*\underline{\mathrm{Hom}}(M_U, N_U) = R^q\P_*(M_U^* \otimes N_U)$ are lisse sheaves on Z. Clearly $L^q = 0$ unless $q = 0, 1$, so the Leray spectral sequence $E_2^{pq} = H^p(Z, L^q) \Rightarrow \mathrm{Ext}^{p+q}_{D(U)}(M_U, N_U)$ becomes degenerate at E_3.

Remark. Certainly, L^q are usual lisse constructible sheaves in constructible situation; in $\mathcal{D}$-module situation they are lisse holonomic modules placed in degree $\dim Z$ in $D(Z)$.

We will need the following lemma (here for an open $Y \subset Z$ we put $\P_Y : U_Y := \P^{-1}(Y) \to Y$):

Lemma 2.1.2. a) There exists an open $Y \subset Z$, a lisse P_{U_Y} in $\mathcal{M}(U_Y)$ and an injection $N_{U_Y} \hookrightarrow P_{U_Y}$ such that the corresponding arrow $R^1\P_{Y*}\,\mathrm{Hom}(M_{U_Y}, N_{U_Y}) \to R^1\P_{Y*}\,\mathrm{Hom}(M_{U_Y}, P_{U_Y})$ is zero.

b) There exists an open $Y' \subset Z$, a lisse $Q_{U_{Y'}}$ in $\mathcal{M}(U_{Y'})$ and an injection $N_{U_{Y'}} \hookrightarrow Q_{U_{Y'}}$ such that the corresponding arrows $H^p(Z, R^0\P_* Hom(M_U, N_U)) \longrightarrow H^p(Y', R^0\P_{Y'*}(Hom(M_{U_{Y'}}, Q_{U_{Y'}}))$ are zero for $p > 0$.

Lemma 2.1.2. $\Longrightarrow$ Lemma 2.1.1: first choose $N_{U_Y} \hookrightarrow P_{U_Y}$ as in 2.1.2. a). Then the Leray spectral sequence shows that the image of $Ext^i_{D(U)}(M_U, N_U)$ in $Ext^i_{D(U_Y)}(M_{U_Y}, P_{U_Y})$ is contained in the image of $H^i(Y, R^0\P_* Hom(M_{U_Y}, P_{U_Y}))$. Now apply 2.1.2. b) to Z replaced by Y, and sheaves M_{U_Y} and P_{U_Y}. We get $Y' \subset Y$ and $P_{U_{Y'}} \hookrightarrow Q_{U_{Y'}}$ such that $H^i(Y, R^0\P_* Hom(M_{U_Y}, P_{U_Y})) \longrightarrow H^i(Y', R^0\P_* Hom(M_{U_{Y'}}, Q_{U_{Y'}}))$ is zero for $i > 0$. This shows that for the composite map $N_{U_{Y'}} \hookrightarrow Q_{U_{Y'}}$ on $V = U_{Y'}$, all the arrows $Ext^i_{D(U)}(M_U, N_U) \to Ext^i_{D(V)}(M_V, O_V)$ are zero for $i > 0$, Q.E.D. $\square$

Proof of 2.1.2. a) First notice that we may easily construct such P along each closed fiber. Namely, consider the canonical element $\alpha \in H^0(Z, L^{1*} \otimes L^1) = H^0(Z, R^1\P_* Hom(\P^* L^1 \otimes M, N))$. If this α came from the global extension $\tilde{\alpha} \in Ext^1(\P^* L^1 \otimes M, N)$ we are done: just take $Y = Z$ and define P_{U_Y} from the extension $0 \to N \to P_{U_Y} \to \P^* L^1 \otimes M \to 0$ of class $\tilde{\alpha}$. If not, consider the obstruction to existense of $\tilde{\alpha}$: the Leray spectral sequence defines the exact sequence

$$\begin{array}{ccccc} Ext^1(\P^* L^1 \otimes M, N) & \to & H^0(Z, R^1\P_* Hom(\P^* L^1 \otimes M, N)) & \xrightarrow{\partial} & H^2(Z, R^0\P_* Hom(\P^* L^1 \otimes M, N)) \\ & & \| & & \| \\ & & H^0(Z, L^{1*} \otimes L^1) & & H^2(Z, L^{1*} \otimes L^0) \end{array}$$

so the obstruction is $\partial(\alpha)$. To kill this obstruction we replace L^1 by a certain extension. To construct this extension we will use the inductive hypothesis applied to L^{0*} and L^{1*}. They say that there exist an open set $Y \subset Z$, a lisse sheaf K_Y on Y and an injective arrow $\varphi : L^{1*}{}_Y \hookrightarrow K_Y$ such that the induced arrow $H^2(Z, L^{1*} \otimes L^0) \xrightarrow{\varphi} H^2(Y, K_Y \otimes L^0_Y)$ is zero. In particular $\varphi(\partial\alpha)$ is zero. Now consider the element $\varphi(\alpha) \in H^0(Y, K_Y \otimes L^1_Y) = H^0(Y, R^1\P_{Y*} Hom(\P^* K_Y \otimes M_Y, N_Y))$. This element comes from certain global $\widetilde{\varphi(\alpha)} \in Ext^1(\P^* K_Y \otimes M_{U_Y}, N_{U_Y})$, since the corresponding obstruction is $\partial\varphi(\alpha) = \varphi(\partial\alpha) = 0$. Now define P_{U_Y} from the extension $0 \to N_{U_Y} \to P_{U_Y} \to \P_* K_Y \otimes M_{U_Y} \to 0$ of class $\widetilde{\varphi(\alpha)}$. It satisfies all the needed properties, since fiberwise it came from the class of α . □

Proof of 2.1.2. b. Apply the inductive hypothesis to Z, a constant sheaf and L^o. We get $Y' \subset Z$, a lisse $Q_{Y'}$ on Y' and an injection $L^o_{Y'} \hookrightarrow Q_{Y'}$ such that the corresponding arrow $H^i(Z, L^o) \longrightarrow H^i(Y', Q_{Y'})$ is zero for $i > 0$. Define $O_{Y'}$ by means of the cocartesian square

$$\begin{tikzcd}
\P^* Q_{Y'} \otimes M_{U_{Y'}} \arrow[r] & O_{U_{Y'}} \\
\P^* L^o_{Y'} \otimes M_{U_{Y'}} \arrow[r] \arrow[u, hook] & N_{U_{Y'}} \arrow[u, hook]
\end{tikzcd}$$

where $\P^* L^o_{Y'} \otimes M_{U_{Y'}} \longrightarrow N_{U_{Y'}}$ is the canonical arrow. The obvious commutative diagram

$$\begin{tikzcd}
R^o\P_*Hom(M_{U_{Y'}}, N_{U_{Y'}}) \arrow[r] & R^o\P_*Hom(M_{U_{Y'}}, O_{U_{Y'}}) \\
L^o_{Y'} \arrow[u, equal] \arrow[r] & Q_{Y'} \arrow[u]
\end{tikzcd}$$

shows that our $N_{U_{Y'}} \hookrightarrow O_{U_{Y'}}$ is what we need. □

So 2.1.1. is proven, and we may pass to the

2.2. Proof of Theorem 1.3. We will also use the induction in dim X; so assume that we have 1.3. for any variety of dimension less then dim X.

First note that the statement of 1.3. is Zaris ki local: let us consider 1.3. in the form 1.4. (iii); assume that M,N are in $\mathcal{M}(X)$ and we have found an affine Zariski covering $\{U_i\}$ of X together with injections $N_{U_i} \hookrightarrow N'_{U_i}$ such that all the maps $\mathrm{Ext}^j_{D(U_i)}(M_{U_i}, N_{U_i}) \longrightarrow \mathrm{Ext}^j_{D(U_i)}(M_{U_i}, N'_{U_i})$ are zero for any $j > 0$; then the injection $N \hookrightarrow N' := \bigoplus_i j_{i*}N'_{U_i}$ kills all $\mathrm{Ext}^j_{D(X)}(M, N)$ (note that $j_{i*}N'_{U_i} \in \mathrm{Ob}\,\mathcal{M}(X)$ since $j_i : U_i \hookrightarrow X$ is affine).

So we may assume X to be affine. Let us prove 1.3. in the form 1.4. (ii): we have to show that for any $M,N \in \mathcal{M}(X)$ one has $\mathrm{Ext}^i_{\mathcal{M}(X)}(M,N) = \mathrm{Ext}^i_{D(X)}(M,N)$.

2.2.1. First assume that the dimensions of supports of M,N are less than dim X. Then for certain $f \in \mathcal{O}(X)$ this supports lie in $Y = f^{-1}(0)$, $\dim Y < \dim X$. We have $\mathrm{Ext}^i_{D(X)}(M,N) = \mathrm{Ext}^i_{D(Y)}(M,N) =$

$= \mathrm{Ext}^i_{M(Y)}(M,N)$ (the first equality follows from adjunction of i_* and i^*, and the second one is the inductive hypothesis). So it remains to prove that the embedding $M(Y) \hookrightarrow M(X)$ induces the isomorphism $I : \mathrm{Ext}^i_{M(Y)}(M,N) \xrightarrow{\sim} \mathrm{Ext}^i_{M(X)}(M,N)$. Let us construct the inverse map to I. Consider the vanishing cycles functor $\Phi_f : M(X) \longrightarrow M(Y)$ (see 1.2.1. b). Since Φ_f is exact, it defines the map $\Phi_{f_*} : \mathrm{Ext}^i_{M(X)}(P, Q) \longrightarrow \mathrm{Ext}^i_{M(Y)}(\Phi_f(P), \Phi_f(Q))$. But $\Phi_f |_{M(Y)}$ is identity functor, so for $M,N \in M(Y)$ we get the arrow $\Phi_{f_*} : \mathrm{Ext}^i_{M(X)}(M,N) \longrightarrow \mathrm{Ext}^i_{M(Y)}(M,N)$ left-inverse to I. It remains to show that it is also right-inverse, i.e. that $I \Phi_{f_*} = \mathrm{id}_{\mathrm{Ext}^i_{M(X)}}$.

At this point it is convenient to use Yoneda's construction of $\mathrm{Ext}^i_{M(X)}$. Let me recall it briefly. Namely, let $E^i_{M(X)}(M,N)$ be the category of acyclic complexes in M of type $0 \longrightarrow N \longrightarrow C^1 \longrightarrow \ldots \longrightarrow C^i \longrightarrow M \longrightarrow 0$, the morphisms in $E^i_{M(X)}(M,N)$ being morphisms of complexes that induce identity maps on the ends M,N. Then $\mathrm{Ext}^i_{M(X)}(M,N)$ is just the set of connected components of $E^i_{M(X)}(M,N)$, i.e. $\mathrm{Ext}^i_{M(X)}$ is the set of equivalence classes of objects of E^i, where two objects are equivalent if you may connect them by a sequence of morphisms.

So for $M,N \in M(Y)$ let $C^\cdot = (N \longrightarrow C^1 \longrightarrow \ldots \longrightarrow C^i \longrightarrow M)$ be an object in $E^i_{M(X)}(M,N)$. We have to show that $C^\cdot$ is equivalent to $\Phi_f(C^\cdot) = (N \longrightarrow \Phi_f(C^1) \longrightarrow \ldots \longrightarrow \Phi_f(C^i) \longrightarrow M)$. Here is the sequence of morphisms in $E^i_{M(X)}(M,N)$ that connects them (it comes from [2]n°3(*)):

$$C^\cdot \longrightarrow C^\cdot \oplus \Xi_f(C^\cdot_U) \longrightarrow C^\cdot \oplus \Xi_f(C^\cdot_U)/j_!(C^\cdot_U) \longleftarrow \Phi_f(C^\cdot).$$

2.2.2. We will use the following remark. Let $j : U \hookrightarrow X$ be an affine embedding. Then the functors j^*, $j_!$ and j_* between $M(U)$ and $M(X)$ are exact and pairs $(j_!, j^*)$, (j^*, j_*) are adjoint. This implies that for any $A \in M(U)$, $B \in M(X)$ one has

$$\mathrm{Ext}^\cdot_{M(X)}(j_!A,B) = \mathrm{Ext}^\cdot_{M(U)}(A,B_U), \quad \mathrm{Ext}^\cdot_{M(X)}(B,j_*A) = \mathrm{Ext}^\cdot_{M(U)}(B_U, A).$$

2.2.3. Consider now the case when $\dim \operatorname{supp} N < \dim X$. Choose an open affine $j : U \hookrightarrow X$ such that $\dim(X-U) < \dim X$ and $X-U \supset \operatorname{supp} N$. The exact sequence of Ext's shows that it suffices to consider the case when M is irreducible. We may also assume that $\operatorname{supp} M \cap U \neq \emptyset$ (otherwise see 2.2.1). Then the canonical map $j_! j^* M \to M$ is epimorphism; let K be its kernel. We have $\operatorname{supp} K \subset X-U$, Hence $\operatorname{Ext}^{\cdot}_{M(X)}(K,N) = \operatorname{Ext}^{\cdot}_{D(X)}(K,N)$ by 2.2.1. Also $\operatorname{Ext}^{\cdot}_{M(X)}(j_! M_U, N) = \operatorname{Ext}^{\cdot}_{D(X)}(j_! M_U, N)$ since by 2.2.2. both parts are zero. Now the exact sequence of Ext's shows that $\operatorname{Ext}^{\cdot}_{M(X)}(M,N) = \operatorname{Ext}_{D(X)}(M,N)$.

2.2.4. Now let we show that $\operatorname{Ext}^{\cdot}_{M(X)} = \operatorname{Ext}^{\cdot}_{D(X)}$ for an irreducible N supported at a generic point ; then the exact sequence of Ext's plus 2.2.3. prove the result for general N. For any open affine $j_U : U \hookrightarrow X$, $\eta \in U$, we have canonical injection $N \hookrightarrow j_*(N_U)$; put $L^U = j_*(N_U)/N$, clearly $\operatorname{supp} L^U \subset \bar{U}-U$. Consider the morphism between the long exact sequences of $\operatorname{Ext}^{\cdot}_{M(X)}$ and $\operatorname{Ext}^{\cdot}_{D(U)}$ for $0 \to N \to j_*(N_U) \to L^U — 0$. By 2.2.3. this morphism is an isomorphism on $\operatorname{Ext}^{\cdot}(M, L^U)$, hence $\mathcal{K}er^{\cdot}(M,N) := \operatorname{Ker}(\operatorname{Ext}^{\cdot}_{M(X)}(M,N) \to \operatorname{Ext}^{\cdot}_{D(U)}(M,N)) = \operatorname{Ker}(\operatorname{Ext}^{\cdot}_{M(X)}(M,j_*N_U) \to \operatorname{Ext}^{\cdot}_{D(X)}(M, j_*N_U))$, the same with Coker. According to 2.2.2., one has $\operatorname{Ext}^{\cdot}(M,j_*N_U) = \operatorname{Ext}^{\cdot}(M_U,N_U)$ for both $\operatorname{Ext}_{M(X)}$, $\operatorname{Ext}_{D(X)}$, therefore $\operatorname{Ker}^{\cdot}(M,N) = \operatorname{Ker}^{\cdot}(M_U, N_U)$. Shrinking U to η we get $\operatorname{Ker}^{\cdot}(M,N) = \operatorname{Ker}^{\cdot}(M_\eta, N_\eta) = 0$ by 2.1.; the same with Coker. This proves Theorem 1.3. $\square$

§3. Direct images as derived functors

3.1. Recall that all the standard functors between categories $D(X)$ come naturally together with their f-liftings. So we may use the construction from A 7. In particular, A 7.1 implies

<u>Lemma 3.1</u>. Let F be a t-exact standard functor between $D(X)$, say, F is Verdier's duality, or F is the nearby cycles functor, or F is the vanishing cycles functor; let $F_M = HF|_M$ be corresponding exact functor between hearts $M(X)$ and $\tilde{F}_M$ the functor between $D^b(M(X))$ induced by F_M. Then $\tilde{F}_M = F$ via the identification established by the main theorem. $\square$

3.2. Now let us pass to the direct image functors. Let $f : X \to Y$ be a morphism of schemes. It determines, under the identification of the main theorem, the exact functors $f_*, f_! : D^b(M(X)) \to D^b(M(Y))$. Assume that f is affine (for a non-affine case, see n° 3.4). Then $f_!$ is left t-exact, and f_* is right t-exact; let $f_{!M} = Hf_!|_{M(X)}$,

$f_{*M} : \mathcal{M}(X) \longrightarrow \mathcal{M}(Y)$ be corresponding functors between $\mathcal{M}$'s. According to A 7. we have natural morphisms $Rf_{!M} \longrightarrow f_!$, $f_* \longrightarrow Lf_{*M}$.

Theorem 3.2. This morphisms are isomorphisms.

Proof. It suffices to treat the case of f_*, the case of $f_!$ follows by duality. The theorem would follow if we show that for any $M \in \mathcal{M}(X)$ there exists $N \twoheadrightarrow M$ such that $H^i f_*(N) = 0$ for any $i < 0$ (hence for any $i \neq 0$). This fact is a particular case of the following more general lemma, valid for arbitrary f (not necessary affine; in full generality this will be needed later; for the aimes of 3.2. you may assume that $U_i = X$)

Lemma 3.3. Let $f : X \to Y$ be any morphism of schemes, $r_\ell : U_\ell \hookrightarrow X$ a finite set of affine open embeddings, and $M \in \operatorname{Ob} \mathcal{M}(X)$. Then there exists $N \twoheadrightarrow M$ such that the perverse sheaves $H^i f|_{U_\ell *}(N_{U_\ell})$ on Y are zero for any U_ℓ and $i < 0$.

Proof. 3.3.1. First assume that X is quasiprojective. Our N will be of the form $j_!(M_V)$ for a certain affine open embedding $j : V \hookrightarrow X$. Consider a diagram

$$\begin{tikzcd} X \arrow[r, hook, "k_X"] \arrow[d, "f"'] & \bar X \arrow[d, "\bar f"] \\ Y \arrow[r, hook, "k_Y"] & \bar Y \end{tikzcd}$$

, where k_X is an affine open embedding and $\bar X$ is projective. Then $k_\ell := k_X r_\ell : U_\ell \hookrightarrow \bar X$ are also affine, therefore $\bar M := k_{X*} M$, $\bar M_\ell := k_{\ell *}(M_{U_\ell})$ belong to $\mathcal{M}(\bar X)$. Find a hyperplane section $H \subset \bar X$ such that the following conditions hold (here $V := \bar X \setminus H \overset{j}{\hookrightarrow} \bar X$).

$$\begin{tikzcd} V \arrow[r, hook, "j"] & \bar X \\ V_\ell := V \cap U_\ell \arrow[u, "k_{\ell V}"] \arrow[r, hook, "j_\ell"] & U_\ell \arrow[u, "k_\ell"'] \end{tikzcd}$$

a_H. The canonical arrow $j_!(\bar M_V) = j_! j^* \bar M \longrightarrow \bar M$ is surjective

b_H. The canonical arrows $j_!(\bar M_{\ell V}) = j_! k_{\ell V *}(M_{V_\ell}) \longrightarrow k_{\ell *} j_{\ell !}(M_{V_\ell})$ are isomorphisms.

Suppose we have found such an H; then set $N = k_X^* j_! \bar M_V = j_{X!} M_{V_X}$ (here $j_X : V_X := V \cap X \hookrightarrow X$); the condition a_H shows that the canonical arrow $N \longrightarrow M$ is surjective, and condition b_H implies that $H^i(f|_{U_\ell *}(N_{U_\ell})) = H^i(\bar f_* k_{\ell *} j_{\ell !} M_{U_\ell})_Y = H^i(\bar f_* j_!(\bar M_\ell)_V)_Y = H^i((\bar f \circ j)_!(\bar M_\ell)_V)_Y$ are zero for $i < 0$ (since $\bar f \circ j$ is affine). Thus 3.3. in our case is proved.

To find H let us rewrite the conditions a_H and b_H as follows (here $H \overset{i_H}{\hookrightarrow} \bar X$):

$$\begin{tikzcd} H \arrow[r, hook, "i_H"] & \bar X \\ H_{U_\ell} \arrow[u, "k_{\ell H}"] \arrow[r, hook, "i_{\ell H}"] & U_\ell \arrow[u, "k_\ell"'] \end{tikzcd}$$

a'_H. The objects $H^a i_H^*(\bar{M}) \in \mathcal{M}(H)$ are zero for $a \neq -1$

b'_H. The canonical arrows $i_H^* k_{\ell *}(M_{U_\ell}) \longrightarrow k_{\ell H *} i_{\ell H}^*(M_{U_\ell})$ are isomorphisms.

(To see that b <=> b' use the octahedron of the commutative diagram

$$\begin{tikzcd} & i_{H*}i_H^* k_{\ell *}(M_{U_\ell}) \arrow[d] \\ k_{\ell *}M_{U_\ell} \arrow[ur] \arrow[r] & i_{H*} k_{\ell H*} i_{\ell H}^*(M_{U_\ell}) \end{tikzcd}$$.)

Now let us consider all the hyperplane sections H simultaneously: they are parametrized by a projective space $\mathcal{P}$, and the above pictures for different H's are the fibers of the global picture over the scheme $\mathcal{P}$. For example we have canonical closed hyperplane section subspace $i : H_{\mathcal{P}} \longrightarrow \bar{X}_{\mathcal{P}} := \bar{X} \times \mathcal{P}$, the object $M_{\mathcal{P}} := P_X^*(M)[\dim \mathcal{P}]$ in $\mathcal{M}(X_{\mathcal{P}})$ and so on.

The statements a'_H and b'_H have obvious analogues $a_{\mathcal{P}}$, $b_{\mathcal{P}}$ in this situation, which are clearly true, since the projection $H_{\mathcal{P}} \longrightarrow \bar{X}$ is smooth. So a_H and b_H for a particular H would follow if we know that the objects from a_H and b_H coincide with the restriction of objects from $a_{\mathcal{P}}$, $b_{\mathcal{P}}$ to the fiber over the closed point "H" of $\mathcal{P}$. But this is true over some Zariski open subset of $\mathcal{P}$ (see [4] th. 1.9. for the constructible case, the holonomic case is quite easy), so we are done (in fact, if the base field is finite, this Zariski open may have no points over our field, so first one has to pass to a certain finite extension, and then use the trace).

3.3.2. To get Lemma 3.3. without quasiprojectivity assumption on X, one proceeds as follows. First, you may assume that for a certain open $r : U \hookrightarrow X$ such that U is affine one has $M = r_! M_U$ (to see this choose an affine open covering $\{U_\nu\}$ of X; then the canonical arrow $\oplus r_{\nu !} M_{U_\nu} \longrightarrow M$ is surjective, so if $N_\nu \longrightarrow r_{\nu !} M_{U_\nu}$ satisfy 3.3, then $N = \oplus N_\nu \longrightarrow M$ also fits 3.3.). Now take

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\P"] \\ U \arrow[ur, "\tilde{r}"] \arrow[r, "r"'] & X \end{tikzcd}$$

, where $\P$ is proper, $\tilde{X}$ is quasiprojective and $\tilde{r}$ is an open embedding. Apply 3.3.1. to $f \circ \P : \tilde{X} \longrightarrow Y$, $\tilde{M} = \tilde{r}_! M_U$ and $\tilde{U}_\ell = \P^{-1}(U_\ell)$. We get an affine $\tilde{j} : \tilde{V} \hookrightarrow \tilde{X}$ such that $\tilde{N} = \tilde{j}_! \tilde{M}_V = (\tilde{r} \circ j)_!(M_V) — M$ (here $j : V := \tilde{V} \cap U \hookrightarrow U$) is surjective and $H^i(f \circ \P)_* (\tilde{N}_{\tilde{U}_\ell})$ are zero for $i < 0$. Consider $N := (r \circ j)_!(M_V) \longrightarrow M$: clearly this arrow is surjective (it coincides with $\tilde{N} \longrightarrow \tilde{M}$ on U and $r_!$ is exact) and $f_*(N_{U_\ell}) = (f \P)_*(\tilde{N}_{\tilde{U}_\ell})$ (since $\P_* \tilde{N} = \P_! \tilde{N} = N$) are acyclic in negative degrees. So we are done. □

3.4. If $f : X \longrightarrow Y$ is an arbitrary morphism of schemes, then f_*

may be neither left, nor right t-exact. To recover f_* (or $f_!$) from f_*° one may proceed as follows. Fix some open covering $\{U_i\}, \bigcup U_i = X$, such that every U_i is affine. We get the functor $f^\circ_{\{U_i\}*} : M(X) \to M(Y)^\Delta$ (= the category of cosimplicial objects in $M(Y)$) by the formula $f^\circ_{\{U_i\}*}(M)^n = \bigoplus_{(U_{i_0},\dots,U_{i_n})} f_*^\circ(M_{U_{i_0}\cap\dots\cap U_{i_n}})$ with obvious face and degeneracy maps (this is f_*° applied to the Cech complex of M). Clearly, $f^\circ_{\{U_i\}*}$ is right exact and 3.3. implies that $f_* = \mathrm{Tot}\, L f^\circ_{\{U_i\}*}$, where $\mathrm{Tot} : D^b(M(Y)^\Delta) \to D^b(M(Y))$ is passing to the total complex functor. One may say this in a more invariant way, without fixing $\{U_i\}$. Consider the category $Sh_{X,Y}$ of $M(Y)$-valued sheaves on X_{Zar}; we have the global section functor $\Gamma : Sh_{X,Y} \to M(Y)$ and its right-derived functor $R\Gamma : D^b(Sh_{X,Y}) \to D^b(M(Y))$. For $M \in M(X)$ let $f_*^\sim(M)$ be the sheaf that corresponds to the presheaf $U \mapsto f_*^\circ(M_U)$; clearly $f_*^\sim : M(X) \to Sh_{X,Y}$ is a right-exact functor of finite cohomology dimension, and 3.3. shows that $f_* = R\Gamma \circ L f_*^\sim$.

Appendix. Filtered categories and realization functor

What follows is a variation on theme [1] (3.1). We deal with the following problem: given a t-category D with the heart $\mathcal{C}$; construct the t-exact functor real : $D^b(\mathcal{C}) \to D$ that induces the identity functor on $\mathcal{C}$. To do this one needs some extra structure on D: namely one should fix a filtered category over D. Here are convenient definitions.

Definition A 1. a) A filtered triangulated category, or f-category for short, is a triangulated category DF together with two strictly full triangulated subcategories DF ($\leqslant 0$) and DF ($\geqslant 0$), an exact automorphism $s : DF \to DF$ (called "shift of filtration"), and morphism of functors $\alpha : Id_{DF} \to s$. The following axioms should hold (here $DF(\leqslant n) := s^n DF(\leqslant 0)$, $DF(\geqslant n) := s^n DF(\geqslant 0)$):

(i) $DF(\geqslant 1) \subset DF(\geqslant 0)$, $DF(\leqslant 1) \supset DF(\leqslant 0)$, $\bigcup_{n\in\mathbb{Z}} DF(\leqslant n) = \bigcup_{n\in\mathbb{Z}} DF(\geqslant n) = DF$

(ii) For any X in DF one has $\alpha_X = s(\alpha_{s^{-1}X})$.

(iii) For any X in $DF(\geqslant 1)$, Y in $DF(\leqslant 0)$ one has $\mathrm{Hom}(X,Y) = 0$ and α induces isomorphisms $\mathrm{Hom}(Y,X) = \mathrm{Hom}(Y, s^{-1}X) = \mathrm{Hom}(sY,X)$.

(iv) For any X in DF there exists a distinguished triangle $A \to X \to B$ with A in $D(\geqslant 1)$ and B in $D(\leqslant 0)$.

b) An f-functor between f-categories is an exact functor that commutes with s and α and conserves $DF(\leqslant 0)$, $DF(\geqslant 0)$.

c) Let D be a triangulated category. An f-category over D is an f-category DF together with equivalence of triangulated categories $i : D \longrightarrow DF(\leqslant 0) \cap DF(\geqslant 0)$. If $\Phi : D_1 \longrightarrow D_2$ is an exact functor between triangulated categories, and DF_i is an f-category over D_i $(i = 1,2)$ then an f-lifting of Φ is an f-functor $\Phi F : DF_1 \longrightarrow DF_2$ such that $i \circ \Phi = \Phi F \circ i$.

Example A 2. Let A be ab abelian category, $D^{\cdot}A$ its derived category, and $DF(A)$ the filtered derived category of complexes with finite decreasing filtration. Then $D^{\cdot}F(A)$ is an f-category over $D^{\cdot}(A)$: take $D^{\cdot}F(A)(\geqslant n) = \{(C^{\cdot}, F) : gr^i_F(C^{\cdot}) = 0 \text{ for } i < n\}$, $D^{\cdot}F(A)(\leqslant n) = \{(C^{\cdot}, F) : gr^i_F(C^{\cdot}) = 0 \text{ for } i > n\}$; $s(C^{\cdot}, F) = (C^{\cdot}, F^{\cdot -1})$; $\alpha_{(C^{\cdot},F)} = id_{C^{\cdot}} : (C^{\cdot}, F) \longrightarrow (C^{\cdot}, F^{\cdot -1})$; $i(C^{\cdot}) = (C^{\cdot}, Tr)$, where Tr is the trivial filtration : $gr^i_{Tr} = 0$ for $i \neq 0$. If $T : A_1 \longrightarrow A_2$ is a left exact functor between abelian categories, and $RT : D^+(A_1) \longrightarrow D^+(A_2)$, $RFT : D^+F(A_1) \longrightarrow D^+F(A)$ are corresponding right derived functors, then RFT is an f-lifting of RT; same for left derived functors.

Note that we have canonical exact functors $\sigma_{\geqslant n} : DF(A) \longrightarrow DF(A)(\geqslant n)$, $\sigma_{\leqslant n} : DF(A) \longrightarrow DF(A)(\leqslant n)$ defined by formulas $\sigma_{\geqslant n}(C^{\cdot}, F) = F^n(C^{\cdot})$ with the induced filtration, $\sigma_{\leqslant n}(C^{\cdot}, F) = C^{\cdot}/F^{n+1}(C^{\cdot})$ with the induced filtration, and also the forgetting of filtration functor $\omega : DF(A) \longrightarrow D(A)$. One may define them for arbitrary f-categories as follows.

Proposition A 3. Let DF, D be as in A 1.c. Then

(i) The inclusion of $DF(\geqslant n)$ in DF has right adjoint $\sigma_{\geqslant n} : DF \longrightarrow DF(\geqslant n)$, and the inclusion of $DF(\leqslant n)$ in DF has left adjoint $\sigma_{\leqslant n} : DF \longrightarrow DF(\leqslant n)$. The functors $\sigma_{\leqslant n}$, $\sigma_{\geqslant n}$ are exact; they preserve subcategories $DF(\leqslant a)$, $DF(\geqslant a)$; there is the unique isomorphism $\sigma_{\leqslant a}\sigma_{\geqslant b} \xrightarrow{\sim} \sigma_{\geqslant b}\sigma_{\leqslant a}$ such that the diagram

$$\begin{tikzcd} & & X \arrow[drr] & & \\ \sigma_{\geqslant b}X \arrow[urr] \arrow[r] & \sigma_{\leqslant a}\sigma_{\geqslant b}X \arrow[rr, "\sim"] & & \sigma_{\geqslant b}\sigma_{\leqslant a}X \arrow[r] & \sigma_{\leqslant a}X \end{tikzcd}$$

commutes. Put $gr^n_F := i^{-1}s^{-n}\sigma_{\leqslant n}\sigma_{\geqslant n} : DF \longrightarrow D$.

(ii) For any X in DF there exists unique morphism $d \in Hom^1(\sigma_{\leqslant 0}X, \sigma_{\geqslant 1}X)$ such that the triangle $\sigma_{\geqslant 1}X \longrightarrow X \longrightarrow \sigma_{\leqslant 0}X \xrightarrow{d} \dots$

is distinguished; this is the only, up to unique isomorphism, triangle (A,X,B) with A in $DF(\geqslant 1)$ and B in $DF(\leqslant 0)$.

(iii) There exists the only, up to unique isomorphism, exact functor $\omega : DF \to D$ such that

1). $\omega|_{DF(\leqslant 0)} : DF(\leqslant 0) \to D$ is right adjoint to $i : D \to DF(\leqslant 0)$

2) $\omega|_{DF(\geqslant 0)} : DF(\geqslant 0) \to D$ is left adjoint to $i : D \to DF(\geqslant 0)$

3) For any X in DF the arrow $\omega(\alpha_X) : \omega(X) \to \omega(sX)$ is an isomorphism.

4) For any A in $DF(\leqslant 0)$, B in $DF(\geqslant 0)$ we have $\omega : Hom(A,B) \xrightarrow{\sim} Hom(\omega A, \omega B)$.

In fact, ω is uniquely determined by properties 1,3 or 2,3. □

One may see that all the standard constructions in usual filtered derived categories may be carried over in the f-category framework. Exercise: do this for [1] (3.1.2.6).

Definition A 4. Let D, DF be as in 1.5.1., and assume that we are given t-structures on D and DF. Say that they are compatible if $i : D \to DF$ is t-exact, and $s(DF^{\leqslant 0}) = DF^{\leqslant -1}$

Proposition A 5. a) Given a t-structure on D, there exists unique t-structure on DF compatible with it, namely $DF^{\leqslant 0} = \{X : gr_F^i(X) \in D^{\leqslant i}\}$, $DF^{\geqslant 0} = \{X : gr_F^i(X) \in D^{\geqslant i}\}$

b) Assume we are given compatible t-structures on D, DF with hearts $\mathcal{C}$, $\mathcal{C}F$ respectively; let $H : D \to \mathcal{C}$ be corresponding cohomology functor. Define the cohomology functor $H_F : DF \to C^b(\mathcal{C})$ by the formula $H_F(X)^i = H^i gr_F^i(X)$, the differential $H_F(X)^i \to H_F(X)^{i+1}$ comes from the distinguished triangle $\omega(\sigma_{\leqslant i+1}\sigma_{\geqslant i+1} \to \sigma_{\leqslant i+1}\sigma_{\geqslant i} \to \sigma_{\leqslant i}\sigma_{\geqslant i})(X)$ in D. Then $H_F|_{\mathcal{C}F} : \mathcal{C}F \to C^b(\mathcal{C})$ is an equivalence of categories, and, via this equivalence, H_F becomes a canonical cohomology functor of our t-structure on DF. □

A 6. Let $\widetilde{real} : C^b(\mathcal{C}) \to D$ be the composition $C^b(\mathcal{C}) \xrightarrow{\alpha} \mathcal{C}F \subset DF \xrightarrow{\omega} D$, where $\alpha = H_F|_{\mathcal{C}F}^{-1}$. One may see that $H \circ \widetilde{real} : C^b(\mathcal{C}) \to$ is a usual cohomology functor, and $\widetilde{real}$ factors (uniquely) through the t-exact functor $real : D^b(\mathcal{C}) \to D$, $real|_{\mathcal{C}} = id_{\mathcal{C}}$. This is the functor we looked for.

A 7. This construction has the following functorial properties. Let D_i, $i = 1,2$, be triangulated categories, and DF_i some f-categories over D_i. Let $\Phi : D_1 \to D_2$ be an exact functor, and $\Phi F : DF_1 \to DF_2$ an f-lifting of Φ. Assume that we are given compatible t-structures on D_i, DF_i.

If Φ is t-exact, then it induces an exact functor $\Phi_{\mathcal{C}} : \mathcal{C}_1 \to \mathcal{C}_2$, hence the t-exact functor $D^b(\mathcal{C}_1) \xrightarrow{D\Phi_{\mathcal{C}}} D^b(\mathcal{C}_2)$. Since, by 5.a, ΦF is also t-exact, we get

Lemma A 7.1. In this situation real $D\Phi_{\mathcal{C}} = \Phi$ real. $\square$

More generally, suppose that Φ is left t-exact. Consider the left exact functor $\Phi_{\mathcal{C}} = H\Phi|_{\mathcal{C}_1} : \mathcal{C}_1 \to \mathcal{C}_2$ and the corresponding graded functor $\Phi^{\cdot}_{\mathcal{C}} : C^b(\mathcal{C}_1) \to C^b(\mathcal{C}_2)$. Clearly, ΦF is also left t-exact, and $\Phi^{\cdot}_{\mathcal{C}} = H_F \circ \Phi F \circ \alpha$, so for any $C^{\cdot}$ in $C^b(\mathcal{C}_1)$ we have the canonical arrow $\alpha\Phi^{\cdot}_{\mathcal{C}}(C^{\cdot}) = \tau_{\leqslant 0}\Phi F\alpha(C^{\cdot}) \to \Phi F\alpha(C^{\cdot})$. Applying ω we get the morphism $\widetilde{\text{real}}\,\Phi^{\cdot}_{\mathcal{C}} \to \Phi\,\widetilde{\text{real}}$ of D_2-valued functors on $C^b(\mathcal{C}_1)$, or, passing to the derived functor $R\Phi^{\cdot}_{\mathcal{C}}: D^b(\mathcal{C}_1) \to \varinjlim D^b(\mathcal{C}_2)$, the morphism $\text{real}\circ R\Phi^{\cdot}_{\mathcal{C}} \to \Phi\circ\text{real}$. This construction has an obvious analogue for the right t-exact functors.

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Complement to §3.

Here we will see how a variant of 3.3.1. gives a very concrete chain complexes that compute $\int$ (:= direct image to a point) of perverse sheaves on projective variety.

These chain complexes will be constructed by means of a pair of transversal flags (F, F') on $\mathbb{P}^n$. Clearly, such pairs are the same as systems of coordinate axes; they are parametrized by $GL(n+1)/$ diagonal matrices. We put $F = (F_n \subset \dots \subset F_0 = \mathbb{P}^n)$, $F' = (F'_n \subset \dots \subset F'_0 = \mathbb{P}^n)$, $i_k : F_k \hookrightarrow \mathbb{P}^n$, $i'_k : F'_k \hookrightarrow \mathbb{P}^n$.

Let M be a perverse sheaf on $\mathbb{P}^n$. Say that (F,F') is in a generic position with respect to M if for any pair (a,b) of indices (we may assume $0 \leq a, b \leq n, a+b \leq n$) the following holds:

(i) The objects $M^a_b := i_{a*} i^!_a i'_{b*} i'^*_b M[a-b]$, $\tilde{M}^a_b := i'_{b*} i'^*_b i_{a*} i^!_a M[a-b]$ of $D(\mathbb{P}^n)$ (supported on $F_a \cap F'_b$) are perverse sheaves

(ii) The natural arrow $\tilde{M}^a_b \to M^a_b$ is isomorphism.

One may see, as in 3.3.1., that a generic (F,F') satisfies this conditions. Assume that our (F,F') is of such kind. Then for any (a,b) we have canonical commentative diagram of perverse sheaves, supported on $F_a \cap F'_b$, with exact rows and columns:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & M^a_b \arrow[u] \arrow[r] & j_* j^* M^a_b \arrow[u] \arrow[r] & M^{a+1}_b \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & j'_! j'^* M^a_b \arrow[u] \arrow[r, "\alpha'"] & \mathcal{J}^a_b(M) \arrow[u, "\beta"] \arrow[r, "\beta'"] & j'_! j'^* M^{a+1}_b \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & M^a_{b+1} \arrow[u] \arrow[r] & j_* j^* M^a_{b+1} \arrow[u, "\alpha"] \arrow[r] & M^{a+1}_{b+1} \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where $j: \mathbb{P}^n \setminus H \hookrightarrow \mathbb{P}^n$, $j': \mathbb{P}^n \setminus H' \hookrightarrow P^n$, H, H' are hyperplanes such that $H \cap F_a \cap F'_b = F_{a+1} \cap F'_b$, $H' \cap F_a \cap F'_b = F_a \cap F'_{b+1}$, and $\mathcal{J}^a_b(M) := j'_! j'^* j_* j^* M = j_* j^* j'_! j'^* M^a_b$ (the property (ii) guarantees that these are the same objects). So the sheaves $\mathcal{J}^a_b(M)$ form a bicomplex with differentials $d = \alpha\beta$, $d' = \alpha'\beta'$. Clearly it is d-acyclic if $b \neq 0$, and d'-acyclic if $a \neq 0$. Let $\mathcal{J}^\cdot(M)$ be the corresponding total complex ($\mathcal{J}^n = \bigoplus_{a-b=n} \mathcal{J}^a_b$). The diagram also shows that it is canonically quasiisomorphic to M (i.e. one has $H^i \mathcal{J}^\cdot(M) = 0$ for $i \neq 0$, $H^0 \mathcal{J}^\cdot(M) = M$): the quasiisomorphism is $M = M^0_0 \to (j_* j^* M^0_0 \to j_* j^* M^1_0 \to \dots) \xleftarrow{\beta} \mathcal{J}^\cdot(M)$ (you may also use the 0's column to get the same quis). Now note that for $i \neq 0$ and any (a,b) $H^i \int \mathcal{J}^a_b(M) = 0$ (see 3.3.1.). So

$\int^0 \mathcal{J}^\cdot(M)$ is chain complex that represents $\int M$.