Homological algebra (Math 613): Problem set 3

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The main goal of this problem set is to explore the relation of sheaf cohomology to singular cohomology. We fix a topological space X. Write $P\mathrm{Ab}(X)$ for the category of abelian presheaves on X, while $\mathrm{Ab}(X) \subset P\mathrm{Ab}(X)$ denotes the full subcategory of abelian sheaves. For basics on sheaves and presheaves, please consult standard sources.

1. Let $f : Y \to X$ be a map of topological spaces.
   (a) Show that there is an induced functor $f_* : \mathrm{Ab}(X) \to \mathrm{Ab}(Y)$ defined by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$.
   (b) Show that f_* admits a left-adjoint f^*.
   (c) Show that f^* is left-exact, and deduce that f_* preserves injectives.

   Now assume f is the inclusion of an open subset.

   (a) Show that f^* coincides with the obvious restriction $\mathrm{Ab}(X) \to \mathrm{Ab}(U)$.
   (b) Show that f^* has an exact left-adjoint $\mathcal{G} \mapsto f_!\mathcal{G}$.
   (c) Show that f_* preserves injectives.

From now on, for an open subset $j : U \hookrightarrow X$, we write $\underline{\mathbf{Z}}_U$ for $j_!\underline{\mathbf{Z}}$.

2. Consider the global sections functor $\Gamma : \mathrm{Ab}(X) \to \mathrm{Ab}$ defined by $\mathcal{F} \mapsto \Gamma(X, \mathcal{F}) := \mathcal{F}(X)$.
   (a) Show that $\oplus_{U \in \mathrm{Op}(X)} \underline{\mathbf{Z}}_U$ is a generator of $\mathrm{Ab}(X)$.
   (b) Show that $\mathrm{Ab}(X)$ is a Grothendieck abelian category.
   (c) Show that Γ is left-exact.

   We write $H^i(X, \mathcal{F})$ for the i-th derived functor of Γ evaluated on $\mathcal{F} \in \mathrm{Ab}(X)$.

   (d) If $\mathcal{F}$ is injective, show that $H^0(X, \mathcal{F})$ is injective, and $H^i(X, \mathcal{F}) = 0$ if $i > 0$.
3. A sheaf $\mathcal{F} \in \mathrm{Ab}(X)$ is called *flasque* if $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective for $V \subset U$.
   (a) Using the canonical inclusions $\underline{\mathbf{Z}}_V \to \underline{\mathbf{Z}}_U$, show that injective sheaves are flasque.
   (b) For any exact sequence
   $$0 \to \mathcal{F} \to I \to \mathcal{Q} \to 0$$
   with $\mathcal{F}$ flasque and I injective, show that $I(U) \to \mathcal{Q}(U)$ is surjective for all U.
   (c) Show that $H^i(X, \mathcal{F}) = 0$ for $i > 0$ if $\mathcal{F}$ is flasque.

Write $C_q(X, \mathbf{Z})$ (resp. $C^q(X, \mathbf{Z})$) for the singular q-chains (resp. q-cochains) on X. The association $U \mapsto C^q(U, \mathbf{Z})$ is a presheaf on X, and let $\mathcal{C}^q$ denote its sheafification. As q varies, this data assembles to give a cochain complex
$$\mathcal{C}^\bullet := \mathcal{C}^0 \to \mathcal{C}^1 \to \mathcal{C}^2 \to \dots$$
in $\mathrm{Ab}(X)$. There is a natural augmentation $\underline{\mathbf{Z}} \xrightarrow{\epsilon} \mathcal{C}^\bullet$ defined by the constant 0-cochain.

4. Let X be a space.

(a) For $U \in \mathrm{Op}(X)$, show that $C_q(U, \mathbf{Z}) \to C_q(X, \mathbf{Z})$ is injective, and $C^q(X, \mathbf{Z}) \to C^q(U, \mathbf{Z})$ is surjective.

(b) Show that the sheaf $\mathcal{C}^q$ is flasque.

(c) For $U \in \mathrm{Op}(X)$, show that $C^q(U, \mathbf{Z}) \to \mathcal{C}^q(U, \mathbf{Z})$ is surjective, and the kernel $C^q(U, \mathbf{Z})_0$ is exactly those q-cochains ϕ such that $\phi|_{U_i} = 0$ for some open cover $\{U_i\}$ of U.

(d) Show that the complex $C^\bullet(X, \mathbf{Z})_0$, defined via (c), is acyclic[1].

(e) When X is locally-contractible, show that the map ϵ above is a quasi-isomorphism. Deduce that $H^i(X, \underline{\mathbf{Z}})$ as defined via sheaves coincides with the i-singular cohomology for such an X.

5. Let $f : X \to Y$ be a map of topological spaces, and fix $\mathcal{F} \in \mathrm{Ab}(X)$.

(a) Show that the sheaf $R^p f_* \mathcal{F} \in \mathrm{Ab}(Y)$ is the sheafification of $U \mapsto H^p(f^{-1}(U), \mathcal{F})$.

(b) Construct a spectral sequence
$$H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

(c) Now assume that Y is locally contractible, and there exists an open cover $\{U_i\}$ of Y such that $f^{-1}(U_i) \simeq U_i \times I$, where I is the unit interval. Show that $H^p(Y, \underline{\mathbf{Z}}) \simeq H^p(X, \underline{\mathbf{Z}})$.

[1] Here you may use the theorem on "small chains" as in, for example, §31 in Munkres's *Elements of algebraic topology*.