Homological algebra (Math 613): Problem set 2

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1. Let $f : K \to L$ be a map of chain complex over an abelian category $\mathcal{A}$. Construct a homotopy-equivalence $\mathrm{cone}(L \to \mathrm{cone}(f)) \simeq K[1]$.
2. Check that any acyclic chain complex $K \in \mathrm{Mod}_R$ with K_i free and 0 for $i \ll 0$ is split.
3. Check that the assumption $K_i = 0$ for $i \ll 0$ above can be dropped when $R = \mathbf{Z}$.
4. Show that $K \in \mathrm{Ch}(\mathrm{Mod}_R)$ is a projective object if and only if K is split and acyclic with each K_i projective.
5. Show that the homotopy category $K(\mathrm{Ab})$ of abelian groups is not abelian.

Fix a cocomplete category $\mathcal{A}$. An object $X \in \mathcal{A}$ is called *compact* if $\mathrm{Hom}(X, -)$ commutes with filtered colimits, i.e., the natural map $\mathrm{Hom}(X, \mathrm{colim}\, Y_i) \leftarrow \mathrm{colim}\, \mathrm{Hom}(X, Y_i)$ is a bijection for any filtered system $\{Y_i\}$ of objects of $\mathcal{A}$. Write $\mathcal{A}^c \subset \mathcal{A}$ for the full subcategory of all compact objects. We say that $\mathcal{A}$ is *compactly generated* if all objects in $\mathcal{A}$ are filtered colimits of objects in $\mathcal{A}^c$.

6. Show that $\mathcal{A}^c$ is closed under finite colimits in $\mathcal{A}$.
7. For the following cocomplete categories $\mathcal{A}$, describe $\mathcal{A}^c$, and determine if the category compactly generated:
   (a) Sets.
   (b) Groups.
   (c) Rings.
   (d) Commutative rings.
   (e) Open subsets of a toppological space X (with morphisms being inclusion).
   (f) Mod_R for a ring R.
   (g) $\mathrm{Ab}^{\mathrm{opp}}$.
8. Given a set of rings $\{R_i\}$, let $R = \prod_i R_i$. Describe the compact objects in Mod_R in terms of compact objects in each Mod_{R_i}.
9. For any small category $\mathcal{C}$, let $\mathrm{Ind}(\mathcal{C})$ be the category *ind-objects* in $\mathcal{C}$, i.e., objects are diagrams $\{A_i\}$, indexed by filtered categories I, and maps are given by $\mathrm{Hom}(\{A_i\}, \{B_j\}) = \mathrm{colim}_j \lim_i \mathrm{Hom}(A_i, B_j)$. Show that if $\mathcal{A}$ is a compactly generated cocomplete category, then $\mathrm{Ind}(\mathcal{A}^c) \simeq \mathcal{A}$.
10. Determine whether the following functors $\mathrm{Ab} \to \mathrm{Ab}$ are exact, left exact, right exact, exact in the middle, or neither:
    (a) $F_1(A) = A/2A$.
    (b) $F_2(A) = \{x \in A \mid 4 \cdot x = 0\}$.
    (c) $F_2 \circ F_1$ and $F_1 \circ F_2$, with F_1 and F_2 as above.
    (d) $F(A) = A \otimes B$ for a fixed abelian group B.
    (e) $F(A) = A^{\otimes n}$.

(f) $F(A) =$ free abelian group on the set A.

(g) $F(A) = A_{\mathrm{tors}}$.

(h) $F(A) = \mathrm{Hom}(A_{\mathrm{tors}}, \mathbf{Q}/\mathbf{Z})$.

(i) Fix a topological space X and $n \in \mathbf{Z}_{\geq 0}$, and let $F(A) = H^n(X, A)$.

(j) Fix an manifold X of dimension n, and let $F(A) = H^n(X, A)$.

11. Let $\mathcal{A}$ be an abelian category, fix $X, Y \in \mathcal{A}$, and $n \geq 1$. A *degree n Yoneda extension* of X by Y is an exact sequence

$$Z_\bullet := 0 \to Y \to Z_1 \to \cdots \to Z_n \to X \to 0.$$

A *map* $Z_\bullet \to Z'_\bullet$ of such extensions is a map of exact sequences which is the identity on the Y and X terms. Two such extensions $Z'_\bullet$ and $Z''_\bullet$ are declared to be equivalent if there are maps $Z'_\bullet \leftarrow Z_\bullet \to Z''_\bullet$ of extensions.

(a) Show that equivalence of extensions is an equivalence relation on the set of all degree n Yoneda extensions of X by Y. The quotient set is denoted $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$.

(b) Show that $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ is covariantly functorial in Y, and contravariantly functorial in X by considering pushouts and pullbacks of extensions.

(c) Show that there is a natural binary operation $+$ on $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ given by setting $[Z_\bullet]+[Z'_\bullet]$ to be the degree n extension obtained by taking the direct sum $W_\bullet := Z_\bullet \oplus Z'_\bullet$, which is an element in $\mathrm{Ext}^n_{\mathcal{A}}(X \oplus Y, Y \oplus Y)$, and composing with the "fold" map $Y \oplus Y \to Y$ and the diagonal map $X \to X \oplus X$.

(d) Let $e_{X,Y}$ be the degree n extension obtained as follows: $Z_1 = Y$, $Z_n = X$ and $Z_i = 0$ for $i \neq 1, n$ if $n \geq 2$, and $Z_1 = X \oplus Y$ if $n = 1$ (and the maps are the obvious ones in both cases). Show that $e_{X,Y}$ is a unit for the operation $+$ defined above.

(e) By tweaking signs, show that $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ is an abelian group under $+$.

(f) For $X, Y, W \in \mathcal{A}$, and $m, n \in \mathbf{Z}_{\geq 0}$, construct a natural map $\mathrm{Ext}^n_{\mathcal{A}}(X, Y) \times \mathrm{Ext}^m_{\mathcal{A}}(Y, W) \to \mathrm{Ext}^{n+m}_{\mathcal{A}}(X, W)$ by splicing extensions together. Show that this operation is bilinear with respect to $+$, and associative.

(g) Now assume $\mathcal{A} = \mathrm{Mod}_R$. Show that $\mathrm{Ext}^n_{\mathcal{A}}(X, -) = 0$ for all $n \geq 1$ if and only if X is projective. Dually, show that $\mathrm{Ext}^n_{\mathcal{A}}(-, Y) = 0$ for all $n \geq 1$ if and only if Y is injective.

(h) Given a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, and $W \in \mathcal{A}$, construct a natural map $\mathrm{Ext}^n_{\mathcal{A}}(X, W) \to \mathrm{Ext}^{n+1}_{\mathcal{A}}(Z, W)$. Using this, show that the family $\{\mathrm{Ext}^n_{\mathcal{A}}(-, W)\}$, together with these "boundary" maps, gives a δ-functor $\mathcal{A}^{\mathrm{opp}} \to \mathcal{A}$.

(i) Now assume $\mathcal{A} = \mathrm{Ab}$. Calculate $\mathrm{Ext}^i_{\mathcal{A}}(X, \mathbf{Z})$ using the exact sequence $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$.