Homological algebra (Math 613): Problem set 1

Bhargav Bhatt

1. Let $\mathcal{C}$ be a category that admits finite products and coproducts, as well as a zero object 0.

   (a) For $X, Y \in \mathcal{C}$, using 0, construct a natural map $X \sqcup Y \to X \times Y$ such that the induced composition $X \to X \sqcup Y \to X \times Y \to X$ is the identity (and similarly for Y).

   Assume now that finite coproducts and products coincide, i.e., the map in (a) is an isomorphism for all X, Y.

   (b) Given maps $f, g : X \to Y$, define a new map $f \oplus g : X \to Y$ through the following diagram:

   $$X \overset{\Delta}{\hookrightarrow} X \times X \overset{\simeq}{\leftarrow} X \sqcup X \overset{f \sqcup g}{\to} Y \sqcup Y \overset{\mathrm{can}}{\to} Y.$$

   Here the second map is the isomorphism coming from (a) by the assumption on $\mathcal{C}$, and the other maps are the usual ones.

   (c) Show that the association $(f, g) \mapsto f \oplus g$ in (b) determines a commutative monoid structure on $\mathrm{Hom}(X, Y)$ for $X, Y \in \mathcal{C}$ that is bilinear with respect to composition. In other words, $\mathcal{C}$ is *semiadditive*.

   (d) Assume now that $\mathcal{C}$ has the structure of an additive category. Then show that the group law on $\mathrm{Hom}(X, Y)$ coming from the additive structure must coincide with the one in (c). In other words, being additive is a *property* of a category, not an extra piece of structure.

2. Recall that a category is complete if it admits all limits. Show that a category is complete if it admits arbitrary products and fibre products.

3. Is the category of torsion abelian groups complete? Is it an abelian category?

4. Prove the snake lemma in an abelian category (no elements!).

Let Ab be the category of abelian groups, and $\mathrm{Ab}^{fg} \subset \mathrm{Ab}$ the full subcategory of finitely generated ones.

5. Classify all full thick[1] abelian subcategories $\mathcal{C} \subset \mathrm{Ab}^{fg}$, as follows:

   (a) For any $A \in \mathcal{C}$, show that A_{tors} and A/A_{tors} are both in $\mathcal{C}$.

   (b) If $\mathbf{Z}^r \in \mathcal{C}$ for some $r > 0$, then $\mathcal{C} = \mathrm{Ab}^{fg}$.

   (c) If $\mathbf{Z}/n \in \mathcal{C}$, then $A \in \mathcal{C}$ for any $A \in \mathrm{Ab}^{fg}$ such that $n^k \cdot A = 0$ for $k \gg 0$.

   (d) For each set S of primes, let $\mathcal{C}_S \subset \mathrm{Ab}^{fg}$ be the subcategory of all A such that $A \otimes \mathbf{Z}[\frac{1}{S}] = 0$; this means $n \cdot A = 0$ for some integer n whose prime factors lie in S. Show that $\mathcal{C}_S$ is abelian.

   (e) Conclude that either $\mathcal{C} = \mathrm{Ab}^{fg}$ or $\mathcal{C} = \mathcal{C}_S$ for some S.

   (f) (*) Find an analogue of this result for Mod_R^{fg}, where R is noetherian. (Gabriel)

6. Give an example of a non-thick abelian subcategory of Ab^{fg}

7. Let $\mathcal{C}$ be the category of *divisible* abelian groups, i.e., those $A \in \mathrm{Ab}$ such that $A = n \cdot A$ for all $n \in \mathbf{Z} - \{0\}$.

   (a) Show that $\mathcal{C}$ is additive, and admits all colimits and limits. In particular, $\mathcal{C}$ admits kernels and cokernels.

[1] An abelian subcatgory $\mathcal{C}$ is thick if for any short exact sequence $0 \to M_1 \to N \to M_2 \to 0$ in the ambient category, if $M_i \in \mathcal{C}$, then $N \in \mathcal{C}$.

(b) Prove that the natural map $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$ has no kernel or cokernel in $\mathcal{C}$. Conclude that $\mathcal{C}$ is not abelian.

8. Show that injective objects in Ab are exactly the divisible abelian groups. Conclude that for any $A \in \mathrm{Ab}$ and $a \in A$, there is a linear map $g : A \to \mathbf{Q}/\mathbf{Z}$ with $g(a) \neq 0$.

9. Fix an associative ring R, an R-module $M \in \mathrm{Mod}_R$.

   (a) Let $F(M) := \oplus_{m \in M} R$ be the free R-module with basis M. Show that there is a functorial epimorphism $\mu_M : F(M) \to M$.

   (b) Let $M^\vee := \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$. Using the injectivity of $\mathbf{Q}/\mathbf{Z}$ as an abelian group, show that there is a functorial injective "evaluation" map $\eta_M : M \to (M^\vee)^\vee$.

   (c) If M is free, show that $M^\vee$ is an injective R-module.

   (d) Set $I(M) := (F(M^\vee))^\vee$. Show that $I(M)$ is injective. Moreover, using η_M and $(\mu_{M^\vee})^\vee$, construct a functorial monomorphism $M \to I(M)$. Conclude that Mod_R has enough injectives.

   The functoriality of the embedding $M \hookrightarrow I(M)$ is quite useful in practice!

10. Let K be a field, and fix $K \in \mathrm{Ch}(\mathrm{Mod}_K)$. Construct a quasi-isomorphism $K \to \oplus_i H_i(K)[i]$. (Given $M \in \mathrm{Mod}_K$, we write $M[j]$ for the chain complex whose only non-zero term is M placed in homological degree j.)