Math 214 — Midterm 2
(Blue Version)

This is an 80-minute exam, but you have the full 110-minute class period to complete it. There are a total of 100 points possible; the value of each problem is marked. Partial credit may be given for significant progress toward the solution to a problem. Except where otherwise indicated, you must show sufficient work to make it clear to me what you are doing and why in order to obtain full credit.

Problem 1 (10 pts)

Compute the determinant of the following matrix by whatever method seems best to you.

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 4 & -1 & 4 & 1 \\ 3 & 5 & 0 & 0 \\ -7 & 3 & -1 & 2 \end{pmatrix}$$

Problem 2 (10 pts)

The set of points in xyz-space with $x+y+z=0$ forms a plane V. Find the matrix which represents orthogonal projection onto V.

Problem 3 (10 pts)

Consider the following matrix.

$$A=\begin{pmatrix} 2 & -1 & 2 \\ 1 & 3 & 5 \\ 2 & 0 & 3 \end{pmatrix}$$

a) Compute $\operatorname{adj} A$.

b) It turns out that $\det A = -1$. (You don't have to verify this fact.) Compute A^{-1}.

Problem 4 (10 pts)

Consider the following linear system.

$$\begin{aligned} 2x+y&=5 \\ x-y&=3 \\ x&=3 \end{aligned}$$

Write down the corresponding normal equation. (You don't need to solve it.)

Problem 5 (10 pts)

Fill in the missing entries so that the matrix is orthogonal.

$$\begin{pmatrix} 3/13 & 4/13 & \\ 4/13 & -12/13 & \\ 12/13 & & -4/13 \end{pmatrix}$$

Problem 6 (15 pts)

Let V be the subspace of $\mathbb{R}^5$ with a basis $\mathcal{A} = \left\{ \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 1 \\ 7 \\ 1 \\ 6 \end{pmatrix} \right\}$.

a) Compute the QR decomposition of $\begin{pmatrix} 2 & 7 \\ 2 & 1 \\ 2 & 7 \\ 2 & 1 \\ 3 & 6 \end{pmatrix}$.

b) Write down an orthonormal basis $\mathcal{B}$ of V.

c) Consider the vectors $\vec{v}, \vec{w}$ whose $\mathcal{A}$-coordinates are $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Express these vectors in $\mathcal{B}$-coordinates.

d) What is the cosine of the angle between $\vec{v}$ and $\vec{w}$?

Problem 7 (15 pts)

Consider the linear transformation $T: P_2 \to P_2$ defined by $T(f(t)) = f(2t+1) - f(t+1)$.

a) Write the matrix representation of T, relative to the basis $\{t^2, t, 1\}$.

b) Write down a basis for the kernel of T.

c) Write down a basis for the image of T.

d) What is the *nullity* (the dimension of the kernel) of T?

e) What is the *rank* (the dimension of the image) of T?

f) What is the determinant of T?

Problem 8 (20 pts)

Indicate whether each item is true or false. Each item is worth 2 points. No explanation necessary, no partial credit possible.

a) **T / F ?** If a 2×2 matrix P represents the orthogonal projection onto a line in $\mathbb{R}^2$, then P is similar to $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

b) **T / F ?** There exists an isomorphism from $P_4$to the space of 2×2 matrices.

c) **T / F ?** The invertible 5×5 matrices form a subspace of the space of 5×5 matrices.

d) **T / F ?** If A and B are any linear transformations from a linear space V to itself, then $\ker(AB)$ contains $\ker B$.

e) **T / F ?** A matrix is orthogonal if and only if it has orthogonal columns.

f) **T / F ?** Every system of linear equations has a (not-necessarily unique) least-squares solution.

g) **T / F ?** If A, B, C are symmetric $n \times n$ matrices, then $ABACABA$ is also a symmetric matrix.

h) **T / F ?** The matrices $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ and $\begin{pmatrix} 3 & -4 \\ 4 & 6 \end{pmatrix}$ are similar.

i) **T / F ?** If $\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{pmatrix}$ is a 4×4 matrix with determinant 10, then the matrix $\begin{pmatrix} \vec{v}_3 & 2\vec{v}_2 & \vec{v}_1 & \vec{v}_4 + 3\vec{v}_2 \end{pmatrix}$ has determinant -60.

j) **T / F ?** A 3×3 matrix has determinant zero if and only if its rows are linearly dependent.