Math 214 W10 — Midterm 1

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Answer Key

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Problem 1 (10 points):

Find the equation of the cubic equation (general form: $f(x) = ax^3 + bx^2 + cx + d$) whose graph passes through the points $(-1,2), (0,3), (1,6), (2,23)$.

Answer.

These four points lead to the following linear system.

$$\begin{aligned} -a + b - c + d &= 2 \\ d &= 3 \\ a + b + c + d &= 6 \\ 8a + 4b + 2c + d &= 23 \end{aligned}$$

This has the unique solution $a = 2, b = 1, c = 0, d = 3$.

$$f(t) = 2t^3 + t^2 + 3$$

Problem 2 (10 points):

Let L be the line spanned by the vector $\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$. Write down the matrix representing the orthogonal projection onto the line L in $\mathbb{R}^4$.

Answer.

$$\begin{pmatrix} 1/4 & -1/4 & 1/4 & -1/4 \\ -1/4 & 1/4 & -1/4 & 1/4 \\ 1/4 & -1/4 & 1/4 & -1/4 \\ -1/4 & 1/4 & -1/4 & 1/4 \end{pmatrix}$$

Problem 3 (10 points):

Is the vector $\vec{x} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ in the linear span of $\vec{v} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$? If so, find real numbers a, b such that $a\vec{v} + b\vec{w} = \vec{x}$.

Answer.

By reducing the matrix $\begin{pmatrix} 2 & 4 & 3 \\ 4 & 3 & 1 \\ 0 & 5 & 5 \end{pmatrix}$ or by any other means, $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = \frac{-1}{2}\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$.

Problem 4 (10 points):

In geometry, the *centroid* of a triangle is given by a simple formula. If the vertices of a triangle are at the points $(x_1,\ y_1)$, $(x_2,\ y_2)$, $(x_3,\ y_3)$, then the centroid of that triangle is at the point $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

We can think of this formula as defining a "centroid map" $T\colon \mathbb{R}^6 \to \mathbb{R}^2$; let $T\begin{pmatrix} u \\ v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}$, where (s, t) is the centroid of the triangle whose vertices (u, v), (w, x), (y, z). This T is a linear transformation.

a) Write down the matrix that represents T.

b) What is the rank of T?

Answer.

T is represented by $\begin{pmatrix} 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \end{pmatrix}$.

The rank is 2.

Problem 5 (10 points):

Alice and Bob are studying a two dimensional subspace of $\mathbb{R}^{100}$ relative to bases $\mathcal{A}$ and $\mathcal{B}$. They are particularly interested in three vectors $\vec{x}\,, \vec{y}\,, \vec{z}\,$.

In Alice's coordinates, we have $[\vec{x}]_{\mathcal{A}} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$, $[\vec{y}]_{\mathcal{A}} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $[\vec{z}]_{\mathcal{A}} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$.

In Bob's coordinates, we have $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$, $[\vec{y}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

How is $\vec{z}$ written in $\mathcal{B}$-coordinates?

Answer.

Note $\begin{pmatrix} 1 \\ 11 \end{pmatrix} = 3\begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 8 \\ 4 \end{pmatrix}$. So $\vec{z} = 3\vec{y} - \vec{x}$, and that will be true in any coordinate system. $[\vec{z}]_{\mathcal{B}} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

(This problem can alternately be done by computing a change-of-basis matrix.

Problem 6 (15 points):

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 3 & 1 \\ 2 & 0 & 2 & 0 & 4 \\ 1 & 0 & 1 & 0 & 7 \\ 2 & 1 & 1 & 3 & 2 \end{pmatrix}$$

and let T be the transformation represented by A.

a) Write down a basis for the kernel of T.

b) Write down a basis for the image of T.

c) What is the *rank* (the dimension of the image) of A?

d) What is the *nullity* (the dimension of the kernel) of A?

Answer.

First, we row reduce the matrix to find rref $A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

A basis of ker T is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

A basis of im T is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 7 \\ 2 \end{pmatrix} \right\}$.

The rank is 3.

The nullity is 2.

Problem 7 (15 points):

Consider the following matrices.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

a) Compute $C = AB$.

b) Compute A^{-1} or explain why A is not invertible.

c) Compute B^{-1} or explain why B is not invertible.

d) Compute C^{-1} or explain why C is not invertible. *(Hint: there is more than one way to do this part.)*

Answer.

$C = \begin{pmatrix} 4 & -1 & 3 \\ 1 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix}$. $A^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$. $B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}$.

$C^{-1} = B^{-1}A^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ -3 & 7 & -5 \\ -2 & 5 & -3 \end{pmatrix}$.

Problem 8 (20 points):

Indicate whether each item is true or false. Each item is worth 2 points. No explanation necessary, no partial credit possible.

a) If A and B are invertible matrices and $AB = BA$, then also $A^{-1}B^{-1} = B^{-1}A^{-1}$.

TRUE

b) If $\vec{v}$ and $\vec{w}$ are in the kernel of a linear transformation T, then $\vec{v} + \vec{w}$ is also in the kernel of T.

TRUE

c) If two matrices have the same rref, then they have the same kernel.

TRUE

d) If V and W are 2-dimensional subspaces of $\mathbb{R}^3$ and $V \neq W$, then the set of vectors in both V and W is a 1-dimensional subspace of $\mathbb{R}^3$.

TRUE

e) If a linear transformation is invertible, then the inverse is linear.

TRUE

f) There exists a system of three equations in four variables what has a unique solution.

FALSE

g) There exists a system of three equations in four variables which has no solutions.

TRUE

h) The kernel of $\begin{pmatrix} 2 & 1 & 0 & 2 \\ 1 & 0 & 0 & 3 \\ 7 & 3 & 0 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 \end{pmatrix}$ is a subspace of $\mathbb{R}^4$.

TRUE

i) If $\vec{u}, \vec{v}, \vec{w}, \vec{x}$ are linearly independent vectors in $\mathbb{R}^n$, then $n \geq 4$.

TRUE

j) If $\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_k\}$ and $\left\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \ldots, \vec{b}_\ell\right\}$ are bases of the same vector space, then $k = \ell$.

TRUE