Math 214 — Final Exam

Blue Version

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Directions: You have 110 minutes to complete this exam. You may not use a calculator, computer, or other electronic device. There are nine problems which are worth 15 points each. Partial credit is possible. For full credit, clear and relevant work must be shown. You choose *eight* of the nine problems to complete. Indicate which problem you are skipping by marking the appropriate box in the upper-right hand corner of that page. If you do not mark any skip box, or if you mark more than one skip box, I will choose a problem to be skipped at random. This is not what you want. There are also ten true/false questions. Each is worth 2 points, just as on the midterms. This exam is worth 140 points, representing 35% of the 400 total points possible in the course.

Problem 1.

Consider the matrix $A=\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.

a) Compute the characteristic polynomial of A.

b) Compute the eigenvalues of A.

c) Compute an invertible matrix S and a diagonal matrix D such that $S^{-1}AS=D$.

Problem 2.

Consider the plane in $\mathbb{R}^3$ spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$.

a) Compute an orthonormal basis of this plane.

b) Compute the matrix representation of the projection onto this plane.

Problem 3.

Solve the following linear system by whatever method seems best to you.

$$\begin{aligned} w+x+y &= 8 \\ w+x+z &= 5 \\ w+y+z &= 19 \\ x+y+z &= 10 \end{aligned}$$

Problem 4.

Consider the matrix $A=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

a) Find an orthogonal matrix S such that S^TAS is diagonal.

b) Is the quadratic form $Q(x, y, z) = 2xy + 2yz + 2zx$ definite, semidefinite, or indefinite?

Problem 5.

Find a 2×2 matrix A such that $A^3 = \begin{pmatrix} -10 & -18 \\ 9 & 17 \end{pmatrix}$.

Problem 6.

Consider the map $T: P_4 \to P_4$ defined by $T(f(x)) = x f'(x)$.

a) Find an eigenbasis of T, and indicate the eigenvalue attached to each eigenpolynomial in the basis.

b) Compute $\det T$.

Problem 7.

Consider the parallelepiped with the following vertices.

$$A(0,0,0), B(2,3,5), C(1,0,-1), D(3,3,4), E(2,-1,-1), F(4,2,4), G(3,-1,-2), H(5,2,3)$$

a) Compute the length AB.

b) The four points A, B, C, D are the vertices of a parallelogram. What is its area?

c) What is the volume of $ABCDEFGH$? (Hint: the parallelepiped is "generated" by $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AE}$.)

Problem 8.

Consider the function $T: P_3 \to P_2$ defined by $T(f(x)) = f'(x) - (x+1) f''(x)$.

a) Compute the matrix representation of T, relative to the customary bases $\{x^3, x^2, x, 1\}$ for P_3 and $\{x^2, x, 1\}$ for P_2.

b) Compute a basis for $\ker T$.

c) Compute a basis for $\operatorname{im} T$.

d) Compute the rank of T.

e) Compute the nullity of T.

Problem 9.

Once again, Alice and Bob are studying one of those linear transformations $T: V \to V$. This time V is a plane in $\mathbb{R}^{42}$, but it is not known to me which plane precisely. Alice is using a basis $\mathcal{A}$, and Bob is using a different basis $\mathcal{B}$. They are especially interested in two vectors $\vec{v}$ and $\vec{w}$.

$$[\vec{v}]_{\mathcal{A}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}; [\vec{w}]_{\mathcal{A}} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}; [\vec{v}]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}; [\vec{w}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

a) Compute the change of basis matrix $S_{\mathcal{A} \to \mathcal{B}}$.

b) Compute the change of basis matrix $S_{\mathcal{B}\to\mathcal{A}}$.

c) In Bob's coordinates, T is represented by $\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$. What matrix represents T in Alice's coordinates?

d) Either Alice or Bob has the good sense to be using orthonormal coordinates. If we know that $\|\vec{v}\| < \|\vec{w}\|$, then what is the cosine of the angle between $\vec{v}$ and $\vec{w}$?

True or False?

Indicate whether the following statements are true or false. Each is worth two points. No explanation is necessary, no partial credit is possible.

a) If $T: \mathbb{R}^4 \to \mathbb{R}^5$ is a linear map and the image of T is a plane, the kernel of T is also a plane.

TRUE or **FALSE**

b) If the reduced row echelon form of a matrix A is the identity, then A is invertible.

TRUE or **FALSE**

c) For any two square matrices A, B of the same size, $AB = BA$.

TRUE or **FALSE**

d) The map $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y-2 \\ 3x-2y+1 \end{pmatrix}$ is a linear function (in the linear algebra sense).

TRUE or **FALSE**

e) The image of a 3×4 matrix is a subspace of $\mathbb{R}^4$.

TRUE or **FALSE**

f) If $\vec{x}$ is any vector in $\mathbb{R}^n$ and V is any subspace of $\mathbb{R}^n$, then $\|\mathrm{proj}_V\vec{x}\| \leq \|\vec{x}\|$

TRUE or **FALSE**

g) The determinant of any orthogonal matrix is 1.

TRUE or **FALSE**

h) If A is a 4×4 matrix, then $\det 2A = 8\det A$.

TRUE or **FALSE**

i) If $\vec{v}$ is an eigenvectore of A, then $\vec{v}$ is also an eigenvector of A^3.

TRUE or **FALSE**

j) Every symmetric matrix is diagonalizable.

TRUE or **FALSE**