Math 214 — Final Exam

Blue Version Answers

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Problem 1.

a) $\mathrm{ch}_A(\lambda) = (1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda)$

b) $1, 2, 3, 4$

c) Each eigenspace is evidently one-dimensional. The 1-eigenspace is spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. The 2-eigenspace is spanned by $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. The 3-eigenspace is spanned by $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. The 4-eigenspace is spanned by $\begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$.

We can take $S = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$; $D = S^{-1}AS = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.

Problem 2.

a) Gram-Schmidt. $\left\{ \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \right\}$

b) $\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} 5/6 & -1/6 & 1/3 \\ -1/6 & 5/6 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$

Problem 3.

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 9 \\ 6 \end{pmatrix}$$

Problem 4.

a) First $\mathrm{ch}_A(\lambda) = -\lambda^3 + 3\lambda + 2 = -(\lambda+1)(\lambda+1)(\lambda+2)$, so the eigenvalues are $-1, -1, 2$.

The (-1)-eigenspace is spanned by $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. The 2-eigenspace is spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Unfortunately neither of these basis is orthonormal, but Gram-Schmidt (on each space individually) gives us an orthonormal eigenbasis $\left\{ \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \right\}$.

Take $S = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$; $S^TAS = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

b) $Q(1,1,0) = 2 > 0$; $Q(1,-1,0) = -2$. This matrix is indefinite.

Problem 5.

First we diagonalize. The eigenvalues are 8 and -1, with eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, respectively. Thus $\begin{pmatrix} -10 & -18 \\ 9 & 17 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$.

We can take $A = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$.

(It turns out that in this case the answer we gave is the only one possible, provided we are confined to matrices of real numbers.)

Problem 6.

a) $T(ax^4 + bx^3 + cx^2 + dx + e) = 4ax^4 + 3bx^3 + 2cx^2 + dx$. In the usual coordinates, the matrix representation is $\begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Fortuitously, the usual basis $\{x^4, x^3, x^2, x, 1\}$ is an eigenbasis, with attached eigenvalues $4, 3, 2, 1, 0$, respectively.

b) The determinant is the product of the eigenvalues, $4 \cdot 3 \cdot 2 \cdot 1 \cdot 0 = 0$.

Problem 7.

a) $\sqrt{2^2 + 3^2 + 5^2} = \sqrt{38}$.

b) $\sqrt{\det\left(\begin{pmatrix} 2 & 3 & 5 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 5 & -1 \end{pmatrix}\right)} = \sqrt{\det\begin{pmatrix} 38 & -3 \\ -3 & 2 \end{pmatrix}} = \sqrt{67}$

c) $\left|\det\begin{pmatrix} 2 & 1 & 2 \\ 3 & 0 & -1 \\ 5 & -1 & -1 \end{pmatrix}\right| = 10$

Problem 8.

a) $\begin{pmatrix} -3 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \end{pmatrix}$

b) $\ker \tilde{T} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}\right\}$; $\ker T = \operatorname{span}\{1, -2x^2 + x\}$

c) $\operatorname{im} \tilde{T} = \operatorname{span}\left\{\begin{pmatrix} -3 \\ -6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$; $\operatorname{im} T = \operatorname{span}\{x^2 + 2x, 1\}$

d) The rank is 2.

e) The nullity is 2.

Problem 9.

a) $S_{\mathcal{A} \to \mathcal{B}} = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}$

b) $S_{\mathcal{B}\to\mathcal{A}} = \begin{pmatrix} -2 & 5 \\ 1 & -2 \end{pmatrix}$

c) $\begin{pmatrix} -2 & 5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -13 & -28 \\ 6 & 13 \end{pmatrix}$

d) If Bob were using orthonormal coordinates, then we would have $\|\vec{v}\| = \sqrt{20} > 1 = \|\vec{w}\|$, which goes against the given information.

It must be that Alice is using orthonormal coordinates, so that the cosine of the angle is $\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|} = \frac{-4}{2\sqrt{5}} = -\frac{2}{5}\sqrt{5}$.

True or False?

a) True

b) True

c) False

d) False

e) False

f) True

g) False

h) False

i) True

j) True