Name:____________________________

Math 214-001/002 W09
Exam 2

This is an 80-minute exam, but you have the full 110 minute class period to complete it. There are a total of 100 points possible; the value of each problem is marked. Partial credit may be given for significant progress toward the solution to a problem. Except where otherwise indicated, you must show sufficient work to make it clear to me what you are doing and why.

Problem 1: (10 points)

Compute the determinant of following matrix by whatever method seems best to you.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Problem 2: (10 points)

Consider the subspace of $\mathbb{R}^5$ spanned by $\begin{bmatrix} 2 \\ 2 \\ 3 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 0 \\ 4 \\ 2 \end{bmatrix}$. Compute an orthonormal basis of this space.

Problem 3: (10 points)

Consider the three-dimensional subspace V of $\mathbb{R}^4$ defined by $w + x + y + z = 0$, which has a basis $\mathcal{B} = \left\{ \vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$.

(a) The vector $\vec{v} = \begin{bmatrix} -3 \\ 1 \\ -2 \\ 4 \end{bmatrix}$ is in the plane V. Give the $\mathcal{B}$-coordinates of $\vec{v}$.

(b) The vector $\vec{w} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}$ is NOT in the space V. Compute $\text{proj}_V \vec{w}$.

Problem 4: (10 points)

Consider the following linear system in three variables x, y, z.

$$\begin{aligned} 3x + y + 2z &= 1 \\ 4x + 3y + \lambda z &= 0 \\ \lambda x + z &= 0 \end{aligned}$$

The coefficient λ varies, so we want to solve for x, y, z in terms of λ.

(a) This system has a unique solution, except for two values of λ. Which two values are these?

(b) Solve the system, assuming λ is not one of the values from the previous part (some or all your answers may involve λ).

Problem 5: (10 points)

Compute the least-squares solution to the following inconsistent system of equations. Please clearly indicate the normal equation.

$$\begin{aligned} x + y &= 1 \\ 2x + 8y &= -2 \\ x + 5y &= 3 \end{aligned}$$

Problem 6: (15 points)

Let V be the plane in $\mathbb{R}^4$ spanned by $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 6 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 7 \\ 4 \\ 12 \\ 6 \end{bmatrix}$. It turns out that the QR decomposition of $\begin{bmatrix} 2 & 7 \\ 3 & 4 \\ 6 & 12 \\ 0 & 6 \end{bmatrix}$ is

$$Q = \begin{bmatrix} 2/7 & 3/7 \\ 3/7 & -2/7 \\ 6/7 & 0 \\ 0 & 6/7 \end{bmatrix} \qquad R = \begin{bmatrix} 7 & 14 \\ 0 & 7 \end{bmatrix}.$$

(You don't need to compute this for yourself.) Let $\mathcal{A}$ be the basis $\vec{v}_1, \vec{v}_2$ and let $\mathcal{B}$ be the basis given by the columns of Q.

(a) Consider the vector $\vec{w}$, which has $\mathcal{A}$-coordinates $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Give the $\mathcal{B}$-coordinates of $\vec{w}$.

(b) Compute the length of $\vec{w}$.

(c) Compute the volume of the parallelepiped generated by $\vec{v}_1$ and $\vec{v}_2$.

Problem 7: (15 points)

Consider the map $T : U^{2\times 2} \to U^{2\times 2}$ defined by $T(M) = M \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} M$.

(a) Write the matrix representation of T, relative to the basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ of $U^{2\times 2}$.

(b) Write down a basis for the kernel of T.

(c) Write down a basis for the image of T.

(d) What is the rank of T?

(e) What is the nullity of T?

(f) What is the determinant of T?

Problem 8: (20 points)

Each item is worth 2 points. No explanation necessary, no partial credit possible.

(a) If A and B are similar matrices, then A and B have the same rank.

True **False**

(b) P_4, the space of all polynomial of degree at most 4, has a basis consisting entirely of fourth-degree polynomials.

True **False**

(c) A linear map $T : V \to W$ is an isomorphism if and only if it is represented by an invertible matrix (relative to some choice of coordinates).

True **False**

(d) If $T : V \to V$ is a linear map from a linear space to itself, $\ker T$ and $\operatorname{im} T$ have nothing in common except the zero element of V.

True **False**

(e) If V is a 3-dimensional subspace of $\mathbb{R}^5$, $V^\perp$ is a plane.

True **False**

(f) If $T : \mathbb{R}^n \to \mathbb{R}^n$ has the property that the angle between $T(\vec{v})$ and $T(\vec{w})$ is always the same as the angle between $\vec{v}$ and $\vec{w}$, then T is an orthogonal transformation.

True **False**

(g) If a matrix has orthonormal rows, it also has orthonormal columns.

True **False**

(h) If A, B are $n \times n$ matrices, $\det A = 7$ and $\det B = 3$, then $\det(AB) = 21$.

True **False**

(i) If A is invertible, A^T is also invertible.

True **False**

(j) If A, B are $n \times n$ matrices, $\det A = 7$ and $\det B = 3$, then $\det(A+B) = 10$.

True **False**