Midterm 2—Answer Key

(typed with great haste, in between answering your numerous emails, by your friendly neighborhood Dr. Cap)

1. Use row operations. Subtract the third row from the fourth, then the second row from the third, and then the first row from the second. The result is $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, which has determinant 1.

2. Use Gram-Schmidt.

   $\vec{u}_1 = \frac{1}{5}\begin{pmatrix} 2 \\ 2 \\ 3 \\ 2 \\ 2 \end{pmatrix}$, $\vec{u}_2 = \frac{1}{\sqrt{6650}}\begin{pmatrix} 2 \\ 27 \\ -72 \\ 27 \\ 2 \end{pmatrix}$

3. The key to this whole problem is the fact that I provided you with an *orthonormal* basis. The maneuvers I am doing would not work properly otherwise.

   a) $\vec{v} \cdot \vec{u}_1 = -3/2 + 1/2 + 1 - 2 = -2$

      $\vec{v} \cdot \vec{u}_2 = -3/2 - 1/2 - 1 - 2 = -5$

      $\vec{v} \cdot \vec{u}_3 = -3/2 - 1/2 + 1 + 2 = 1$

      The coordinates are $\begin{pmatrix} -2 \\ -5 \\ 1 \end{pmatrix}$.

   b) $\text{proj}_V \vec{w} = (\vec{w} \cdot \vec{u}_1)\vec{u}_1 + (\vec{w} \cdot \vec{u}_2)\vec{u}_2 + (\vec{w} \cdot \vec{u}_3)\vec{u}_3 = -3\vec{u}_1 + 1\vec{u}_2 + 2\vec{u}_3 = \begin{pmatrix} 0 \\ -3 \\ 1 \\ 2 \end{pmatrix}$.

4.

   a) Our system can be expressed as $\begin{pmatrix} 3 & 1 & 2 \\ 4 & 3 & \lambda \\ \lambda & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. The determinant of the coefficient matrix is $\lambda^2 - 6\lambda + 5$. Thus the system will have a unique solution, except when $\lambda = 1$ or $\lambda = 5$.

   b) Use Cramer's Rule. $x = \frac{3}{\lambda^2 - 6\lambda + 5}$, $y = \frac{\lambda^2 - 4}{\lambda^2 - 6\lambda + 5}$, $z = \frac{-3\lambda}{\lambda^2 - 6\lambda + 5}$

5. Our initial system is $\begin{pmatrix} 1 & 1 \\ 2 & 8 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$.

   The normal system is $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 8 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 8 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 8 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, which evaluates to $\begin{pmatrix} 6 & 22 \\ 22 & 90 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Evidently a solution to this equation is $x = 0, y = 0$.

6.

   a) R is the change of basis matrix from $\mathcal{A}$ to $\mathcal{B}$, so, in $\mathcal{B}$-coordinates,
      $\vec{w} = \begin{pmatrix} 7 & 14 \\ 0 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 56 \\ 21 \end{pmatrix}$.

   b) We can compute lengths with orthonormal coordinates. $\|\vec{w}\| = \sqrt{56^2 + 21^2}$.

   c) There are various ways to do this computation. The shortest is to remember that the volume of such a parallelipiped is the product of the diagonal entries in R. The volume is 49.

7.

a) $T\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 3a & 3a+4b+c \\ 0 & 4c \end{pmatrix}$.

T is represented by the matrix $\begin{pmatrix} 3 & 0 & 0 \\ 3 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$.

b) The matrix of T is invertible, so the kernel is zero.

c) The image is all of $U^{2\times 2}$, so any basis will do, such as $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

d) The rank is 3.

e) The nullity is 0.

f) $\det T = 48$.

8.

a) True

b) True

c) True

d) False

e) True

f) False

g) False (would be true if I had also said the matrix was square)

h) True

i) True

j) False