Math 214 — Midterm 1

Blue Version Answers

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Problem 1:

Let d, s, m be the numbers of sacks of Dragon, Serenity, and Miracle Tea that Datura buys, respectively.

$$\begin{aligned} d+s+m&=9\\ 2d+3s+5m&=25\\ 25d+30s+20m&=230 \end{aligned}$$

This system has a unique solution $(d, s, m) = (6, 1, 2)$. Datura bought six sacks of Dragon Tea, one sack of Serenity Tea, and two sacks of Miracle Tea.

Problem 2:

We use the projection formula. Let L be the line spanned by $\begin{pmatrix} 3\\ 4\\ 12 \end{pmatrix}$.

$$\mathrm{proj}_L\begin{pmatrix} -1\\ 1\\ 1 \end{pmatrix} = \frac{\begin{pmatrix} -1\\ 1\\ 1 \end{pmatrix}\cdot\begin{pmatrix} 3\\ 4\\ 12 \end{pmatrix}}{\begin{pmatrix} 3\\ 4\\ 12 \end{pmatrix}\cdot\begin{pmatrix} 3\\ 4\\ 12 \end{pmatrix}}\begin{pmatrix} 3\\ 4\\ 12 \end{pmatrix} = \frac{13}{169}\begin{pmatrix} 3\\ 4\\ 12 \end{pmatrix} = \begin{pmatrix} 3/13\\ 4/13\\ 12/13 \end{pmatrix}$$

Problem 3:

There are infinitely many solutions. What all solutions have in common is that each column consists of cnumbers in arithmetic progression and that the five columns are not all a multiple of a common vector. An example follows.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1\\ 1 & 2 & 3 & 4 & 5\\ 1 & 3 & 5 & 7 & 9\\ 1 & 4 & 7 & 10 & 13 \end{pmatrix}$$

Problem 4:

Since T is compatible with scalar multiples,

$$T\begin{pmatrix} 1\\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 21\\ 15\\ 12 \end{pmatrix} = \begin{pmatrix} 7\\ 5\\ 4 \end{pmatrix} \qquad T\begin{pmatrix} 0\\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 10\\ 12\\ 14 \end{pmatrix} = \begin{pmatrix} 5\\ 6\\ 7 \end{pmatrix}.$$

This gives the matrix representing T, column-by-column: $\begin{pmatrix} 7 & 5\\ 5 & 6\\ -4 & 7 \end{pmatrix}$.

Problem 5:

So many possibilities...

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{B}_3 = \left\{ \begin{pmatrix} 9 \\ 1 \\ \sqrt[3]{42} \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{19} \\ 0 \end{pmatrix}, \begin{pmatrix} \pi \\ 6 \\ 1 \end{pmatrix} \right\}$$

Problem 6:

First, we need the rref of A. $\text{rref}(A) = \begin{pmatrix} 1 & 0 & -2 & 5 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

a) The solutions of $A\vec{x} = \vec{0}$ have the form $\begin{pmatrix} 2r-5s \\ -r+s \\ r \\ s \\ 0 \end{pmatrix}$; a basis for $\ker A$ is $\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

b) The third and fourth columns are redundant. A basis of im A is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$. (Indeed the image is all of $\mathbb{R}^3$, so any basis of $\mathbb{R}^3$ will do.)

c) The rank is 3.

d) The nullity is 2.

Problem 7:

a) $C = AB = \begin{pmatrix} -10 & 10 & 3 \\ 16 & -5 & 3 \\ -5 & 2 & 1 \end{pmatrix}$.

b) $A^{-1} = \begin{pmatrix} 1 & -4 & -15 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

c) $B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 7 & -2 & 1 \end{pmatrix}$.

d) $C^{-1} = B^{-1}A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 7 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -4 & -15 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -4 & -15 \\ -1 & 5 & 18 \\ 7 & -30 & -110 \end{pmatrix}$

Problem 8:

a) False b) True c) True d) False e) True f) True g) True h) True i) False j) True