Math 214 — Final Exam

Paper Version

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Directions: You have 110 minutes to complete this exam. You may not use a calculator, computer, or other electronic device. You may use two 3×5 notecards (front and back) but no other reference materials of any kind. There are nine problems which are worth 15 points each. Partial credit is possible. For full credit, clear and relevant work must be shown. You choose *eight* of the nine problems to complete. Indicate which problem you are skipping by marking the appropriate box in the upper-right hand corner of that page. If you do not mark any skip box, or if you mark more than one skip box, I will choose a problem to be skipped at random. This is not what you want. There are also ten true/false questions. Each is worth 2 points, just as on the midterms. This exam is worth 140 points, representing 35% of the 400 total points possible in the course.

Problem 1

Consider the matrix $A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & 0 \\ 0 & 2 & 2 \end{pmatrix}$.

a) Compute the characteristic polynomial of A.

b) Compute the eigenvalues of A.

c) Write down an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

Problem 2

Write down a 3×3 matrix which is *not* similar to any diagonal matrix. (A complete answer will include an explanation of why the matrix is not similar to a diagonal matrix.)

Problem 3

Compute the inverse of the matrix $\begin{pmatrix} 2 & 1 & 1+\mu \\ 5 & 3 & 1 \\ 3 & 2 & 1-\mu \end{pmatrix}$. (Obviously, your answer will involve μ.)

Problem 4

Let V be the subspace of $\mathbb{R}^4$ spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Give a basis for $V^{\perp}$.

Problem 5

a) Consider the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. Is there an orthogonal matrix S such that $S^T A S$ is diagonal? If so, write down a diagonal matrix which is orthogonally similiar to A. (Note that I am not asking you to compute S.)

b) Is the form $Q(x,y) = 2x^2 + 3y^2 + 2z^2 + 2xy + 2yz$ positive definite, semidefinite, or indefinite?

Problem 6

Solve the following system by whatever method seems best to you.

$$\begin{aligned} w + 2x + 3y + 4z &= 20 \\ w + x + 2y + 3z &= 14 \\ w + x + y + 2z &= 11 \\ w + x + y + z &= 10 \end{aligned}$$

Problem 7

Consider the function $T\colon P_3 \to P_3$ defined by $T(f(x)) = x f'(x) - f(x-1)$.

a) Compute the matrix representation of T, relative to the customary basis $\{x^3, x^2, x, 1\}$.

b) Compute a basis for $\ker T$.

c) Compute a basis for $\operatorname{im} T$.

d) Compute the rank of T.

e) Compute the nullity of T.

Problem 8

Compute a formula for A^t, where A is the matrix $\left(\begin{smallmatrix} 1 & 2 \\ 4 & 3 \end{smallmatrix}\right)$. (Your answer should give each entry of the matrix in terms of t.)

Problem 9

Meredith is studying a three-dimensional subspace V of $\mathbb{R}^5$; she has a basis $\mathcal{B}$ for V, and for convenience she usually writes vectors in their $\mathcal{B}$-coordinates.

For example, the vector $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{pmatrix}$ has coordinates $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$, and the vector $\vec{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ has coordinates $[\vec{y}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. What are the $\mathcal{B}$-coordinates of the vector $\begin{pmatrix} 2 \\ 4 \\ 5 \\ 3 \\ 1 \end{pmatrix}$?

True or False?

a) If $A = SDS^T$, where D is diagonal and S is orthogonal, then A is symmetric.

TRUE or **FALSE**

b) If the columns of A are linearly independent, then the system $A\vec{x} = \vec{b}$ has a unique least-squares solution.

TRUE or **FALSE**

c) If A is a 3×3 matrix with eigenvalues $2, 7, 13$, then $\det A = 22$.

TRUE or **FALSE**

d) If A is an invertible matrix, then A^T is also invertible.

TRUE or **FALSE**

e) The solutions of the system $A\vec{x} = \vec{0}$ are precisely the elements of $\ker A$.

TRUE or **FALSE**

f) If A is a square matrix, then A and A^T have the same eigenvalues.

TRUE or **FALSE**

g) Every 5×5 matrix has at least one (real) eigenvalue.

TRUE or **FALSE**

h) The characteristic polynomial of $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ is $\lambda^2 - 7\lambda - 2$.

TRUE or **FALSE**

i) Every subspace of $\mathbb{R}^5$ has an orthonormal basis.

TRUE or **FALSE**

j) The matrix $\begin{pmatrix} 12/13 & -5/13 \\ 5/13 & 12/13 \end{pmatrix}$ represents a rotation of $\mathbb{R}^2$ (in the standard coordinates).

TRUE or **FALSE**