Math 214 — Final Exam

Paper Version Answers

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Problem 1.

a) $\mathrm{ch}_A(\lambda) = (1-\lambda)(2-\lambda)(3-\lambda) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$

b) $1, 2, 3$

c) Each eigenspace is evidently one-dimensional. The 1-eigenspace is spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. The 2-eigenspace is spanned by $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. The 3-eigenspace is spanned by $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

We can take $S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$; $D = S^{-1}AS = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Problem 2.

An example is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. The only eigenvalue is 1, but the 1-eigenspace is the line spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. There aren't "enough" eigenvectors to form a basis.

Problem 3.

$$\begin{pmatrix} 1-3\mu & 1+3\mu & -2-3\mu \\ -2+5\mu & -1-5\mu & 3+5\mu \\ 1 & -1 & 1 \end{pmatrix}$$

Problem 4.

Lower-Tech Solution. $V^\perp = \left\{ \vec{x} \in \mathbb{R}^4 : \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \vec{x} = 0 \right\}$. Thus we are looking for the solutions of the system.

$$x_1 + x_2 + x_3 = 0$$
$$x_2 + x_3 + x_4 = 0$$

This is a totally standard system of two equations in four variables, and we can solve in the Chapter 1 way by Gauss-Jordan elimination to get a basis for $V^\perp$: $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$.

Higher-Tech Solution. Extend $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ to a basis of $\mathbb{R}^4$ in an arbitrary fashion, e.g. $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. Apply Gram-Schmidt to convert this into an orthonormal basis $\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{15}} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \frac{1}{\sqrt{15}} \begin{pmatrix} -1 \\ 3 \\ -2 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$. The first two vectors in this list are a basis of V, and the latter two are a basis of $V^\perp$. We don't really need the scaling factors, so we may as well take the basis $\begin{pmatrix} -1 \\ 3 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$.

Problem 5.

a) There is such an S, because all symmetric matrices are orthogonally diagonalizable. The characteristic polynomial is $(4-\lambda)(2-\lambda)(1-\lambda)$, so the eigenvalues are 1, 2, 4. This matrix is orthogonally similar to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

b) This Q is represented by the matrix in the previous part, so it is equivalent to the form $u^2+2v^2+4w^2$. Positive-definite.

Problem 6.

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

Problem 7.

a) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix}$

b) $\ker \tilde{T} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$; $\ker T = \operatorname{span}\{x+1\}$

c) $\operatorname{im} \tilde{T} = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 3 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}$; $\operatorname{im} T = \operatorname{span}\{2x^3+3x^2-3x+1, x^2+2x-1, -1\}$

d) The rank is 3.

e) The nullity is 1.

Problem 8.

$$\frac{1}{3}\begin{pmatrix} 2(-1)^t + 5^t & (-1)^{t+1} + 5^t \\ -2(-1)^t + 2\cdot 5^t & (-1)^t + 2\cdot 5^t \end{pmatrix}$$

Problem 9.

$\begin{pmatrix} 2 \\ 4 \\ 5 \\ 3 \\ 1 \end{pmatrix} = 2\vec{x} - \vec{y}$, so the $\mathcal{B}$-coordinates of the vector $\begin{pmatrix} 2 \\ 4 \\ 5 \\ 3 \\ 1 \end{pmatrix}$ are $2\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix}$.

True or False?

a) True

b) True

c) False

d) True

e) True

f) True

g) True

h) True

i) True

j) True