

Expander Graphs and their Applications  
Draft - not for distribution

Shlomo Hoory  
University of British Columbia  
Vancouver, BC  
shlomoh@cs.ubc.ca

Nathan Linial  
The Hebrew University  
Jerusalem, Israel  
nati@cs.huji.ac.il

Avi Wigderson  
Institute of Advanced Study  
Princeton, NJ  
avi@ias.edu

January 1, 2006



# An Overview

A major consideration we had in writing this survey was to make it accessible to mathematicians as well as computer scientists, since *expander graphs*, the protagonists of our story come up in numerous and often surprising contexts in both fields. A glossary of some basic terms and facts from computer science can be found at the end of this article.

But, perhaps, we should start with a few words about graphs in general. They are, of course, one of the prime objects of study in Discrete Mathematics. However, graphs are among the most ubiquitous models of both natural and human-made structures. In the natural and social sciences they model relations among species, societies, companies, etc. In computer science, they represent networks of communication, data organization, computational devices as well as the flow of computation, and more. In Mathematics, Cayley graphs are useful in Group Theory. Graphs carry a natural metric and are therefore useful in Geometry, and though they are “just” one-dimensional complexes they are useful in certain parts of Topology, e.g. Knot Theory. In statistical physics, graphs can represent local connections between interacting parts of a system, as well as the dynamics of a physical process on such systems.

The study of these models calls, then, for the comprehension of the significant structural properties of the relevant graphs. But are there nontrivial structural properties which are *universally* important? The property of being an *expander* seems significant in many of these mathematical, computational and physical contexts. It is not surprising that expanders are useful in the design and analysis of communication networks. What is less obvious is that expanders have surprising utility in other computations settings such as in the theory of error correcting codes, and the theory of pseudorandomness. In mathematics, we will encounter e.g., their role in the study of metric embeddings, and in particular in work around the Baum-Connes Conjecture [Val03]. Expansion is closely related to the convergence rates of Markov Chains, and so they play a key role in the study of Monte-Carlo algorithms in statistical mechanics and in a host of practical computational applications. The list of such interesting and fruitful connections goes on and on with so many applications we will not even be able to mention. This universality of expanders is becoming more evident as more connections are discovered. It transpires that expansion is a fundamental mathematical concept, well deserving to be thoroughly investigated on its own.

At least with a hindsight, one reason that expanders are so ubiquitous, is that their very definition can be given in at least three languages: combinatorial/geometric, probabilistic and algebraic. Combinatorially, expanders are graphs which are highly connected; to disconnect a large part of the graph, one has to sever many edges. Equivalently, using the geometric notion of isoperimetry, every set of vertices has a (relatively) very large boundary. From the probabilistic viewpoint, one considers the natural random walk on a graph, in which we have a token on a vertex, that moves at every step to a random neighboring vertex, chosen uniformly and independently. Expanders are graphs for which this process converges to its limiting distribution as rapidly as possible. Algebraically, one can consider the Laplace operator on the graph, and its spectrum. From this perspective, expanders are graphs in which the first positive eigenvalue is bounded away from zero.

The study of expanders leads in different directions. There are structural problems - what are the best bounds on the various expansion parameters, and how do they relate to each other and to other graph invariants. There are problems concerning explicit constructions - How to efficiently generate expanders with given parameters. These are extremely important for applications. There are algorithmic problems - given a graph, test if it is an expander with given parameters. Finally, understanding the relation of expansion with other mathematical notions, and the application of expanders to practical and theoretical problems.

In the past four decades, a great amount of research has been done on these topics, resulting in a wide-ranging body of knowledge. In this survey, we could not hope to cover even a fraction of it. We have tried to make our presentation as broad as possible, touching on the various research directions mentioned above. Even what we do cover is of course

incomplete, and we try to give the relevant references for more comprehensive coverage. We have also tried to mention in each chapter related research which we are not covering at all, and reference some of this as well.

The selection of material naturally reflects our interests and biases. It is rather diverse, and can be read in different orders, according to the reader's taste and interests.

This article evolved from lecture notes of a course on expanders taught at the Hebrew University, Israel in 2003 by Nati Linial and Avi Wigderson. We are greatly indebted to the scribes of the course notes: Ran Gilad-Bachrach, Danny Harnik, Boaz Barak, Udi Wieder, Eran Ofek, Erez Waisbard, Yael Vinner-Dekel, Yishai Beeri, David Statter, Eyal Bigman, Tamir Hazan, Elon Portugaly, Ariel Elbaz, Yuval Filmus, Michal Igell, Eyal Rozenman, Danny Gutfreund, and Yonatan Bilu. Also, we acknowledge that the proof that Margulis construction is an expander, is taken (with slight changes) from course notes of Ravi Boppana, with Mukesh Dalal as scribe.

We are also grateful for the careful reading of this manuscript by Mark Goresky and Eyal Rozenaman.

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# Chapter 1

## The Magical Mystery Tour

We begin our discussion with three fundamental problems from three different domains. At first sight these problems seem to have very little to do with expander graphs, or even graph theory, but as we shall see, they can all be solved using expander graphs.

### 1.1 Three Motivating Problems

#### 1.1.1 Hardness results for linear transformations

The **P vs. NP** problem is arguably the most important open problem in theoretical computer science. Despite its great significance and despite intensive research efforts, very little progress has been made. But interesting aspects of computational complexity can be investigated in other, more restricted contexts. For example, we may consider evaluating polynomials over a field using only the field's arithmetic, or even evaluating linear transformations using only addition and multiplication by scalars from the field. Valiant [Val76] considered the following natural problem:

**Problem 1.1.** *Let  $A$  be an  $n \times n$  matrix over the field  $\mathcal{F}$ . What is the least number of gates in a circuit that computes the linear transformation  $x \mapsto Ax$ ? Each gate is specified by two field elements  $a$  and  $b$ . Such a gate receives two inputs  $x$  and  $y$  and outputs  $ax + by$ .*

Aside from its profound theoretical importance, certain instances of this question have far-reaching technological significance. Consider the matrix  $a_{r,s} = \omega^{rs}$  ( $n - 1 \geq r, s \geq 0$ ), where  $\omega$  is a primitive  $n$ -th root of unity. The transformation  $x \mapsto Ax$  is the Discrete Fourier Transform which is fundamental to many modern technologies involving signal processing, machine learning, etc. As observed by Cooley and Tukey [CT65], there is circuit realizing this linear transformation (the so-called Fast Fourier Transform (FFT)) with only  $O(n \log n)$  gates. Therefore the least number of gates in such a circuit is between  $O(n \log n)$  and  $n$  (which are required just to input the vector  $x$ ). This may seem like a small gap in our knowledge, but it is rather significant. The technological implications of a Very Fast Fourier Transform, i.e an  $O(n)$ -sized circuit that computes the transform (should one exist) are hard to overestimate. On the other hand, it would be a great theoretical breakthrough to establish a matching lower bound of  $\Omega(n \log n)$ , or even rule out the existence of such a circuit with only  $O(n)$  gates.

For every field  $\mathcal{F}$ , it is fairly easy to show that for most  $n \times n$  matrices  $A$ , every circuit realizing  $A$  must have  $\Omega(n^2 / \log n)$  gates. This is based on a counting argument that compares the number of circuits with a given number of gates and the number of  $n \times n$  matrices over the field. As is often the case in computational complexity, despite this abundance of computationally hard functions, we are unable to exhibit any *specific, explicit* linear transformations  $A$  that requires asymptotically more than  $O(n)$  gates. In an attempt to solve this problem, Valiant [Val76] conjectured that super regular transformation are “hard” in this sense.

**Definition 1.2 (Super Regular Matrix).** A matrix  $A$  is *Super Regular* if every square sub-matrix of  $A$  has full rank.

---

<sup>1</sup>This problem is interesting for finite as well as for infinite fields.

Valiant considered the graph layout of a circuit which realizes the linear transformation corresponding to a super regular matrix. His main observation was that this graph must be a *Super Concentrator*:

**Definition 1.3 (Super Concentrator).** Let  $G = (V, E)$  be a graph and let  $I$ , and  $O$  be two subsets of  $V$  with  $n$  vertices each called the input and output sets respectively. We say that  $G$  is a *Super Concentrator* if for every  $k$  and every  $S \subseteq I$  and  $T \subseteq O$  with  $|S| = |T| = k$ , there exist  $k$  vertex disjoint paths in  $G$  from  $S$  to  $T$ .

It is a simple exercise to show that indeed the underlying graph of any circuit for a super regular matrix is a super concentrator (with inputs and outputs retaining their meaning in both). Valiant conjectured that any super concentrator must have  $\gg n$  edges. That would have implied that any circuit which computes a super regular matrix must have  $\gg n$  gates. However, Valiant himself disproved the conjecture and presented super concentrators with  $O(n)$  edges. As you might have guessed, this is where expanders come into the picture.

We note that this construction can actually be used to give a super regular matrix which has a linear sized circuit, which seems to put this approach to rest. This is not quite so, and Valiant's ideas were later realized, as follows: If we consider circuits with more than two inputs per gate, but where the circuit's depth is restricted, then super-linear lower bounds for depth-limited super concentrators were proven [DDPW83]. Subsequently the desired superlinear lower bounds for this restricted computational model were derived [Lok01, RS03].

Even though this approach did not yield strong lower bounds on circuit sizes, these attempts have brought forward the importance of sparse super concentrators in network theory and other areas. Valiant's idea has eventually had a major impact on the field.

We now skip to a totally different problem.

### 1.1.2 Construction of good error correcting codes

One of the most fundamental problems in communication is noise. Suppose that Alice has a message of  $k$  bits which she would like to deliver to Bob over some (noisy) communication channel. The problem is that noise in the channel may corrupt the message so that Bob receives a message that differs from the one sent by Alice.

In his groundbreaking paper "A Mathematical Theory of Communication" [Sha48], Claude Shannon laid the foundations for Information Theory and the mathematical theory of communication. The problem of communicating over a noisy channel occupies a central part of this theory.

**Problem 1.4 (communication over noisy channel).** *Alice and Bob communicate over a noisy channel. Up to a fraction of  $p$  of the bits sent through the channel may be altered. What is the smallest number of bits that Alice can send, assuming she wants to communicate an arbitrary  $k$ -bit message, so that Bob should be able to unambiguously recover the original message?*

To solve this problem, Shannon suggested creating a dictionary (or code)  $C \subseteq \{0, 1\}^n$  of size  $|C| = 2^k$ , and using a bijective mapping ("an encoding")  $\varphi : \{0, 1\}^k \rightarrow C$ . To send a message  $x \in \{0, 1\}^k$ , Alice transmits the  $n$ -bit encoded message  $\varphi(x) \in C$ . It is assumed that Bob receives a string  $y \in \{0, 1\}^n$  that is a corrupted version of the actually sent message  $\varphi(x) \in C$ . Bob finds the codeword  $z \in C$  that is closest to  $y$  (the metric used is the Hamming distance -  $d_H(u, v)$  is the number of coordinates  $i$  where  $u_i \neq v_i$ ). He concludes that the message actually sent was  $\varphi^{-1}(z)$ . If the distance between every two words in  $C$  is greater than  $2pn$ , it is guaranteed that indeed  $z = \varphi(x)$ , and Bob correctly infers the message sent by Alice.

The problem of communicating over noisy channel is thus reduced to the problem of finding a good dictionary. Namely, a set  $C$  of  $n$ -bit strings of largest possible cardinality subject to the condition that every two strings in  $C$  are at a large Hamming distance.

**Definition 1.5 (the rate and distance of a dictionary).** Let  $C \subseteq \{0, 1\}^n$  be a dictionary. Its **rate** and (normalized) **distance** are defined by:

$$R = \frac{\log |C|}{n},$$

$$\delta = \frac{\min_{c_1 \neq c_2 \in C} d_H(c_1, c_2)}{n}.$$

As we saw before, the distance of a dictionary controls its power to overcome noise. A code's rate measures its efficiency in channel utilization. At this point we can refine our problem and ask:

**Problem 1.6 (refined communication problem).** *Is it possible to design arbitrarily large dictionaries  $\{C_k\}$  of size  $|C_k| = 2^k$ , with  $R(C_k) \geq R_0$ , and  $\delta(C_k) \geq \delta_0$  for some absolute constants  $R_0, \delta_0 > 0$ ? Moreover, can we make these codes explicit, and efficiently encodable and decodable?*

This problem and its relatives (optimizing the code's parameters, and the algorithms' efficiency, in this and other error models and communication settings) is the subject of Coding Theory - a rich and active field initiated by Shannon's work (see e.g. [MS77a, MS77b] and Sudan's notes [Sud00] for complexity theoretic aspects of the field). It took over 20 years of research until even the basic Problem 1.6 was resolved, but below we present a simple solution to this problem using expander graphs. However, before we do that, let us present our third motivating problem.

### 1.1.3 Deterministic Error Amplification for RP

The field of probabilistic algorithms burst into existence within Theoretical Computer Science, with the fast primality tests of Rabin [Rab80] and of Solovay and Strassen [SS77]. Given a  $k$ -bit integer  $x$ , and a string  $r$  of  $k$  random bits, these algorithms efficiently compute a function  $f(x, r)$  with the following property. If  $x$  is prime then  $f(x, r) = 1$  for all choices of  $r$ . Otherwise, if  $x$  is composite,  $f(x, r) = 1$  with probability smaller than  $1/2$  over a randomly chosen  $r$ . If  $f = 1$  the algorithm declares  $x$  a prime, and otherwise declares it to be composite. It never fails on primes, and for every composite  $x$  its probability of failure is at most  $1/2$ ,

The error bound  $1/2$  may not be very satisfactory, and one would like to reduce it to some desired level. A very simple way to reduce our failure probability is to apply the same algorithm repeatedly with new randomly chosen  $r$ 's. Repeating it (say)  $d$  times, will reduce the probability of error to below  $2^{-d}$ . On the other hand, the running time, and the number of random bits used, increase by a factor of  $d$ . Is there a way to reduce the error "deterministically", without using more random bits, or at least using less than the obvious procedure above? We will see several answers to this question in these notes, and this chapter contains an initial advance on the problem. The importance of minimizing the number of random bits may not be evident, but we can assure the reader that it is a basic theoretical problem, and moreover, that getting your hands on good random bits is a nontrivial practical problem.

The above-mentioned primality testing algorithms belong to the class **RP** of Randomized Polynomial-Time algorithms. It is in this general setting that we discuss our problem. We say that a language  $\mathcal{L} \subseteq \{0, 1\}^*$  is in the class **RP**, if there exists a randomized algorithm  $A$  with a polynomial (in  $|x|$ ) running time, such that if  $x \in \mathcal{L}$  then  $A(x, r) = 1$  (with certainty), whereas if  $x \notin \mathcal{L}$ , the probability of  $A(x, r) = 1$  is smaller than  $1/16$  (The definition remains unchanged with any constant  $< 1$  that we choose. The constant  $1/16$  was chosen for notational convenience). Note again that  $r$  is a uniformly chosen random string of  $k$  bits, with  $k$  polynomially dependent on the length  $|x|$  of the input  $x$ . In this case we say that  $\mathcal{L} \subseteq \{0, 1\}^*$  has a (1-sided error) randomized polynomial time membership algorithm.

**Problem 1.7 (Saving Random Bits).** *Assume that  $\mathcal{L} \subseteq \{0, 1\}^*$  has a (1-sided error) randomized polynomial time membership algorithm. How many random bits are needed in order to reduce the probability of error to be  $\leq \epsilon$ ? (Note that we seek a bound that should apply to every input).*

## 1.2 Magical Graphs

In the previous section we presented three seemingly unrelated problems. We now introduce a new object - a "Magical Graph" that will enable us to solve all these problems. Its properties combine several notions of "expansion" (a "combinatorial isoperimetric inequality") to fit our three applications.

**Definition 1.8 (Magical Graph).** Let  $G = (L, R, E)$  be a bipartite graph. The vertex set consists of  $L$  and  $R$ , two disjoint subsets, henceforth the left and right vertex sets. We say that  $G$  is an  $(n, m; d)$ -**magical graph** if  $|L| = n$ ,  $|R| = m$ , and every left vertex has  $d$  neighbors and the following two properties hold (where  $\Gamma(S)$  denotes the set of neighbors of a set  $S$  in  $G$ ):

1.  $|\Gamma(S)| \geq \frac{5d}{8} \cdot |S|$  for every  $S \subseteq L$  with  $|S| \leq \frac{n}{10d}$ .

2.  $|\Gamma(S)| \geq |S|$  for every  $S \subseteq L$  with  $\frac{n}{10d} < |S| \leq \frac{n}{2}$ .

As observed by Pinsker [Pin73] (for other but related expansion properties), such graphs exist. The proof is by a probabilistic argument and it implies that, in fact, most graphs are magical.

**Lemma 1.9.** *There exists a constant  $n_0$  such that for every  $d \geq 32$  and  $n \geq n_0, m \geq 3n/4$  there exists an  $(n, m; d)$ -magical graph.*

*Proof.* Let  $G$  be a random bipartite graph with  $n$  vertices on the left, and  $m$  vertices on the right, where each left vertex connects to a randomly chosen set of  $d$  vertices on the right. We claim that with high probability  $G$  is a magical graph. We start by proving that the first property holds with high probability.

Let  $S \subseteq L$  have cardinality  $s = |S| \leq \frac{n}{10d}$ , and let  $T \subseteq R$  have cardinality  $t = |T| < \frac{5d}{8} \cdot |S|$ . Let  $X_{S,T}$  be an indicator random variable for the event that all the edges from  $S$  go to  $T$ . It is clear that if  $\sum X_{S,T} = 0$ , where the sum is over all choices of  $S$  and  $T$  as above, then the first property holds. The probability of the event  $X_{S,T}$  is  $(t/m)^{sd}$  and therefore using a union bound and the inequality  $\binom{n}{k} \leq (ne/k)^k$ , we get that:

$$\begin{aligned} \Pr[\sum_{S,T} X_{S,T} > 0] &\leq \sum_{S,T} \Pr[X_{S,T} = 1] = \sum_{S,T} (t/m)^{sd} \\ &\leq \sum_{s=1}^{n/10d} \binom{n}{s} \binom{m}{5ds/8} \left(\frac{5ds}{8m}\right)^{sd} \\ &\leq \sum_{s=1}^{n/10d} \left(\frac{ne}{s}\right)^s \left(\frac{8me}{5ds}\right)^{5ds/8} \cdot \left(\frac{5ds}{8m}\right)^{sd} < 1/10. \end{aligned}$$

The last inequality follows since the  $s$ -th term is bounded by  $20^{-s}$ .

Similarly, we bound the probability of violating the second property by an analogous expression, which is simpler to bound. For every  $S \subset L$  with cardinality  $\frac{n}{10d} < s = |S| \leq \frac{n}{2}$ , and  $T \subset R$  with  $t = |T| < |S|$ , let  $Y_{S,T}$  be an indicator random variable for the event that all the edges from  $S$  go to  $T$ . As before, we would like to prove that the probability of the event  $\sum Y_{S,T} = 0$  is small.

$$\begin{aligned} \Pr[\sum_{S,T} Y_{S,T} > 0] &\leq \sum_{S,T} \Pr[Y_{S,T} = 1] = \sum_{S,T} (t/n)^{sd} \leq \sum_{s=n/10d}^{n/2} \binom{n}{s} \binom{m}{s} (s/m)^{sd} \\ &\leq \sum_{s=1}^{n/2} [(ne/s) \cdot (me/s) \cdot (s/m)^d]^s < 1/10. \end{aligned}$$

As before, the last inequality follows, by noting that for all  $s$  the quantity in square brackets is bounded by  $10^{-4}$ . Therefore, most graphs are  $(n, m; d)$ -magical graphs.  $\square$

We now turn to solve the three problems presented above. We note that Lemma 1.9 is existential, whereas we need explicit constructions of magical graphs to resolve our three problems. The issue of explicit constructions is an important aspect of this field and of this article, but at present we show how to solve these problems using the existence of magic graphs as a "black box".

## 1.3 The Three Solutions

### 1.3.1 A Super Concentrator with $O(n)$ edges

We will see how magical graphs allow us to construct super concentrators. These graphs exhibit incredibly high connectivity despite the fact that they have only  $O(n)$  edges. There is a long and still ongoing search for super

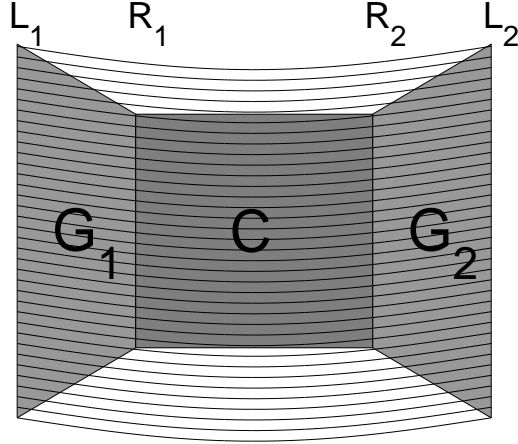


Figure 1.1: Constructing a super concentrator

concentrators with  $n$  input and output vertices and  $Kn$  edges with  $K$  as small as possible. This “sport” has motivated quite a few important advances in this area. The current “world record” holders are Alon and Capalbo [AC04].

If  $G$  is an  $(n, 3n/4; d)$ -magical graph, then  $|\Gamma(S)| \geq |S|$  for every  $S \subset L$  with  $|S| \leq \frac{n}{2}$ . By Hall’s marriage theorem (e.g., [Die97, Theorem 2.1.2]), for every  $S \subseteq L$  of size  $|S| \leq \frac{n}{2}$  there is a perfect matching from  $S$  to  $\Gamma(S)$ .

We use this fact to recursively construct a super concentrator  $C'$  with  $n$  vertices on each side. For  $n$  below  $n_0$ , we simply observe that a complete bipartite graph is a super concentrator with  $n^2$  edges.

For  $n \geq n_0$  we construct a super concentrator  $C'$  with  $n$  inputs and outputs, using three building blocks: (i) Two copies  $G_1 = (L_1, R_1, E_1)$  and  $G_2 = (L_2, R_2, E_2)$  of our magical graph, where  $|L_i| = n$ , and  $|R_i| = 3n/4$ . (ii) A super concentrator  $C$  connecting the input set  $R_1$  to the output set  $R_2$ . The input, output sets have size  $3n/4$  and therefore  $C$  exists by induction. (iii) A perfect matchings between  $L_1$  and  $L_2$ . The input and output sets of our graph are  $L_1$  and  $L_2$  respectively. This is illustrated in figure 1.1.

We need to prove that the graph we have constructed  $C'$  is indeed a super concentrator, and derive an upper bound on the number of its edges. Let  $S$  be set of input vertices and  $T$  a set of output vertices such that  $|S| = |T| = k$ .

If  $k \leq n/2$  then  $|\Gamma_{G_1}(S)| \geq |S|$  and  $|\Gamma_{G_2}(T)| \geq |T|$ , since  $G_1, G_2$  are magic graphs. Hence, by Hall’s theorem there exists a perfect matching from  $S$  to  $\Gamma_{G_1}(S)$  and from  $\Gamma_{G_2}(T)$  to  $T$ . Let  $S' \subseteq \Gamma_{G_1}(S)$  be the set of vertices matched to vertices in  $S$  and likewise for  $T'$  and  $T$ . Since  $C$  is a super concentrator, the sets  $S'$  and  $T'$  can be connected by  $k$  disjoint paths. Consequently,  $S$  and  $T$  can be connected by disjoint paths in  $C'$ .

If the two sets  $S$  and  $T$  are big, i.e.  $|S| = |T| = k > n/2$  then there must exist at least  $k - n/2$  vertices in  $S$  that are matched to vertices in  $T$  by direct matching edges of (iii) above. Therefore we can delete the matched vertices from  $S$  and  $T$ , and reduce the problem to the previous case of  $k \leq n/2$ . It follows that  $C'$  is a super concentrator.

We still need to provide an upper bound on the number of edges  $e(n)$  in our  $n$ -inputs graph  $C'$ . We obtain the following recursion:

$$e(n) \leq \begin{cases} 2nd + n + e(3n/4) & \text{for } n > n_0 \\ n^2 & \text{for } n \leq n_0 \end{cases} .$$

Solving this recursion yields  $e(n) \leq Kn$ , where  $K$  is a constant that depends only on  $n_0$  and  $d$ . Therefore we obtained a super concentrator with  $O(n)$  edges as required.

A word about algorithms to construct such graphs: Suppose that we have an algorithm which constructs magical graphs of left size  $n$  in time  $t(n)$ . It should be clear that the above recursive construction yields an algorithm that constructs a superconcentrator with input/output size  $n$  in time  $O(t(n))$ .

### 1.3.2 Construction of good error correcting codes

We now turn to Shannon’s problem concerning communicating over a noisy channel, and present a solution due to Sipser and Spielman [SS96]. We observe a simple but useful property of magical graphs. Let  $G$  be such a graph with  $n$  left vertices and  $3n/4$  right vertices. We show that for every non-empty  $S \subset L$  with  $s = |S| \leq \frac{n}{10d}$  there exists a vertex  $u \in R$  with exactly one neighbor in  $S$ . Namely,  $|\Gamma(u) \cap S| = 1$ . To see this consider  $e(S, \Gamma(S))$ , the number of edges between  $S$  and  $\Gamma(S)$ . Clearly,  $e(S, \Gamma(S)) = d \cdot |S| = ds$ . On the other hand, since  $|\Gamma(S)| \geq 5ds/8$ , the average number of neighbors in  $S$  that a vertex in  $\Gamma(S)$  has is at most  $8/5 < 2$ . But every vertex in  $\Gamma(S)$  has at least one neighbor in  $S$ , so there must be some (indeed, many) vertices in  $\Gamma(S)$  with exactly one neighbor in  $S$ .

We use the magical graph  $G$  to construct a code  $C \subset \{0, 1\}^n$  with rate at least  $1/4$ , and distance at least  $1/10d$ . To this end, we represent the magical graph  $G = (R, L, E)$  by a matrix  $A$  with row set  $R$  and column set  $L$ , where  $a_{ij}$  equals 1 or 0 depending on whether or not the  $i$ -th vertex in  $R$  is adjacent to the  $j$ -th vertex in  $L$ . The code is defined as the right kernel of  $A$ , viz.  $C = \{x \in \{0, 1\}^n \mid Ax = 0\}$ . (Here calculations are done over the field with two elements). Clearly  $C$  is a linear subspace of  $\{0, 1\}^n$  of dimension  $\geq n/4$ . Hence  $|C| \geq 2^{n/4}$ , yielding the claimed lower bound on the code’s rate.

To prove a lower bound on the distance, first observe that since  $C$  is a linear code (i.e. a linear subspace of  $\{0, 1\}^n$ ) the smallest distance between two of its codewords equals the smallest weight of a non-zero codeword. Let  $x \neq 0$  be an  $n$ -bit vector with support  $S = \{j \in L : x_j = 1\}$ . If  $|S| < \frac{n}{10d}$  then, as we saw, there is some  $i \in R$  with  $|\Gamma(i) \cap S| = 1$ . It follows that the  $i$ -th coordinate in  $Ax$  is 1, and so  $x$  is not in the right kernel of  $A$  and cannot be a codeword. It follows that the normalized distance of  $C$  is at least  $1/10d$ .

The above construction is a special case of a so-called LDPC (for Low Density Parity Check) code. This idea was first suggested by Gallager [Gal63], and has inspired work by Pinsker and others in Russia. Indeed, this was one of the earlier motivations to search for explicit expanders. After being nearly dormant for about 20 years, LDPC codes regained prominence in the 90’s, and are now believed to give simultaneously the best coding parameters as well as best algorithmic performance in various settings. For a survey of this fascinating field, see Richardson and Urbanke [RU].

We note that only fairly recently [CRVW02], did the art of explicit constructions of expanding graphs reach the level that makes the above simple argument feasible. It should also be mentioned that this construction not only yields codes with linear distance but also linear time iterative decoding. We will review these “lossless expanders” in Chapter 10.

As in the previous application, the time complexity of constructing the magical graph dominates the time to construct the (parity check matrix of the) appropriate code. This is yet another reason to seek efficient algorithms to construct these graphs. The next application calls for an even more concise and efficient description of these graphs.

### 1.3.3 Deterministic Error Amplification for RP

Our last problem revolves around deciding membership in a language  $\mathcal{L} \in \mathbf{RP}$ , with a given bound on the algorithm’s error probability. The solution we present is due to Karp, Pippenger, and Sipser [KPS85]. It carries out *dependent* sampling of random strings using magical graphs.

As we explained above, we have to decide whether a given  $k$ -bit string  $x$  belongs to  $\mathcal{L}$  or not. By assumption, there is a polytime algorithm that upon receiving  $x$  and a random  $k$ -bit string  $r$ , calculates a function  $f(x, r)$  such that  $f(x, r) = 1$  whenever  $x \in \mathcal{L}$ , but  $f(x, r) = 1$  with probability at most  $1/16$  (the probability is over the choice of  $r$ ) when  $x \notin \mathcal{L}$ .

To reduce our probability of error we will be considering several strings  $r$ . However, our goal is to reduce our failure probability below some set threshold while we utilize as few as possible such strings  $r$ . In other words, fix some  $x \notin \mathcal{L}$  and let  $B = \{r \in \{0, 1\}^k \mid f(x, r) = 1\}$ , be the set of strings  $r$  that are “bad” for us in that they fail us on input  $x$ . We’d like to make it as likely as possible that at least one of the  $r$ ’s we consider lies outside of  $B$ . The only information we have about the set  $B \subseteq \{0, 1\}^k$  is that it’s not too big,  $|B| \leq n/16$  where  $n = 2^k$ .

For any given integer  $d$ , we offer an algorithm for the membership problem that evaluates  $f$  only  $d$  times and fails with probability  $\epsilon \leq \frac{1}{10d}$ . The algorithm is rather simple. We fix an  $(n, n; d)$ -magical graph  $G$  with  $n = 2^k$ , where each vertex in  $R$  and each vertex in  $L$  is associated with a unique  $k$ -bit string. To decide whether a given  $x$  is in  $\mathcal{L}$  we sample a  $k$ -bit string  $r$  which we consider as a vertex in  $L$ . Let  $r_1, \dots, r_d \in R$  be the (strings associated with) the

$d$  neighbors of  $r$ . Our algorithm outputs 1 iff  $f(x, r_1) = f(x, r_2) = \dots = f(x, r_d) = 1$ .

Clearly this algorithm fails iff  $x \notin \mathcal{L}$  and  $r_1, \dots, r_d \in B$ , i.e.  $\Gamma(r) \subseteq B$ . Let  $S \subset L$  be the set of left vertices that satisfy this condition, i.e.  $S = \{v \in L \mid \Gamma(v) \subseteq B\}$ . So we fail iff  $r \in S$ . But  $|S| \leq \frac{n}{10d}$  or else  $|\Gamma(S)| > 5|S|/8 \geq n/16$  (this is the moment of magic here...). This upper bound on  $|S|$  means that we fail with probability at most  $\frac{1}{10d}$  while using only the original  $k$  random bits. We can reduce the probability of error arbitrarily by increasing  $d$  appropriately. A key point is that we have reached this reduction in error probability *without using any additional random bits*.

Here are a few comments on this algorithm.

Note that unlike the previous two examples, the size  $n$  of the graph used is *exponential* in the natural size of the problem considered (the parameter  $k$  here). This means that for an efficient implementation of the new algorithm, our encoding of the magical graph must be much more efficient than in the previous applications. Specifically, given the name of a vertex (a  $k$ -bit string), we must be able to generate its  $d$  neighbors in time  $\text{poly}(k)$ , which is far *smaller* than the size of the graph. We will later see that even this level of “explicitness” is achievable.

Next, with the  $d$  (dependent) samples we use here, we can reduce the error to  $O(\frac{1}{d})$ . This is much inferior to the exponential decay of the error as a function of  $d$  when we “waste” random bits and make  $d$  independent samples. We will later see that (other) dependent sampling via expanders (which uses only a few more random bits than the solution above) can achieve such an exponential decay as well.

Another comment regards the 1-sided errors. Many probabilistic algorithms err on all inputs - those in the language  $\mathcal{L}$  and those which are not. Again, we will later see that appropriate modification of dependent sampling via expanders can achieve nearly optimal error reduction in such situations as well.

These problems and results have developed into a whole subfield of Theoretical Computer Science. Two excellent surveys of these issues are [Gol97] and [Sha04].



## Chapter 2

# Graph Expansion & Eigenvalues

### 2.1 Definitions and Notation

Let us introduce some conventions now. Unless we say otherwise, a graph  $G = (V, E)$  is undirected and  $d$ -**regular** (all vertices have the same degree  $d$ , namely each vertex is incident with exactly  $d$  edges). Self loops and multiple edges are allowed. The number of vertices  $|V|$  is denoted by  $n$ . Unlike the previous chapter, graphs need not be bipartite. We denote by  $E(S, T) = \{(u, v) | u \in S, v \in T, (u, v) \in E\}$  the set of edges from  $S$  to  $T$  for  $S, T \subset V$ . Note that here, we think of every undirected edge as a pair of directed edges, so  $E(S, T)$  is a set of *directed* edges. It will also be convenient to define  $E(S)$  as the set of edges both vertices of which belong to  $S$

#### Definition 2.1.

1. The **Edge Boundary** of a set  $S$ , denoted  $\partial S$ , is  $\partial S = E(S, \bar{S})$ . This is the set of edges emanating from the set  $S$  to its complement.
2. The **Expansion Ratio** of  $G$ , denoted  $h(G)$ , is defined as:

$$h(G) = \min_{\{S \mid |S| \leq \frac{n}{2}\}} \frac{|\partial S|}{|S|}.$$

Later on we will encounter other interesting notions of expansion in graphs. Of particular importance is vertex expansion, where we count the number of neighboring vertices of vertex sets  $S$ , rather than the number of outgoing edges. See Chapters 4 and 10 for more on this.

#### Definition 2.2. Family Of Expander Graphs

A family of graphs  $\{G_i\}_{i \in \mathbb{N}}$  is called a family of expander graphs if there exists an integer  $d$  and  $\epsilon > 0$  such that

- Each  $G_i$  is a  $d$ -regular graph on  $n_i$  vertices. The integers  $\{n_i\}$  are increasing, but not too fast (e.g.  $n_{i+1} \leq n_i^2$  will do).
- $h(G_i) \geq \epsilon$  for all  $i$ .

Issues concerning the *explicit construction* of mathematical objects are fundamental to all of computer science, and expander graphs are no exception. There are two natural levels of efficiency to be considered in the construction of such graphs, which we have already seen in the examples of the previous chapter. In the first we require that an  $n$ -vertex graph should be generated “from scratch” in time polynomial in  $n$ . In the stronger version we demand that the neighborhood of any given vertex should be computable in time that is polynomial in the description length of the vertex (usually polynomial in  $\log n$ ).

The technicalities of these definitions may seem odd to the uninitiated reader, but they reflect a very natural need. Expander graphs are to be used by various algorithms. The algorithms’ performance will depend on our efficiently in obtaining the relevant information of the expanders being used.

### Definition 2.3.

1. A family of expander graphs is called **Mildly Explicit** if there is an algorithm that generates the  $j$ -th graph in the family,  $G_j$  in time polynomial in  $j$  (I.e.,  $G_j$  is computed in time  $< Aj^B$  for some constants  $A, B > 0$ .)
2. It is called **Very Explicit** if there is an algorithm that on input an integer  $i$ , a vertex  $v \in V(G_i)$  and  $k \in \{1, \dots, d\}$  computes the  $k$ -th neighbor of vertex  $v$  in the graph  $G_i$ . This algorithm's run time should be polynomial in its input length (the number of bits needed to express the triple  $(i, v, k)$ .)

## 2.2 Examples of Expander Graphs

1. A family of 8-regular graphs  $G_m$  for every integer  $m$ . Its vertex set is  $V_m = \mathbb{Z}_m \times \mathbb{Z}_m$ . The neighbors of the vertex  $(x, y)$  are  $(x+y, y), (x-y, y), (x, y+x), (x, y-x), (x+y+1, y), (x-y+1, y), (x, y+x+1), (x, y-x+1)$ , (all operations are mod  $m$ ).

This family of graphs, due to Margulis [Mar73] is the first explicitly constructed family of expander graphs. Margulis' proof of expansion was based on representation theory and did not provide any specific bound on the expansion ratio  $h$ . This was remedied by Gaber and Galil [GG81] who derived such a bound using harmonic analysis. That Margulis' graphs are expanders is shown in Chapter 8. Note that this family is very explicit.

2. A family of 3-regular  $p$ -vertex graphs for every prime  $p$ . Here  $V_p = \mathbb{Z}_p$ , and a vertex  $x$  is connected to  $x+1, x-1$ , and to its inverse  $x^{-1}$  (operations are mod  $p$ , and we define the inverse of 0 to be 0).

Here, the proof of expansion depends on a deep result in Number Theory - The Selberg 3/16 theorem (see discussion in Lubotzky [Lub94].) Note that it is only mildly explicit, since we are at present unable to generate large primes deterministically. See [Gra05] for a survey about the Agrawal-Kayal-Saxena polytime primality testing.

## 2.3 The Spectrum of a Graph

The **Adjacency Matrix** of an  $n$ -vertex graph  $G$ , denoted  $A = A(G)$ , is an  $n \times n$  matrix whose  $(u, v)$  entry is the number of edges in  $G$  between vertex  $u$  and vertex  $v$ . Being real and symmetric, the matrix  $A$  has  $n$  real eigenvalues which we denote by  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . We can also associate with it an orthonormal system of eigenvectors  $v_1, \dots, v_n$  with  $Av_i = \lambda_i v_i$ . We often refer to the eigenvalues of  $A(G)$  as the **Spectrum** of the graph  $G$ . The spectrum of a graph encodes a lot of information about it. Here are some simple illustrations how certain properties of a  $d$ -regular graph are reflected in its spectrum:

- $\lambda_1 = d$ , and the corresponding eigenvector is  $v_1 = \bar{1}/\sqrt{n}$ .
- The graph is connected iff  $\lambda_1 > \lambda_2$
- The graph is bipartite iff  $\lambda_1 = -\lambda_n$

As seen in the next theorem, the graph's second eigenvalue is closely related to its expansion parameter.

### Theorem 2.4.

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}$$

This Theorem is due to Cheeger [Che70] and Buser [Bus82] in the continuous case (see Chapter 4 for more on this). In the discrete case, it was proved by Dodziuk [Dod84], and independently by Alon-Milman [AM85], and Alon [Alo86]. More concretely we see that  $d - \lambda_2$ , also known as the **Spectral Gap**, provides an estimate on the expansion of a graph. In particular, a  $d$ -regular graph has an expansion ratio  $h(G)$  bounded away from zero iff its spectral gap  $d - \lambda_2$  is bounded away from zero. A proof of this theorem appears in Chapter 4. The following Lemma shows that a small second eigenvalue in a graph implies that its edges are "spread out" - a hallmark of random graphs.

## 2.4 The Expander Mixing Lemma

Given a  $d$ -regular graph  $G$ , we denote by  $\lambda = \lambda(G) = \max(|\lambda_2|, |\lambda_n|)$ . In words,  $\lambda$  is the largest absolute value of an eigenvalue other than  $\lambda_1 = d$ .

**Lemma 2.5. Expander Mixing Lemma** *Let  $G$  be a  $d$ -regular graph with  $\lambda(G) = \lambda$ . Then for all  $S, T \subseteq V$ :*

$$\left| |E(S, T)| - \frac{d|S||T|}{n} \right| \leq \lambda \sqrt{|S||T|}.$$

A word of interpretation is in place here. The left hand side measures the deviation between two quantities: One is  $|E(S, T)|$ , the number of edges between the two sets; The other is the expected number of edges between  $S$  and  $T$  in a random graph of edge density  $d/n$ , namely  $d|S||T|/n$ . A small  $\lambda$  (or large spectral gap) implies that this deviation (or *discrepancy* as it's sometimes called) is small, and that our graph is nearly random in this sense.

When the spectral gap of  $G$  is much smaller than  $d$ , the upper and lower bounds in Theorem 2.4 differ substantially. This makes one wonder whether the spectral gap can be captured more tightly by any combinatorial invariant of the graph. A positive answer and a converse to the Expander Mixing Lemma was found recently by Bilu and Linial [BL]. We will not prove this result here and refer the reader to the original paper.

**Lemma 2.6 (Bilu-Linial [BL]). Converse of the Expander Mixing Lemma** *Let  $G$  be a  $d$ -regular graph and suppose that*

$$\left| |E(S, T)| - \frac{d|S||T|}{n} \right| \leq \rho \sqrt{|S||T|},$$

*holds for every two disjoint sets  $S, T$  and for some positive  $\rho$ . Then  $\lambda \leq O(\rho \cdot (1 + \log(d/\rho)))$ . The bound is tight.*

*Proof.* (of the Expander Mixing Lemma)

Let  $1_S$  and  $1_T$  be the characteristic vectors of  $S$  and  $T$  (i.e., the  $v$ -th coordinate of the vector  $1_S$  is 1 if  $v \in S$  and zero otherwise). We expand these vectors in the orthonormal basis of eigenvectors  $v_1, \dots, v_n$ , viz.,  $1_S = \sum_i \alpha_i v_i$  and  $1_T = \sum_j \beta_j v_j$ . Recall that  $v_1 = \bar{1}/\sqrt{n}$ . Then

$$|E(S, T)| = 1_S A 1_T = (\sum_i \alpha_i v_i) A (\sum_j \beta_j v_j).$$

Since the  $v_i$ 's are orthonormal eigenvectors this equals  $\sum_i \lambda_i \alpha_i \beta_i$ . Since  $\alpha_1 = \langle 1_S, \frac{\bar{1}}{\sqrt{n}} \rangle = \frac{|S|}{\sqrt{n}}$ ,  $\beta_1 = \frac{|T|}{\sqrt{n}}$ , and  $\lambda_1 = d$ :

$$|E(S, T)| = d \frac{|S||T|}{n} + \sum_{i=2}^n \lambda_i \alpha_i \beta_i.$$

By definition of  $\lambda$ :

$$\left| |E(S, T)| - d \frac{|S||T|}{n} \right| = \left| \sum_{i=2}^n \lambda_i \alpha_i \beta_i \right| \leq \sum_{i=2}^n |\lambda_i \alpha_i \beta_i| \leq \lambda \sum_{i=2}^n |\alpha_i \beta_i|.$$

Finally, by Cauchy-Schwartz:

$$\left| |E(S, T)| - d \frac{|S||T|}{n} \right| \leq \lambda \|\alpha\|_2 \|\beta\|_2 = \lambda \|1_S\|_2 \|1_T\|_2 = \lambda \sqrt{|S||T|}.$$

□

In what follows, it is sometimes convenient to consider the normalized second eigenvalue  $\lambda(G)/d$ . A  $d$ -regular graph  $G$  on  $n$  vertices is called an  **$(n, d)$  graph** and an  **$(n, d, \alpha)$ -graph** if  $\lambda(G) \leq \alpha d$ . Regular graphs with small  $\alpha$  have a number of significant properties, some of which we collect below:

- An *independent set* in a graph is a set of vertices  $S$  no two of which are adjacent, i.e. with  $|E(S, S)| = 0$ . It is an immediate consequence of the expander mixing lemma that an independent set in an  $(n, d, \alpha)$ -graph has cardinality is at most  $\alpha n$ .

- A  $k$ -coloring of a graph  $G = (V, E)$  is a mapping  $c : V \rightarrow \{1, \dots, k\}$ , such that  $c(x) \neq c(y)$  for any two adjacent vertices  $x, y$ . The chromatic number of  $G$ , denoted  $\chi(G)$ , is the smallest  $k$  for which  $G$  has a  $k$ -coloring. Now the set  $c^{-1}(j)$  is an independent set in  $G$  for every  $k \geq j \geq 1$ . Consequently,  $\chi(G) \geq 1/\alpha$  for an  $(n, d, \alpha)$ -graph  $G$ .
- The distance  $d_G(x, y)$  between vertices  $x$  and  $y$  in a graph  $G = (V, E)$  is the length of the shortest path between them. The diameter of  $G$  is defined as  $\max_{x, y} d_G(x, y)$ . Also  $B(x, r) = \{y | d_G(x, y) \leq r\}$ , is the ball or radius  $r$  around  $x$ . We claim that an  $(n, d, \alpha)$ -graph  $G$  has diameter  $O(\log n)$ . That certainly follows if we show that  $|B(x, r)| > n/2$  for every vertex  $x$  and some  $r \leq O(\log n)$ . This, in turn follows from  $G$ 's expansion properties. Namely, we show that  $|B(x, r+1)| \geq (1+\epsilon)|B(x, r)|$  for some fixed  $\epsilon > 0$  as long as  $|B(x, r)| \leq n/2$ . The expander mixing lemma implies that  $|E(S, S)|/|S| \leq d \cdot (|S|/n + \alpha)$  for every set  $S$ . Therefore,  $|E(S, \bar{S})|/|S| \geq d \cdot ((1-\alpha) - |S|/n)$ . But  $S$  has at least  $|E(S, \bar{S})|/d$  neighbors outside of itself, so the claim follows with  $\epsilon = 1/2 - \alpha$ .

## 2.5 How Big Can the Spectral Gap be?

The question in the title has to be qualified, since the answer depends on the relationship between  $d$  and  $n$ . We are mostly interested in  $d$  fixed and large  $n$ . To illustrate how things may differ when  $d$  grows with  $n$ , consider the complete graph on  $n$  vertices  $K_n$  where every two vertices are adjacent and so  $d = n - 1$ . Clearly the adjacency matrix of  $K_n$  is  $J - I$  where  $J$  is the all-ones matrix and  $I = I_n$  is the identity matrix. The spectrum of  $K_n$  is  $[n - 1, -1, -1, \dots, -1]$ , and  $\lambda = 1$ .

For the range we are interested in  $n \gg d$ , the question was answered by N. Alon and R. Boppana (see A. Nilli [Nil91]):

**Theorem 2.7 (Alon-Boppana).** *For every  $d$ -regular graph:*

$$\lambda \geq 2\sqrt{d-1} - o_n(1)$$

The  $o_n(1)$  term is a quantity that tends to zero for every fixed  $d$  and  $n \rightarrow \infty$ . More on this and a proof of this theorem appear in Chapter 5. Here is a very easy but somewhat weaker statement:

**Claim 2.8.** *For every  $d$ -regular graph  $G$ :*

$$\lambda \geq \sqrt{d} \cdot (1 - o_n(1))$$

*Proof.* Let  $A$  be the adjacency matrix of  $G$ . It is not hard to see that  $\text{trace}(A^k)$  is the number of all walks of length  $k$  in  $G$  that start and end in the same vertex. In particular, all the diagonal entries in  $A^2$  are  $\geq d$ . (Just move back and forth along any edge incident with the vertex in question). Consequently,  $\text{trace}(A^2) \geq nd$ . On the other hand:

$$\text{trace}(A^2) = \sum_i \lambda_i^2 \leq d^2 + (n-1)\lambda^2.$$

It follows that  $\lambda^2 \geq d \cdot \frac{n-d}{n-1}$ , as claimed. □

## Chapter 3

# Random Walks on Expander Graphs

A key property of the random walk on an expander graph is that it converges rapidly to its limit distribution. This fact has numerous important consequences at which we can only hint. In many theoretical and practical computational problems in science and engineering it is necessary to draw samples from some distribution  $\mathcal{F}$  on a (usually finite but huge) set  $V$ . Such problems are often solved by so-called "Monte-Carlo" algorithms. One considers a graph  $G$  on vertex set  $V$ , so that the limit distribution of the random walk on  $G$  is  $\mathcal{F}$ . A clever choice of  $G$  can guarantee that (i) It is feasible to efficiently simulate this random walk and (ii) The distribution induced on  $V$  by the walk *converges rapidly* to  $\mathcal{F}$ . Among the fields where this methodology plays an important role are Statistical Physics, Computational Group Theory and Combinatorial Optimization. We should mention approximation algorithms for the permanent of matrices, and for the volume of convex bodies in high dimension as prime examples for the latter. Excellent surveys on the subject are [JS96, Jer03]. In particular, and as we briefly mention in Section 4.5, some of this theory extends to the more general context of time-reversible Markov Chains.

The main principle behind the topics we survey here is that the set of vertices visited by a length  $t$  random walk on an expander graph "looks like" (in some respects) a set of  $t$  vertices sampled uniformly and independently. The computational significance of this is that the number of random bits required in order to generate a length  $t$  walk on a (constant-degree) graph is significantly smaller than the number of random bits that are needed in order to independently sample  $t$  random vertices. We exhibit two applications of this idea: (i) A randomness-efficient error reduction procedure for randomized algorithms, and (ii) A strong hardness-of-approximation result for the maximum clique problem. Other computational applications of these ideas that we will not go into include derandomization of probabilistic space-bounded algorithms (see e.g. Nisan-Zuckerman [NZ96], and Impagliazzo-Nisan-Wigderson [INW94]).

### 3.1 Rapid Mixing of Walks

A *walk* on a graph  $G = (V, E)$  is a sequence of vertices  $v_1, v_2, \dots \in V$  such that  $v_{i+1}$  is a neighbor of  $v_i$  for every index  $i$ . When  $v_{i+1}$  is selected uniformly at random from among  $v_i$ 's neighbors, independently for every  $i$ , this is called a *random walk* on  $G$ . We usually initiate this random process by selecting the first vertex  $v_1$  from some *initial probability distribution*  $\pi_1$  on  $V$ . Clearly this induces a sequence of probability distributions  $\pi_i$  on  $V$  so that the probability that  $v_i = x \in V$  equals  $\pi_i(x)$  for every  $i$  and  $x$ . It is well known that for every finite connected non-bipartite graph  $G$ , the distributions  $\pi_i$  converge to a limit, or *stationary* distribution. Moreover, it is easy to see that if  $G$  is regular, then this distribution is the uniform distribution on  $V$ .

This subsection deals with the speed of this convergence. There are several interesting ways to measure the distance between  $\pi_i$  and the limit distribution and we will consider several norms and entropy measures. The main thrust is that in expanders the distance to the limit shrinks substantially with *every* step of the random walk, and that this condition characterizes expander graphs. Now we turn to make this statement quantitative.

We start with some definitions and notations:

We recall that an  $(n, d)$ -graph is a  $d$ -regular graph  $G$  on  $n$  vertices. It is called an  $(n, d, \alpha)$ -graph if  $|\lambda_2(G)|, |\lambda_n(G)| \leq \alpha d$ , where  $d = \lambda_1(G) \geq \dots \geq \lambda_n(G)$  is the spectrum of  $G$ ,

A vector  $\vec{p} \in \mathbb{R}^n$  is called a *probability vector* if its coordinates are nonnegative and  $\sum_{i=1}^n \vec{p}_i = 1$ . The probability vector that corresponds to the uniform distribution on  $\{1, \dots, n\}$  is denoted by  $\vec{u} = (1, \dots, 1)/n$ . In this section we show that a random walk on the vertices of an expander converges rapidly to the stationary distribution.

**Definition 3.1.** A random walk on a finite graph  $G = (V, E)$  is a discrete-time stochastic process  $(X_0, X_1, \dots)$  taking values in  $V$ . The vertex  $X_0$  is sampled from some initial distribution on  $V$  and  $X_{i+1}$  is chosen uniformly at random from the neighbors of  $X_i$ .

If  $G$  is a  $d$ -regular graph with adjacency matrix  $A$ , then its *normalized adjacency matrix* is defined as  $\hat{A} = \frac{1}{d}A$ . Here are some simple comments on this random walk.

- The random walk on  $G = (V, E)$  is a Markov Chain with state set  $V$ , and transition matrix  $\hat{A}$ .
- $\hat{A}$  is real, symmetric, and double stochastic; i.e. every column and every row sums up to 1.
- If  $\hat{\lambda}_1, \dots, \hat{\lambda}_n$  are the eigenvalues of  $\hat{A}$ , then  $\hat{\lambda}_1 = 1$  and  $\max\{|\hat{\lambda}_2|, |\hat{\lambda}_n|\} \leq \alpha$ .
- The corresponding eigenvectors are the same eigenvectors of  $A$ .
- Consider an experiment where we sample a vertex  $x$  from some probability distribution  $\vec{p}$  on  $V$  and then move to a random neighbor of  $x$ . This is equivalent to sampling a vertex from the distribution  $\hat{A}\vec{p}$ .
- The matrix  $\hat{A}^t$  is the transition matrix of the Markov chain defined by random walks of length  $t$ . i.e.  $(\hat{A}^t)_{ij}$  is the probability a random walk starting at  $i$  is at  $j$  after  $t$  steps.
- The stationary distribution of the random walk on  $G$  is the uniform distribution. Namely,  $\vec{u}\hat{A} = \hat{A}\vec{u} = \vec{u}$ . (This uses the symmetry of  $\hat{A}$ .)

### 3.1.1 Convergence in $L_1$ and $L_2$ norms

The inner product of  $\vec{u}, \vec{v} \in \mathbb{R}^n$  is denoted by  $\langle \vec{u}, \vec{v} \rangle$ . The  $l_1, l_2$  and  $l_\infty$  norms are denoted as usual by

- $\|\vec{u}\|_1 = \sum_{i=1}^n |\vec{u}_i|$
- $\|\vec{u}\|_2 = \sqrt{\langle \vec{u}, \vec{u} \rangle} = \sqrt{\sum_{i=1}^n \vec{u}_i^2}$
- $\|\vec{u}\|_\infty = \max_{1 \leq i \leq n} |\vec{u}_i|$ .

We now observe that if  $G$  is an  $(n, d, \alpha)$ -graph, and if  $\alpha < 1$ , then regardless of the initial distribution  $\vec{p}$ , the random walk converges in  $l_1$  exponentially fast to its limit (uniform) distribution. This will follow (via Cauchy-Schwartz) from a similar bound on  $l_2$ , which in turn follows from the fact that in  $l_2$  the distance to the uniform distribution shrinks by a factor of  $\alpha$  at each step.

**Theorem 3.2.**  $\|\hat{A}^t \vec{p} - \vec{u}\|_1 \leq \sqrt{n} \cdot \alpha^t$  for any distribution vector  $\vec{p}$ .

Why use the  $l_1$  norm to measure for the distance between two probability distributions  $p, q$ ? A natural and commonly used metric is the **total variation distance**  $\max_B |\Pr_p[B] - \Pr_q[B]|$  and it is not difficult to check that this equals  $\frac{1}{2}\|p - q\|_1$ . In other words, if the  $l_1$  distance is small, then the two distributions  $p$  and  $q$  assign nearly equal probabilities to *every* event in the probability space. This theorem follows immediately from the analogous  $l_2$  bound below.

**Theorem 3.3.** For every probability vector  $p$ ,  $\|\hat{A}^t \vec{p} - \vec{u}\|_2 \leq \|\vec{p} - \vec{u}\|_2 \alpha^t \leq \alpha^t$  for any distribution vector  $\vec{p}$ .

Obviously it suffices to prove this bound for  $t = 1$  (shrinkage per step) and use induction.

**Lemma 3.4.** For every probability vector  $p$ ,  $\|\hat{A}\vec{p} - \vec{u}\|_2 \leq \alpha\|\vec{p} - \vec{u}\|_2 \leq \alpha$

*Proof.* The uniform distribution  $\vec{u}$  is invariant under the action of  $\hat{A}$ . Also,  $\vec{p} - \vec{u}$  is orthogonal to  $\vec{u}$ , and thus shrinks in  $l_2$ -norm by a factor  $\alpha$  under the action of  $\hat{A}$ . Consequently

$$\|\hat{A}\vec{p} - \vec{u}\|_2 = \|\hat{A}(\vec{p} - \vec{u})\|_2 \leq \alpha\|\vec{p} - \vec{u}\|_2 \leq \alpha.$$

where the last inequality follows easily from the fact that  $\vec{p}$  is a probability distribution.  $\square$

### 3.1.2 Convergence in entropy.

Another important perspective of a random walk is offered by the *entropy* of the associated probability distributions. The entropy of probability distributions is a fundamental concept in the theory of communication, capturing the amount of "information", or alternatively "uncertainty" that it carries. When we take a random step, we "inject" more randomness into our distribution, indeed precisely the  $\log d$  random bits that are needed to specify which of the  $d$  neighbors of the current vertex we move to next. One expects this injection to increase the amount of "randomness" in the distribution, namely its entropy. This is indeed always true, in every regular graph, and expanders are those graphs for which the increase is significant.

This entropy viewpoint will be extremely important when we explain the zigzag product and its use in combinatorial constructions of various expanders in Chapters 9, 10. In the same way that different norms capture different aspects of the probability distributions, there are several variations on the theme of entropy that do this. If  $p$  is a probability distribution on  $[n]$ , then we define its

- *Shannon entropy*:  $H(p) = -\sum_{i=1}^n p_i \log(p_i)$ .
- *Rényi 2-entropy*:  $H_2(p) = -2 \log(\|p\|_2)$ .
- *Min entropy*:  $H_\infty(p) = -\log(\|p\|_\infty)$ .

To see the connection between these two quantities, note that if  $p$  is a probability distribution on  $[n]$ , then  $\max p_i \geq \sum p_i^2 \geq \max p_i^2$ . It follows that:

**Proposition 3.5.**

$$H_\infty(X) \leq H_2(X) \leq 2H_\infty(X).$$

Here are some simple and useful properties that are common to all three, which the reader is invited to prove. As above,  $p$  is a probability distribution on an  $n$ -element set, and we denote our "generic" entropy by  $\tilde{H}$ .

- $\tilde{H}(p) \geq 0$  with equality iff the distribution is concentrated on a single element.
- $\tilde{H}(p) \leq \log n$  with equality iff the distribution is uniform.
- For any doubly stochastic matrix  $X$ ,  $\tilde{H}(Xp) \geq \tilde{H}(p)$ , with equality iff  $p$  is uniform.

The last item shows that entropy increases with every step of the random walk on a regular graph. Making this quantitative, depends on the choice of entropy measure. Below we do so for the Rényi 2-entropy in terms of the spectral bound  $\alpha$ , which (not surprisingly) is just a restatement of the  $l_2$  bound from the previous section. However, as noted above  $H_2$  and  $H_\infty$  are very close to each other, and it is the latter we use in Chapter 10, so this interpretation will be important for us.

Before doing so, we remark that for Shannon entropy  $H$ , the precise relation between the increase in  $H$  and the spectral constant  $\alpha$  is still unknown. However, one can define an analogous "entropy constant" which governs the increase "per step" in entropy. It is called the Log-Sobolev constant, and there are known quantitative relations between it and the spectral constant (much like the relations between edge expansion and the spectral constant of the previous chapter). Using the Log-Sobolev constant to analyze the mixing time of random walks is a powerful method, but it is beyond the scope of this survey. For more on this, see e.g. [MT].

Let us write the distribution as  $p = \frac{1}{n}1_n + f$ , for some  $f \perp 1$ . We let  $\mu$  capture how close  $p$  is to the uniform distribution, via  $\mu = \|f\|/\|p\| \leq 1$  (e.g.  $\mu = 0$  iff  $p$  is uniform). Then

$$\|\hat{A}p\|^2 = \left\| \frac{1}{n}1_n + \hat{A}f \right\|^2 = \left\| \frac{1}{n}1_n \right\|^2 + \|\hat{A}f\|^2 \leq ((1 - \mu^2) + \alpha^2 \mu^2) \|p\|^2,$$

Hence

$$H_2(\hat{A}p) \geq H_2(p) - \log((1 - \mu^2) + \alpha^2 \mu^2) = H_2(p) - \log(1 - (1 - \alpha^2)\mu^2).$$

It follows that the distribution's 2-entropy never decreases and is, in fact, strictly increasing as long as the distribution  $p$  is not uniform. It is also clear that for better expanders ( $\alpha$  smaller) the 2-entropy grows faster.

## 3.2 Random walks resemble independent sampling

Suppose that we are given an  $(n, d, \alpha)$ -graph  $G = (V, E)$  where the vertices in some subset  $B \subseteq V$  are “bad”. All we know about the set  $B$  is its cardinality  $|B| = \beta n$ . We wish to sample at least one vertex outside of  $B$ . We can certainly sample, uniformly at random  $t + 1$  vertices  $x_0, \dots, x_t$  from  $V$  and fail with probability  $\Pr[\forall i x_i \in B] \leq \beta^{t+1}$ . This approach uses  $(t + 1) \log n$  random bits, and we will show that similar performance can be achieved with substantially fewer random bits. Namely, that if we choose a random starting vertex and carry out a random walk of length  $t$  from it, then our chance of failure i.e., the probability that the whole random walk is confined to  $B$  is exponentially small in  $t$  as well.

To get started, let us reinterpret the expander mixing lemma as the case  $t = 1$  of this approach. Recall that the lemma says that:

$$\left| \frac{d|S||T|}{n} - |E(S, T)| \right| \leq \alpha d \sqrt{|S||T|} \leq \alpha dn,$$

for every two subsets  $S, T \subseteq V(G)$  in a  $(n, d, \alpha)$ -graph  $G$ . It is useful to rewrite this as:

$$\left| \frac{|S||T|}{n^2} - \frac{|E(S, T)|}{dn} \right| \leq \alpha.$$

Now consider two experiments in which we sample an ordered pair of vertices  $(i, j)$  in  $G$ , and consider it a success if  $i \in S$  and  $j \in T$ . In the first version of this experiment both vertices are selected uniformly at random from  $V = V(G)$ , so our probability of success is  $|S||T|/n^2$ . In the second version we randomly pick  $i \in V$ , and then  $j$  is selected uniformly among the neighbors of  $i$ . In this version of the experiment, our probability of success is  $|E(S, T)|/dn$ . The Expander Mixing Lemma says that despite the different nature of the sample spaces in the two experiments, their success probabilities differ only by a small constant  $\alpha$ . We now turn to the complete argument that pertains to longer random walks.

Let  $G = (V, E)$  be an  $(n, d, \alpha)$ -graph, and  $B \subset V$  of cardinality  $|B| = \beta n$ . We carry out the following experiment: We pick  $X_0 \in V$  uniformly at random and start from it a random walk  $X_0, \dots, X_t$  on  $G$ . We denote by  $(B, t)$  the event that this random walk is confined to  $B$ ; i.e. that  $\forall i X_i \in B$ .

**Theorem 3.6 (Ajtai-Komlós-Szemerédi [AKS87], Alon-Feige-Wigderson-Zuckerman [AFWZ95]).**

$$\Pr[(B, t)] \leq (\beta + \alpha)^t.$$

Let  $P = P_B$  be a projection on the subspace of coordinates belonging to  $B$ . In matrix notation  $P_{ij} = 1$  if  $i = j \in B$  and 0 otherwise. We need the following simple observation:

**Lemma 3.7.**  $\Pr[(B, t)] = \|(P\hat{A})^t P\vec{u}\|_1$

*Proof.* To calculate the  $(x, y)$  entry in the matrix  $(\hat{A})^t$ , we should sum the probabilities of walks of length  $t$  that start at vertex  $x$  and end at  $y$ . In the same calculations for the matrix  $(P\hat{A})^t P$  only walks confined to  $B$  are to be considered, since all other contributions are eliminated by the matrices  $P$  in the product. But  $\|(P\hat{A})^t P\vec{u}\|_1$  is just  $\frac{1}{n}$  of the total sum of these entries and the conclusion follows.  $\square$

We also need the following lemma.

**Lemma 3.8.** For any nonnegative vector  $v$ :

$$\|P\hat{A}P\vec{v}\|_2 \leq (\beta + \alpha) \cdot \|\vec{v}\|_2$$

*Proof.* First note that there is no loss in assuming that  $\vec{v}$  is supported on  $B$ , i.e.  $P\vec{v} = \vec{v}$ . Otherwise we may replace  $\vec{v}$  by  $P\vec{v}$ . This leaves the l.h.s. unchanged, and does not increase the r.h.s, since  $P$  is a contraction in  $l_2$ . Also, by linearity of both sides we may assume that  $\sum v_i = 1$  and so  $\vec{v}$  can be expressed as:  $P\vec{v} = \vec{v} = \vec{u} + \vec{z}$  where  $\vec{z}$  is orthogonal to  $\vec{u}$ . It follows that

$$P\hat{A}P\vec{v} = P\hat{A}\vec{u} + P\hat{A}\vec{z} = P\vec{u} + P\hat{A}\vec{z}$$

and hence

$$\|P\hat{A}P\vec{v}\|_2 \leq \|P\vec{u}\|_2 + \|P\hat{A}\vec{z}\|_2.$$

The claim follows by showing that  $\|P\vec{u}\|_2 \leq \beta \cdot \|\vec{v}\|_2$  and  $\|P\hat{A}\vec{z}\|_2 \leq \alpha \cdot \|\vec{v}\|_2$ .

Since  $\sum v_i = 1$ , and the support of  $\vec{v}$  has at most  $\beta n$  coordinates, Cauchy-Schwartz yields  $1 = \sum v_i \leq \sqrt{\beta n} \cdot \|\vec{v}\|_2$ . But, since  $\|P\vec{u}\|_2 = \sqrt{\beta/n}$ , we obtain

$$\|P\vec{u}\|_2 \leq \beta \cdot \|\vec{v}\|_2.$$

As for the other term,  $\|\hat{A}\vec{z}\|_2 \leq \alpha \|\vec{z}\|_2$  since  $z$  is orthogonal to  $\vec{u}$  and therefore a combination of eigenvectors of  $\hat{A}$  with eigenvalues  $\leq \alpha$ . But  $\|P\hat{A}\vec{z}\|_2 \leq \|\hat{A}\vec{z}\|_2$  since  $P$  is a contraction in  $l_2$ . Also  $\vec{v}$  is the sum of  $\vec{z}$  and the orthogonal vector  $\vec{u}$  whence  $\|\vec{z}\|_2 \leq \|\vec{v}\|_2$ . It follows that  $\|P\hat{A}\vec{z}\|_2 \leq \alpha \cdot \|\vec{v}\|_2$ , as needed.  $\square$

Now we use the two lemmas to prove Theorem 3.6:

*Proof.* (Theorem 3.6)

$$\begin{aligned} \|(P\hat{A})^t P\vec{u}\|_1 &\leq \sqrt{n} \cdot \|(P\hat{A})^t P\vec{u}\|_2 \\ &= \sqrt{n} \cdot \|(P\hat{A}P)^t \vec{u}\|_2 \\ &\leq \sqrt{n} \cdot (\beta + \alpha)^t \|\vec{u}\|_2 \\ &= (\beta + \alpha)^t. \end{aligned}$$

$\square$

The probability that  $t + 1$  uniformly and independently sampled vertices all land in a set  $B$  of density  $\beta$  is  $\beta^{t+1}$ . Is it true that  $\Pr[(B, t)]$  is very close to this bound for a sufficiently good expander? Theorem 3.6 falls a factor of  $\beta$  short of this bound. It also does not provide a comparable lower bound. The following result fills this gap.

**Theorem 3.9** ([AFWZ95]). *If  $\beta > 6\alpha$ . Then,*

$$\beta \cdot (\beta + 2\alpha)^t \geq \Pr[(B, t)] \geq \beta \cdot (\beta - 2\alpha)^t.$$

It is also possible to derive ‘‘time dependent’’ versions of the upper bound through simple adaptations of the proof of Theorem 3.6. Define the graph  $G^k$  as a graph on the same vertex set  $V = V(G)$ , where the number of edges connecting vertices  $x, y \in V$  is the number of length- $k$  walks in  $G$  from  $x$  to  $y$ . It is easy to check that if  $G$  is an  $(n, d, \alpha)$ -graph, then  $G^k$  is an  $(n, d^k, \alpha^k)$ -graph. Note that the above proof holds even if at different times steps are taken according to different graphs. In particular we can take steps on  $G^k$  which is equivalent to making  $k$  steps in  $G$ . This leads to:

**Theorem 3.10.** *For every subset  $K \subset \{0, \dots, t\}$ , and vertex subset  $B$  of density  $\beta$ :*

$$\Pr[X_i \in B \text{ for all } i \in K] \leq (\beta + \alpha)^{|K|-1}.$$

Occasionally we have to deal with a situation where the excluded set varies between time steps. This is addressed in the following theorem:

**Theorem 3.11.** *Let  $B_0, \dots, B_t$  be vertex sets of densities  $\beta_0, \dots, \beta_t$  in an  $(n, d, \alpha)$ -graph  $G$ . Let  $X_0, \dots, X_t$  be a random walk on  $G$ . Then:*

$$\Pr[X_i \in B_i \text{ for all } i] \leq \prod_{i=0}^{t-1} (\sqrt{\beta_i \beta_{i+1}} + \alpha).$$

This follows by a small adaptation of the previous arguments. We let  $P_i$  be the projection on  $B_i$ , and note that  $\Pr[X_i \in B_i \text{ for all } i] = \|\prod_{i=1}^t (P_i \hat{A}) P_0 \vec{u}\|_1$ . Also  $\|P_{i+1} \hat{A} P_i \vec{v}\|_2 \leq (\sqrt{\beta_i \beta_{i+1}} + \alpha) \cdot \|\vec{v}\|_2$ .

As before, this simple approach seems to give away a factor of  $\sqrt{\beta_0 \beta_t}$ , which is important for certain applications. For further discussion, see Alon-Feige-Wigderson-Zuckerman [AFWZ95], or Bilu-Hoory [BH04].

## 3.3 Applications

### 3.3.1 Amplifying the success probability of probabilistic algorithms

We now return to the computational problem of deciding membership in a language  $\mathcal{L}$  that belongs to the complexity class **RP**. Let  $A$  be an algorithm to decide membership in  $\mathcal{L}$  that is randomized with a one sided error. To decide whether a given input  $x$  belongs to  $\mathcal{L}$ , the algorithm samples a string  $r \in \{0, 1\}^k$  and computes in polynomial time a function  $f = f(x, r)$ . If  $x \in \mathcal{L}$  then  $f(x, r) = 1$  for any  $r$ . For  $x \notin \mathcal{L}$  the probability (over choices of  $r$ ) that  $f(x, r) = 1$  is at most  $\beta$ . Again our goal is to reduce the probability of error to below a given threshold, without a substantial increase in the number of random bits that are required. To this end, we construct an  $(n, d, \alpha)$ -graph  $G = (V, E)$ , with  $V = \{0, 1\}^k$ . For a given input  $x$  let  $B_x = B \subseteq \{0, 1\}^k$  be the set of all strings  $r$  for which algorithm  $A$  errs on input  $x$ . We now introduce another algorithm  $A'$  for the membership problem.

1. Pick a vertex  $v_0 \in V$  uniformly at random.
2. Start from it a length  $t$  random walk, say  $(v_0, v_1, \dots, v_t)$ .
3. Return  $\bigwedge_{i=0}^t A(x, v_i)$

By Theorem 3.6

$$\Pr[A' \text{ fails}] = \Pr[\forall i v_i \in B] \leq (\beta + \alpha)^t.$$

Compared with the algorithm from Chapter 1, the new algorithm achieves an exponential reduction in error probability, while the number of random bits used is only  $m + t \log d = m + O(t)$ .

What if our basic algorithm can err on every input not just when  $x \notin \mathcal{L}$ ? We show that the same trick applies as well to algorithms which make two-sided errors, to similarly amplify the probability of success. We say that the language  $\mathcal{L}$  belongs to the complexity class **BPP** if there is a polynomial time randomized algorithms  $A$  to decide whether a given input  $x$  belongs to  $\mathcal{L}$ . It is assumed that for every  $x$  (either in or out of  $\mathcal{L}$ )  $A$  errs with probability  $\leq \beta \leq \frac{1}{10}$ . To reduce our probability of error we can run  $A$  on  $t$  independently sampled random strings and take a *majority* vote. It is a simple consequence of the Chernoff bound that the resulting error probability decreases exponentially with  $t$ . To save on randomness we again use expander walks.

As before we assume that  $A$  uses  $k$  random bits and we employ an  $(n, d, \alpha)$ -graph on vertex set  $V = \{0, 1\}^k$ . Again we let  $B_x = B \subset V$  be the collection of all random strings for which the algorithm  $A$  errs on input  $x$ . Our modified algorithm  $A'$  works as follows:

1. Pick a vertex  $v_0 \in V$  uniformly at random.
2. Start from it a length  $t$  random walk, say  $(v_0, \dots, v_t)$ .
3. Return *majority* $\{A(x, v_i)\}$ .

$A'$  fails iff a majority of the  $v_i$ 's belong to  $B$ . Fix a set of indices  $K \subset \{0, 1, \dots, t\}$  of cardinality  $|K| \geq (t + 1)/2$ . By Theorem 3.10

$$\Pr[v_i \in B \text{ for all } i \in K] \leq (\beta + \alpha)^{|K|-1} \leq (\beta + \alpha)^{(t-1)/2}.$$

We will assume that  $\alpha + \beta \leq 1/8$ , and apply the union bound on  $K$ , to deduce that:

$$\Pr[A' \text{ fails}] \leq 2^t \cdot (\beta + \alpha)^{(t-1)/2} = O\left(2^{-t/2}\right).$$

So here too we achieve an exponential reduction of the error probability using only  $m + O(t)$  random bits. In the following table we collect the main parameters of the various techniques presented for error reduction:

Method	Error Probability	No. of random bits
Randomized algorithm $A$	$1/10$	$m$
$t$ independent repetitions of $A$	$2^{-t}$	$t \cdot m$
Sampling a point and its neighbors in an $(n, t, 1/\sqrt{t})$ -graph.	$1/t$	$m$
A random walk of length $t$ on an $(n, d, 1/40)$ -graph	$2^{-t/2}$	$m + O(t)$

## Further progress and reading

The exact form of the exponential decay in error using expander walks and its dependence on the spectral gap was found by Gillman [Gil98], and is a natural optimal generalization of the Chernoff bound for independent sampling.

It is easy to see that a good approximation to the probability of hitting any set (event) in the space actually gives a good sampler for the average of any real function, and indeed expander walks are used for that purpose (for a good survey on randomness efficient samplers, see [Gol97]).

Recently it was shown by Wigderson and Xiao [WX05] that expander walks can actually serve as randomness efficient samplers even for matrix-valued functions, extending the Chernoff-like bound for such functions of [AW02].

### 3.3.2 Hardness of approximating maximum clique size

We now turn to other applications of random walks on expanders to computational complexity, namely to the hardness of approximation of the clique problem. We first give the necessary background on hardness of approximation.

Let us recall that a *clique* in a graph  $G$  is a subset  $S \subseteq V(G)$  every two of which are adjacent to each other. The *clique number*  $\omega(G)$  of a graph  $G$  is defined as the largest cardinality of a clique in  $G$ . An important computational problem is estimating this parameter. No efficient (or even subexponential time) algorithm is known for this problem. The discussion below explains why it is unlikely that such an algorithm exists.

Among the great achievements of theoretical computer science in the 70's and the 80's was the discovery of numerous natural decision problems that are **NP**-hard. The determination of the clique number is among these problems. We are given a graph  $G$  and an integer  $k$  and are asked to determine whether  $\omega(G) \geq k$ . The fact that this problem is **NP**-hard means that if there is a polynomial-time algorithm to solve the clique then **P** = **NP**. This is considered a very unlikely conclusion, as it implies that *every* problem in **NP** has a polynomial time algorithm. That conclusion is considered so unlikely, since hundreds of these problems have been attacked for decades by many scientists and engineers for their practical importance, in vain.

A major challenge that emerged from these pioneering studies in computational complexity is: For hard *optimization* problems, how difficult is it to find an *approximate* solution? A breakthrough in the study of this fundamental problem was made with the proof of the **PCP** Theorem, in [AS98, ALM<sup>+</sup>98]. The connection between such theorems and hardness of approximation was established in [FGL<sup>+</sup>91]. We are unable to go into this fascinating subject at any length and refer the reader to surveys by Sudan [Sud04] and by Arora-Lund [AL96].

As mentioned above, in a classical paper, Karp [Kar72] showed that it is **NP**-hard to determine exactly the clique number of a given graph. An early triumph of the **PCP** theory was the proof that it is **NP**-hard even to approximate the clique number to within any constant factor.<sup>1</sup>

**Theorem 3.12 (Feige-Goldwasser-Lovasz-Safra-Szegedy [FGL<sup>+</sup>91]).** *There are two constants  $1 > \delta_1 > \delta_2 > 0$  such that it is **NP**-hard to decide for a given  $n$ -vertex graph  $G$  whether  $\omega(G) \leq \delta_2 n$  or  $\omega(G) \geq \delta_1 n$ .*

In this section we will show that even obtaining a *very rough* approximation for  $\omega(G)$  is **NP**-hard. That is, we will show that it is **NP**-hard to approximate  $\omega(G)$  even within a factor of  $n^\epsilon$  for some fixed  $\epsilon > 0$ . Specifically we show the following theorem.

**Theorem 3.13.** *If there exists a polynomial-time algorithm  $A$  whose output on every  $n$ -vertex graph  $G$  satisfies  $n^{-\epsilon} \leq A(G)/\omega(G) \leq n^\epsilon$ , then **NP** = **P**. Here  $\epsilon > 0$  is some absolute constant.*

This theorem was proved by [ALM<sup>+</sup>98]. The proof we present will follow [AFWZ95]. It assumes an algorithm  $A$  as in the theorem, and creates from it an efficient algorithm  $B$  for the problem of approximating clique to within a constant factor. Such a conversion is called a *reduction*. If the resulting algorithm  $B$  is deterministic, then we can use Theorem 3.12 to conclude that **P** = **NP**.

We will first present a *probabilistic reduction* (due to Berman and Schnitger [BS92]) between the two problems, namely an algorithm  $B$  which is probabilistic polynomial time. Note that with this weaker reduction, the assumption of Theorem 3.12 only implies a probabilistic polynomial time algorithm for all problems in **NP** (or **RP** = **NP** in complexity theoretic lingo). After describing this probabilistic reduction, we will show how to eliminate the randomness from the algorithm  $B$ , resulting in a deterministic algorithm  $B'$ . For this we will again employ walks on expander

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<sup>1</sup>In fact, the simplest form of the **PCP** Theorem is almost equivalent to this statement.

graphs. This will prove the conclusion  $\mathbf{P} = \mathbf{NP}$ , and thus the fact that approximating the clique number to within  $n^\epsilon$  is  $\mathbf{NP}$ -hard.

We should note that a much stronger hardness result is known about approximating the clique number, due to Håstad [Hås99] (which proof requires much more advanced techniques). He showed that efficiently approximating  $\omega(G)$  to within  $n^{1-\delta}$ , for any  $\delta > 0$ , implies that  $\mathbf{NP} = \mathbf{RP}$  (via a probabilistic reduction). This was very recently derandomized by Zuckerman [Zuc05], again via more difficult techniques than mere expander walks, to yield that even such approximation is actually  $\mathbf{NP}$ -hard. Note that since a factor- $n$  approximation is trivial, this result is surprisingly tight.

### The probabilistic reduction

The randomized reduction of [BS92] yields a version of Theorem 3.13 where the same assumptions lead to a weaker conclusion:

**Lemma 3.14 (Theorem 3.13; weak version).** *If there exists a polynomial-time algorithm  $A$  whose output on every  $n$ -vertex graph  $G$  satisfies  $n^{-\epsilon} \leq A(G)/\omega(G) \leq n^\epsilon$ , then  $\mathbf{NP} \subseteq \mathbf{RP}$ . Here  $\epsilon > 0$  is some absolute constant.*

The proof follows by converting the algorithm  $A$  into a probabilistic polynomial-time algorithm  $B$  that can solve the decision problem considered in Theorem 3.12. This shows that  $\mathbf{NP} \subseteq \mathbf{RP}$ .<sup>2</sup>

In order to apply algorithm  $B$  to given a  $n$ -vertex graph  $G = (V, E)$ , we consider a graph  $H$ , with vertex set  $V^t$ , where  $t = \log n$ . The vertices  $(v_1, \dots, v_t)$  and  $(u_1, \dots, u_t)$  in  $H$  are adjacent if the subgraph of  $G$  induced by the set  $\{v_1, \dots, v_t\} \cup \{u_1, \dots, u_t\}$  is a clique. Whether  $\omega(G)$  is below  $\delta_2 n$  or above  $\delta_1 n$ , this is significantly amplified in  $H$ . The amplification is so strong that a random subset of  $m = \text{poly}(n)$  vertices in  $H$  tends to behave very differently with respect to the clique number of the induced graph.

So here is what algorithm  $B$  does on input  $G = (V, E)$ .

1. Generate  $m$  vertices from  $V^t$  and compute the subgraph  $H'$  of  $H$  induced on this set of vertices.
2. Apply algorithm  $A$  to  $H'$ .
3. Algorithm  $B$  returns 1 if  $A(H') > \frac{1}{2}\delta_1^t m$ , and otherwise it returns 0.

We need the following simple combinatorial observation.

**Claim 3.14.1.** *Every clique in  $H$  is contained in a clique of the form  $S^t$  where  $S$  is an inclusion-maximal clique in  $G$ . In particular,  $\omega(H) = \omega(G)^t$ .*

*Proof.* Clearly if  $S$  is a clique in  $G$  then the set  $S^t$  is a clique in  $H$ , and in particular  $\omega(H) \geq \omega(G)^t$ . On the other hand, consider some clique  $S'$  in  $H$ . Let  $S \subseteq V(G)$  be the set of those vertices in  $G$  that appear as an entry in any of the  $t$ -tuples in  $S'$ . Clearly  $S$  forms a clique in  $G$ , and  $|S'| \leq |S|^t$ , whence also  $\omega(H) \leq \omega(G)^t$ .  $\square$

We need to show two things:

1. If  $\omega(G) \geq \delta_1 n$  then almost surely  $\omega(H') \geq \frac{1}{2}\delta_1^t m$ ,
2. If  $\omega(G) \leq \delta_2 n$  then almost surely  $\omega(H') \leq 2\delta_2^t m$ .

With a proper choice of  $m = \text{poly}(n)$  and  $t = \log n$  this proves the Lemma.

For the first claim, consider a clique  $Q$  in  $H$  of size  $\omega(H) = \omega(G)^t \geq (\delta_1 n)^t$ . The expected number of vertices from  $Q$  in  $H'$  is  $|Q| \cdot \frac{|V(H')|}{|V(H)|} \geq \delta_1^t m$ . By the Chernoff bound,<sup>3</sup> with high probability this intersection is at least  $\frac{1}{2}\delta_1^t m$  as stated.

<sup>2</sup>This, in fact only shows  $\mathbf{NP} \subseteq \mathbf{BPP}$ , but it a standard fact (using the self-reducibility of  $\mathbf{NP}$ -complete problems), that the inclusion  $\mathbf{NP} \subseteq \mathbf{BPP}$  actually implies the stronger conclusion  $\mathbf{NP} \subseteq \mathbf{RP}$ .

<sup>3</sup>This is an upper bound on the tail of a binomial distribution. Let  $Z$  be  $\text{Binomial}(N, p)$ , i.e. the sum of  $N$  independent 0-1 random variables that are one with probability  $p$ . Then  $\Pr[|Z - E[Z]| < \Delta \cdot E[Z]] < 2 \exp(-Np\Delta^2/3)$ , for all  $1 > \delta > 0$ .

For the other claim we need to show that it is very unlikely that the  $m$  vertices we sample from  $H$  include a large clique. For this analysis it suffices to consider subsets of *inclusion-maximal* cliques  $Q$  in  $H$ , of which, by Claim 3.14.1, there are at most  $2^n$ . The cardinality of  $Q$  does not exceed  $(\delta_2 n)^t$  and so we expect to sample  $\frac{|Q|m}{n^t} < \delta_2^t m$  vertices from  $Q$ . We consider it a *failure* if we sample more than  $2 \cdot \delta_2^t m$  vertices from  $Q$ . Again by Chernoff's bound, our failure probability does not exceed  $\exp(-\Omega(m\delta_2^t))$ . As mentioned, there are at most  $2^n$  inclusion-maximal cliques in  $H$ , and so our total probability of failure is still  $o(1)$ , provided that  $m\delta_2^t \gg n$ . This can be guaranteed by a proper choice of  $m = \text{poly}(n)$ .

### The deterministic reduction

Again we assume there exists a polynomial-time algorithm  $A$  distinguishing between the two cases of Theorem 3.13. This time we use  $A$  to derive a *deterministic* polynomial-time algorithm  $B'$  that can distinguish between the two cases of Theorem 3.12, whence  $\mathbf{NP} = \mathbf{P}$ .

The only difference between Algorithm  $B'$  and Algorithm  $B$ , is that Algorithm  $B'$  uses a *derandomized sampling* to construct the graph  $H'$ . This is done as follows. We introduce some  $(n, d, \alpha)$ -expander  $\mathcal{G}$  on the same vertex set as  $G$ . In order to select the vertices in  $H'$ , we no longer take a random sample of  $t$ -tuples from  $V(G)^t$ . Rather we consider all  $t$ -tuples representing a *length*  $(t-1)$  walk in the graph  $\mathcal{G}$ . The resulting graph  $H'$  has  $m = nd^{t-1}$  vertices. Since  $d$  is fixed and  $t = \Theta(\log n)$ , it follows that  $m$  is polynomial in  $n$ . We are already familiar with the idea that length  $(t-1)$  random walks on  $\mathcal{G}$  should behave like random  $t$ -tuple in  $|V|^t$ . We need to establish this principle in the present context as well.

**Claim 3.15.** *If  $\omega(G) \leq \delta_2 n$ , then  $\omega(H') \leq (\delta_2 + 2\alpha)^t m$*

*Proof.* As before, a clique in  $H'$  corresponds to all length- $(t-1)$  walks in  $\mathcal{G}$  that are confined to some clique in  $G$ . Consequently,  $\omega(H')/m$  is the largest probability that such a walk is confined to some clique  $S$  in  $G$  (i.e., the maximum of such a probability over all choices of a clique  $S$ ). By assumption  $|S| \leq \omega(G) \leq \delta_2 n$ , and our claim follows now by Theorem 3.9.  $\square$

The complementary statement that we need is:

**Claim 3.16.** *If  $\omega(G) \geq \delta_1 n$ , then  $\omega(H') \geq (\delta_1 - 2\alpha)^t m$ .*

*Proof.* Let  $S$  be a clique in  $G$  of cardinality  $|S| \geq \delta_1 n$ . By Theorem 3.9 a random walk of length  $(t-1)$  in  $\mathcal{G}$  remains confined to  $S$  with probability  $(\delta_1 - 2\alpha)^t$ . The conclusion follows.  $\square$

The rest of the proof follows as above.



## Chapter 4

# A Geometric View of Expander Graphs

An attractive feature of expander graphs is that they can be viewed from many different angles. As we saw already, these combinatorial objects have a variety of applications in the design of algorithms and in computational complexity. The relationship between expansion and the spectral gap adds an algebraic perspective to the picture. Probabilistic considerations arise in the study of rapidly mixing random walks on graphs. In the present chapter we investigate some geometric aspects of expander graphs.<sup>1</sup>

### 4.1 The Classical Isoperimetric Problem

The fact that a given graph  $G$  is a good expander is equivalent to the statement that  $G$  satisfies a certain discrete isoperimetric inequality. Let us quickly recall the grandmother of all isoperimetric inequalities which goes back to the ancient Greeks:

**Problem 4.1.** *Of all simple closed curves in the plane of a given length, which curve encloses the greatest area?*

Already the ancient Greeks had no doubt about the correct answer, namely that the optimal curve is the circle. Proving this statement was a different matter altogether. The first rigorous proof was claimed by Jacob Steiner (1841), based on the so-called *Steiner Symmetrization*. A flaw in Steiner's argument was pointed out by Weierstrass, but several valid proofs were found shortly afterwards. Some of these proofs do rely on Steiner's original ideas. As we show below, analogous ideas are very useful in the study of isoperimetric inequalities on graphs and in particular in recent work [LL05] on the Margulis expander (Viz. Section 2.2).

So here is the idea of Steiner Symmetrization. We first observe that there is no loss of generality in considering only convex domains, for the convex hull of a nonconvex closed planar domain  $K$  has a larger area and a smaller circumference than  $K$  itself. So given that  $K$  is a compact convex set, let us symmetrize it as follows. Let  $l$  be some line that divides  $K$  into two parts  $K_1, K_2$  of equal area. If the circumference of  $K_1$  is not more than that of  $K_2$ , then we consider  $K'$ , the union of  $K_1$  with its reflection around  $l$ . This transformation preserves the area and does not increase the circumference. Consequently the circumference of an optimal<sup>2</sup>  $K$  must be invariant under this operation. With a little extra work, one shows that an optimal  $K$  must, in fact, be invariant under such a symmetrization step. From here it's only a short route to proving that the optimal  $K$  is a disc. Let us point out that the subtle points concerning the existence of an optimum do not bother us in the realm of finite graphs. For a general survey of geometric isoperimetric problems, see the book by Burago and Zalgaller [BZ88]. Siegel [Sie] provides a historical review.

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<sup>1</sup>Just for the record - this is only a partial list of topics that are related to expander graphs. E.g., most of the known explicit constructions of expander graphs depend on deep ideas from number theory and representation theory.

<sup>2</sup>... should the optimum exist. This subtle issue was pointed out by Weierstrass.

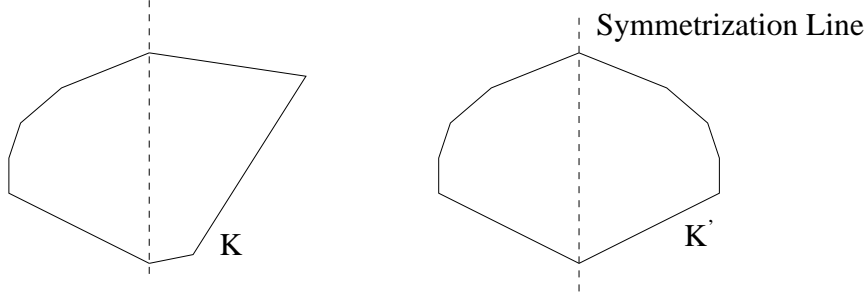


Figure 4.1: Steiner Symmetrization

## 4.2 Graph Isoperimetric problems

In the spirit of the classical isoperimetric problem, one can consider analogous problems in graphs rather than in Euclidean spaces. In this analogy the “area” of a set of vertices  $S$  is its cardinality  $|S|$ . There are two main natural discrete analogs for the “circumference” of  $S$ . We can consider the number of edges going out from  $S$  or the number of vertices outside  $S$  that have a neighbor in it. This leads us to define the two basic isoperimetric parameters for a given  $n$ -vertex graph  $G$  and an integer  $n > k \geq 1$ :

**Definition 4.2.** *The edge isoperimetric parameter:*

$$\Phi_E(G, k) = \min_{SCV} \{|E(S, \bar{S})| : |S| = k\}$$

**Definition 4.3.** *The vertex isoperimetric parameter<sup>3</sup>:*

$$\Phi_V(G, k) = \min_{SCV} \{|\Gamma(S) \setminus S| : |S| = k\}$$

It is desirable to completely understand these parameters for natural families of graphs (see below). It is also of great interest to determine or estimate these parameters for given  $G$  and  $k$ . In this generality, these computational problems are difficult (co-NP-hard). Much (and still ongoing) research was dedicated to the search for efficient algorithms to approximate these quantities. Deep and surprising connections were found between these problems and the subject of Metric Embeddings, see Chapter 13.

We illustrate these concepts with an important family of graphs for which the two basic isoperimetric problems are completely solved. The  $d$ -dimensional discrete cube  $Q_d$ , is a graph on vertex set  $V(Q_d) = \{0, 1\}^d$ . Two vertices  $v_1, v_2 \in \{0, 1\}^d$  are adjacent if these two vectors differ in exactly one coordinate. The graph of the  $d$ -dimensional cube plays a major role in many parts of discrete mathematics, and both the vertex and the edge-isoperimetric inequalities on the  $d$ -cube are useful in a variety of applications.

### 4.2.1 Example: The discrete cube

As stated above, both the vertex and edge-isoperimetric parameters for the  $d$ -cube are known for every  $d$  and  $2^d > k > 0$ . To reduce the technicalities we state these results only for certain values of  $k$  and in an asymptotically tight form. A thorough survey of these and related problems can be found in [Bol86].

- $\Phi_E(Q_d, k) \geq k(d - \log_2 k)$ . This bound holds with equality when  $k$  is a power of two,  $k = 2^l$ . In this case equality holds for  $S$  that is the set of vertices of an  $l$ -dimensional subcube.

<sup>3</sup>This parameter counts the number of vertices in  $\bar{S}$  with a neighbor in  $S$ . Other interesting variants may be defined, e.g. (i) The number of vertices in  $S$  with a neighbor in  $\bar{S}$ , or (ii) The more symmetric  $|\{x \in S | x \text{ has some neighbors in } \bar{S}\}| + |\{y \in \bar{S} | y \text{ has some neighbors in } S\}|$ . However, the present definition seems to be the most useful for applications.

- If  $k = \binom{d}{0} + \binom{d}{1} + \dots + \binom{d}{r}$  (for some integer  $r$ ), then the vertex isoperimetric parameter  $\Phi_V(Q_d, k) = \binom{d}{r+1}$ . Equality is achieved for the set  $S$  that is a ball of radius  $r$  around some vertex  $v_0$ , namely  $\{v \in \{0, 1\}^d : d_H(v, v_0) \leq r\}$ . (Recall that  $d_H(u, v)$ , the Hamming distance between  $u, v \in \{0, 1\}^d$  is the number of coordinates on which  $u, v$  differ.)

### 4.3 The Margulis construction

Margulis [Mar73] gave the first explicit construction of an infinite family of expander graphs. Chapter 8 is devoted to this construction and its analysis. Here we study it from a geometric point of view, by considering its infinite analog in the way suggested by Gabber and Galil [GG81]. The vertex set of this infinite graph is the unit square (or torus)  $I \times I$ , where  $I$  is the half-open interval  $[0, 1)$ . The edges are defined by two linear transformations:

$$T(x, y) \rightarrow (x + y, y) \bmod 1, \quad S(x, y) \rightarrow (x, x + y) \bmod 1.$$

The neighbors of a point  $(x, y)$  are the points:  $T(x, y), S(x, y), T^{-1}(x, y), S^{-1}(x, y)$ . Thus the graph is 4-regular. The expansion property of this graph is described by the following theorem:

**Theorem 4.4. (Margulis [Mar73], Gabber-Galil [GG81])**

*There exists an explicit  $\epsilon > 0$  such that for any measurable set  $A \subset I \times I$  of Lebesgue measure  $\mu(A) \leq \frac{1}{2}$ ,*

$$\mu(\Gamma(A) \cup A) \geq (1 + \epsilon)\mu(A),$$

where  $\Gamma(A) = S(A) \cup T(A) \cup S^{-1}(A) \cup T^{-1}(A)$  is the neighbor set of  $A$ .

It is natural to conjecture which sets are extremal for the isoperimetric problem in this graph.

**Conjecture 4.5. (Linal)** *For every  $A \subset [0, 1]^2$  of Lebesgue measure  $\mu(A) \leq 1/2$ ,*

$$\mu(S(A) \cup T(A) \cup S^{-1}(A) \cup T^{-1}(A)) \geq 2\mu(A).$$

If true, this bound is clearly tight. If  $A = \{(x, y) : |x| + |y| < t\}$ , then  $\Gamma(A) = \{(x, y) : |x|, |y| < t\}$ , and  $\mu(\Gamma(A)) = 2\mu(A)$ .

The analogous (and weaker) version of this conjecture concerning the transformations  $\tilde{T}(x, y) \rightarrow (x + y, y)$ , and  $\tilde{S}(x, y) \rightarrow (x, x + y)$  (no mod 1 here, namely the ground set is no longer the unit torus but rather the Euclidean plane  $\mathbb{R}^2$ ) was recently proved by Linal and London [LL05]. Their proof is very brief and completely elementary. It is based on the idea of symmetrization described above.

#### 4.3.1 The Discrete Laplacian

In classical vector analysis, the Laplace operator is defined as  $\Delta(f) = \text{div}(\text{grad}(f))$ . It turns out that it has an analogue, the discrete Laplacian which is natural and useful for several reasons. To present this analogy, let us begin by introducing the discrete analogs for the gradient, and divergence operators in graphs. The correct definition for the Laplacian will then be apparent. Given an undirected graph  $G = (V, E)$ , we fix an arbitrary orientation of the edges. (The specific choice of the orientation does not affect anything in our discussion.) Let  $K$  be the  $V \times E$  incidence matrix of  $G$  where the entry

$$K_{u,e} = \begin{cases} +1 & \text{if the edge } e \text{ exits the vertex } u \\ -1 & \text{if the edge } e \text{ enters the vertex } u \\ 0 & \text{otherwise} \end{cases}$$

We now define:

**The gradient:** Let  $f : V \rightarrow \mathbb{R}$  be a function on the vertices of  $G$  which we view as a row vector indexed by  $V$ . The gradient operator maps  $f$  to  $fK$ , a vector indexed by  $E$ . The gradient measures the change of  $f$  along the edges of the graph. If  $e$  is the edge from  $u$  to  $v$ , then  $(fK)_e = f_u - f_v$ .

**The divergence:** Let  $g : E \rightarrow \mathbb{R}$  be a function on the edges of  $G$ . The divergence operator maps  $g$ , considered as a column vector indexed by  $E$ , to  $Kg$ , a vector indexed by  $V$ . If we think of  $g$  as describing a flow, then its divergence at a vertex is the net outbound flow. Namely,

$$(Kg)_v = \sum_{e \text{ exits } v} g_e - \sum_{e \text{ enters } v} g_e.$$

**The Laplacian:** In order to maintain the analogy with the real Laplacian, the discrete Laplacian should map  $f$  to  $KK^t f$ , where  $f : V \rightarrow \mathbb{R}$ . The matrix  $L = L_G = KK^t$  is accordingly called the (discrete) Laplacian of  $G$ .<sup>4</sup> A simple calculation shows that  $L$  is the following symmetric matrix with rows and columns indexed by  $V$ :

$$L_{u,v} = \begin{cases} -1 & (u,v) \in E \\ \text{deg}(u) & u = v \end{cases}$$

One can easily deduce the following equality:

$$fLf^T = fKK^T f^T = \|fK\|^2 = \sum_{(u,v) \in E} (f(u) - f(v))^2, \quad (4.1)$$

where  $\|\cdot\|$  denotes the  $l_2$  norm.

In particular:

**Proposition 4.6.** *For every graph  $G$ , the matrix  $L_G$  is positive semidefinite. Its smallest eigenvalue is zero, and the corresponding eigenfunction is the constant function.*

Though it is possible to develop some of the theory for general (irregular) graphs, the regular case is simpler to state and analyze:

**Lemma 4.7.** *The Laplacian of a  $d$ -regular graph  $G$  satisfies:*

- $L = L_G = dI - A_G$ , where  $A_G$  is  $G$ 's adjacency matrix.
- The spectrum of  $L$  is in  $[0, +2d]$  (since the spectrum of  $A_G$  is in  $[-d, +d]$ ).
- The smallest eigenvalue of  $L$  is zero.
- The spectral gap of  $G$ , namely  $\lambda_1(A_G) - \lambda_2(A_G)$  equals the smallest positive eigenvalue of  $L$ .

## 4.4 The Cheeger constant and inequality

Many of the things we do here for graphs had been studied previously in the geometric framework of Riemannian manifolds. In particular, the Cheeger constant that we introduce now, captures a notion of "expansion" in this geometric context. The geometry underlying this theory is that of an  $n$ -dimensional Riemannian manifold. We skip any formal definitions here, and refer the reader to Buser's book [Bus82]. Suffices it to say that this is a space that looks locally like  $\mathbb{R}^n$ . It also carries a differentiable structure with a smoothly varying notion of inner product among tangent vectors. This allows us to carry out the familiar operations from calculus, compute volumes and distances. We now turn to define the continuous analog of edge expansion for Manifolds: the Cheeger constant.

**Definition 4.8.** The **Cheeger constant** of a compact  $n$ -dimensional Riemannian manifold  $M$  is:<sup>5</sup>

$$h(M) = \inf_A \mu_{n-1}(\partial A) / \min(\mu_n(A), \mu_n(M \setminus A)),$$

where  $A$  runs over all open subsets of  $M$ , and  $\partial A$  is  $A$ 's boundary. The  $n$  and  $n-1$  dimensional measures are denoted  $\mu_n$  and  $\mu_{n-1}$ .

<sup>4</sup>In the graph theory literature it is also called the Tutte Matrix of  $G$ , and appears in contexts such as the Matrix-Tree Theorem and randomized algorithms for matchings in graphs.

<sup>5</sup>The original definition of Cheeger's constant was in a slightly different context, and did not involve  $A^c$ .

The analogy with the definition of edge expansion should be obvious: We partition  $M$  into  $A$  and its complement, and consider the ratio between two quantities: (i) The  $((n - 1)$ -dimensional) measure of  $A$ 's boundary, (ii) The minimum between  $A$ 's measure and the measure of  $A$ 's complement. These measures are  $n$ -dimensional.

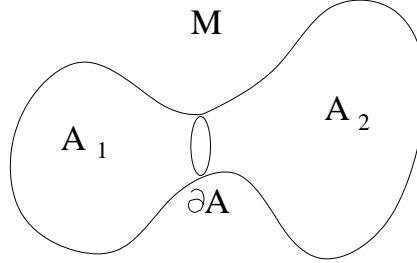


Figure 4.2: The sets  $A$ ,  $A^c$  and  $\partial A$  for some Riemannian manifold  $M$

An intuitive demonstration of the definition is illustrated in figure 4.2. Here is a “dictionary” to move between the theory for graphs and the geometric counterpart.

$$\begin{array}{ccc}
 M & & G \\
 \partial A & & E(S, \bar{S}) \\
 A, M \setminus A & \iff & S, \bar{S} \\
 \mu_{n-1}(\partial A) & & |E(S, \bar{S})| \\
 \min(\mu_n(A), \mu_n(M \setminus A)) & & \min\{|S|, |\bar{S}|\}
 \end{array}$$

As hinted above, it is possible to develop everything in differential calculus in the broader context of Riemannian manifolds. In particular, associated with any Riemannian manifold  $M$  is the Laplacian, a linear differential operator defined on real functions  $f : M \rightarrow \mathbb{R}$ . It is defined in the familiar way via  $\Delta(f) = \text{div}(\text{grad}(f))$ . If  $\Delta f = \lambda f$ , we say that  $f$  is an eigenfunction of the Laplacian with eigenvalue  $\lambda$ . It can be shown that all eigenvalues of  $\Delta$  are non-negative, and that its lowest eigenvalue is zero, corresponding to the constant eigenfunction.<sup>6</sup> A fundamental theme in this area is the connection between expansion (Cheeger constant) and the spectrum of the Laplacian.

**Theorem 4.9 (Cheeger [Che70]).** *Let  $M$  be a compact Riemannian manifold, and let  $\lambda$  be the smallest positive eigenvalue of its Laplacian. Then  $\lambda \geq h^2/4$ .*

Below we will derive the discrete analog of this theorem.

## 4.5 Expansion and the spectral gap

We turn to prove a discrete analogue of Theorem 4.9, namely the qualitative equivalence between expansion and spectral gaps in graphs. This theorem was already stated without proof in Chapter 2. We restate the theorem and prove it here. Recall first the definition of (edge) expansion:

**Definition 4.10.** The edge expansion ratio of a graph  $G = (V, E)$ , is

$$h(G) = \min_{S \subseteq V, |S| \leq |V|/2} \frac{|E(S, \bar{S})|}{|S|}.$$

**Theorem 4.11.** *Let  $G = (V, E)$  be a finite, connected,  $d$ -regular graph and let  $\lambda$  be its second eigenvalue. Then*

$$\frac{d - \lambda}{2} \leq h(G) \leq \sqrt{2d(d - \lambda)}$$

<sup>6</sup>The standard definition of eigenvalues and eigenfunctions of the Laplacian is usually made through the variational principle. For compact manifolds, the infimum value of each Rayleigh quotient is attained by a corresponding eigenfunction.

This theorem was proved by Dodziuk [Dod84], and independently by Alon-Milman [AM85], and Alon [Alo86].

This theorem was also generalized in several ways, all beyond the scope of this manuscript. We just mention the most useful generalization, to general *reversible* Markov chains. For these, one can define an weighted analog of edge expansion called *conductance*, and give similar bounds on it in terms of the spectral gap of the chain. This was done by Jerrum and Sinclair [JS89], and had a huge impact on the analysis of convergence of "Monte Carlo" algorithms, used extensively in statistical physics and optimization (see [JS96] for a survey).

The proof of Theorem 4.11 is given in the next two subsections.

### 4.5.1 Large spectral gap implies high expansion

The proof is similar to that of the Expander Mixing Lemma 2.5, but using the fact that the two sets are complementary. The first eigenvector of a regular graph is the constant  $\vec{1}$ , so we can prove that  $\lambda \geq d - 2h(G)$ , by exhibiting a vector  $f \perp \vec{1}$  for which  $fAf^t/\|f\|^2 \geq d - 2h(G)$ . The vector we consider is  $f = |\bar{S}|1_S - |S|1_{\bar{S}}$ , where  $1_X$  denotes the characteristic vector of the set  $X$ . Here  $S$  is a set satisfying  $h(G) = |E(S, \bar{S})|/|S|$ , and  $|S| \leq n/2$ . Let us evaluate **the Rayleigh quotient**:

$$\begin{aligned} \|f\|^2 &= |\bar{S}|^2|S| + |S|^2|\bar{S}| = |S||\bar{S}|(|S| + |\bar{S}|) = n|S||\bar{S}|, \\ fAf^t &= 2(|E(S)||\bar{S}|^2 + |E(\bar{S})||S|^2 - |S||\bar{S}||E(S, \bar{S})|). \end{aligned}$$

Since  $G$  is  $d$ -regular, we can substitute

$$\begin{aligned} 2|E(S)| &= d|S| - |E(S, \bar{S})| \\ 2|E(\bar{S})| &= d|\bar{S}| - |E(S, \bar{S})|. \end{aligned}$$

We put it all together to conclude that

$$\lambda \geq \frac{fAf^t}{\|f\|^2} = \frac{nd|S||\bar{S}| - n^2|E(S, \bar{S})|}{n|S||\bar{S}|} = d - \frac{n|E(S, \bar{S})|}{|S||\bar{S}|} \geq d - 2h(G).$$

The last inequality follows from the fact that  $h(G) = |E(S, \bar{S})|/|S|$ , and  $|\bar{S}| \geq n/2$ .

### 4.5.2 High expansion implies large spectral gap

This is the more difficult (and interesting) direction. Let  $g$  be the eigenvector associated with  $\lambda_2$ . Based on the knowledge of  $g$ , we seek a cut with relatively few edges. Had it been the case that the function  $g$  takes only two values, the obvious thing is to partition the vertices according to the two values of  $g$ . In this idealized case, we need not give away a square in the inequality. For a general  $g$ , we partition according to the *sign* of  $g$  (Since  $g \perp \vec{1}$  it has both positive and negative entries). We then need to bound the edge expansion of the corresponding cut.

We define  $f = g^+$ , and  $V^+ = \text{supp}(f)$ , i.e.  $f_v = \max(g_v, 0)$ , and  $V^+ = \{v : f_v > 0\}$ . Without loss of generality,  $V^+$  contains at most  $n/2$  vertices, (or else we consider  $-g$  which is also an eigenvector with the same eigenvalue  $\lambda$ ). The result is obtained by considering the Rayleigh quotient  $fLf^T/\|f\|^2$ , and proving that (i)  $fLf^T/\|f\|^2 \leq d - \lambda$ , and (ii)  $h^2/2d \leq fLf^T/\|f\|^2$ .

We start by proving (i). First observe that for  $x \in V^+$  we can write

$$\begin{aligned} (Lf)_x &= df_x - \sum_{y \in V} a_{xy}f_y = dg_x - \sum_{y \in V^+} a_{xy}g_y \\ &\leq dg_x - \sum_{y \in V} a_{xy}g_y = (Lg)_x = (d - \lambda) \cdot g_x. \end{aligned}$$

As  $f_x = 0$  for  $x \notin V^+$  we obtain that

$$fLf^T = \sum_{x \in V} f_x \cdot (Lf)_x \leq (d - \lambda) \sum_{x \in V^+} g_x^2 = (d - \lambda) \sum_{x \in V} f_x^2 = (d - \lambda)\|f\|^2.$$

It remains to prove (ii). To this end, we introduce yet another quantity:

$$B_f = \sum_{(x,y) \in E} |f_x^2 - f_y^2|,$$

and prove that

$$h \cdot \|f\|^2 \leq B_f \leq \sqrt{2d} \cdot \|fK\| \cdot \|f\|, \quad (4.2)$$

which clearly yields (ii), as  $\|fK\|^2 = fLf^T$ .

For ease of notation, we label the  $n$  vertices of  $G$  by  $1, \dots, n$ , so that  $f_1 \geq f_2 \geq \dots \geq f_n$ , and denote  $[i] = \{1, \dots, i\}$ . The upper and lower bounds on  $B_f$  are proved in the following two lemmas. The upper bound is obtained using the Cauchy-Schwartz inequality, while the lower bound is obtained by considering the expansion of level sets,  $[i]$  for  $i \in V^+$ .

**Lemma 4.12.**  $B_f \leq \sqrt{2d} \cdot \|fK\| \cdot \|f\|$ .

*Proof.* Using the Cauchy-Schwartz inequality, we have

$$\begin{aligned} B_f &= \sum_E |f^2(x) - f^2(y)| = \sum_E |f(x) + f(y)| \cdot |f(x) - f(y)| \\ &\leq \sqrt{\sum_E (f(x) + f(y))^2} \cdot \sqrt{\sum_E (f(x) - f(y))^2}. \end{aligned}$$

The required result is obtained by evaluating the two factors:

$$\begin{aligned} \sqrt{\sum_E (f(x) - f(y))^2} &= \|fK\| \\ \sqrt{\sum_E (f(x) + f(y))^2} &\leq \sqrt{2 \sum_E (f^2(x) + f^2(y))} = \sqrt{2d \sum_V f^2(x)} = \sqrt{2d} \cdot \|f\|. \end{aligned}$$

□

**Lemma 4.13.**  $B_f \geq h \cdot \|f\|^2$ .

*Proof.* We rewrite  $B_f$  in term of the values of  $f$  and the sizes of cuts  $E([i], \overline{[i]})$  for  $i \in V^+$ . Then we use expansion, and the assumption that  $|V^+| \leq n/2$ , to give a lower bound on the number of edges in these cuts.

$$\begin{aligned} B_f &= \sum_{(x,y) \in E, x < y} (f_x^2 - f_y^2) = \sum_{(x,y) \in E, x < y} \sum_{i=x}^{y-1} f_i^2 - f_{i+1}^2 \\ &= \sum_{i=1}^{n-1} (f_i^2 - f_{i+1}^2) \cdot |E([i], \overline{[i]})| = \sum_{i \in V^+} (f_i^2 - f_{i+1}^2) \cdot |E([i], \overline{[i]})| \\ &\geq h \sum_{i \in V^+} (f_i^2 - f_{i+1}^2) \cdot i = h \sum_{i \in V^+} f_i^2 = h \cdot \|f\|^2. \end{aligned}$$

The last equality is obtained by collapsing the telescopic sum and observing the  $f_{i+1} = 0$  for  $i = |V^+|$ . □

## 4.6 Three perspectives of expansion and how they compare

We are now in a position to offer the reader a broader view of some of the main questions in the field. As we saw, expansion is defined in **combinatorial** terms (and this definition comes in several different flavors). This is closely related to the **spectral** theory of graphs. Finally, rapidly mixing random walks provide a **probabilistic** perspective.

In each of these three frameworks we consider mostly four types of questions:

- **Extremal:** How large/small can the pertinent expansion parameters be?
- **Typical:** How are these parameters distributed over random graphs?
- **Explicit construction:** Can you construct graphs for which these parameters (nearly) attain their optimum?
- **Algorithmic:** Given a graph, can you efficiently evaluate/estimate its expansion parameters?

It then becomes natural to consider some **comparative** problems: What can you conclude, say, about combinatorial-type expansion parameters from spectral information etc.?

Here are some pointers to the present article where we either explain what is known about such question, or provide some further pointers into the literature.

### 4.6.1 Extremal problems

Here the most satisfactory answer comes from the spectral realm. The Alon-Boppana Theorem 5.3 tells us precisely how large the spectral gap can be in an  $(n, d)$ -graph.

The largest edge expansion of an  $(n, d)$ -graph can be  $(1+o(1))\frac{d}{2}$ , and there is no  $d \geq 3$  for which the exact number is known. See Section 5.1.1. More interesting (and often more difficult) questions concern the expansion of smaller sets in the graph and in particular the edge isoperimetric parameter  $\Phi_E(G, k) = \min_{S \subset V} \{|E(S, \bar{S})| : |S| = k\}$ . Some discussion of this problem is to be found in Chapter 5. Closely related questions concern vertex expansion and discrepancy. Some of these problems are address in Chapter 10.

### 4.6.2 Typical behavior

Here the situation reverses. It is relatively not hard to analyze the (vertex/edge) expansion in random graphs. The typical behavior of the spectrum is harder to understand and Chapter 7 is dedicated to an exposition of this fascinating story and the still lingering mysteries.

### 4.6.3 Explicit constructions

We have already mentioned the Margulis construction to which Chapter 8 is dedicated. The so-called Ramanujan Graphs due to Lubotzky-Phillips-Sarnak [LPS88] and Margulis [Mar88] are mentioned briefly in Section 5.3, but are otherwise not discussed at depth here. We do survey some more combinatorial approaches to the problem, viz. Section 6.4 and Chapters 10 and 12.

### 4.6.4 Algorithms

The exact determination of  $h(G)$ , given  $G$ , is difficult. This fact and the approximate version of the problem are briefly discussed in Section 13.6. Likewise, we lack good estimates for the vertex isoperimetric parameter of a given graph, or for the edge expansion of sets of a given size in a graph. These are among the most significant open questions in the theory. Not only are there no general good explicit bounds for these parameters, even the following question is quite nontrivial: Find explicit constructions of graphs in which even small sets of vertices expand well. We will have quite a bit to say about this problem in Chapter 10. Direct estimates of the expansion, even for specific families of graphs are even harder to come by and [LL05] is one of very few exceptions.

On the other hand, standard algorithms in linear algebra can be used to efficiently compute the spectrum of a given graph.

### 4.6.5 Comparisons

The following two examples show that both the upper and the lower bound in Theorem 4.11 can be tight.

1. **The lower bound is tight:** Our discussion of  $G = Q_d$ , the  $d$ -dimensional cube, in Section 4.2.1 yields that  $h(G) = 1$  (the bound is attained for the  $d - 1$  dimensional subcube). On the other hand, the spectral gap is  $d - \lambda = 2$ .

2. **The upper bound is tight:** The  $n$ -vertex cycle: This is the 2-regular graph  $C_n$  on the vertices  $\{0, \dots, n-1\}$ , where vertex  $i$  is adjacent to  $i+1$  and  $i-1 \pmod n$ . Here  $h(C_n) = \Theta(1/n)$  is attained on a half cycle, while  $d - \lambda = \Theta(1/n^2)$ .

For the above estimates of the spectral gap see [Lov93]. Both graphs are Cayley graphs of Abelian groups and so their spectrum can be computed using methods from Chapter 12.

As mentioned above, for random graphs, expansion is more accessible than spectral gap. On the other hand, eigenvalues are easily computable, while expansion is not. It is interesting to ask how well one theory reflects on the other when we seek (nearly) optimal graphs. Graphs with very large spectral gap are very good expanders: When  $\lambda = o(d)$ , the lower bound in Theorem 4.11 yields  $h(G) \geq (\frac{1}{2} - o(1))d$ . On the other hand, for  $d$  large, an  $(n, d)$ -graph  $G$  can have  $h(G) \geq \Omega(d)$  while the spectral gap is small. Here is an illustration how this can happen: Pick a small  $\delta > 0$ , and construct an  $(n, (\delta \cdot d))$ -graph  $G$  with  $h(G) = \Omega(\delta \cdot d)$ . Now add to it a collection of disjoint cliques of size  $(1 - \delta)d + 1$  each. Clearly  $h(G)$  does not decrease, but the spectral gap is at most  $\delta d$ . Finally, we mention that Lemma 2.6 shows the near equivalence of discrepancy and spectral gap.

Connections with rapid mixing of random walks are discussed in Section 3.1.



## Chapter 5

# Extremal Problems on Spectrum and Expansion

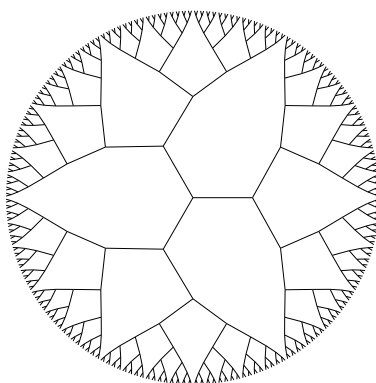


Figure 5.1: The 3-regular infinite tree

Much research in modern Discrete Mathematics revolves around the study of *extremal problems*, and this is the topic of the present chapter. Here are several natural extremal problems about expansion and spectra:

- What is the largest expansion ratio (or vertex expansion etc.) of a  $(n, d)$  graph?
- Recall, more generally, the edge and vertex isoperimetric parameters of a graph

$$\Phi_E(G, k) = \min_{S \subset V} \{|E(S, \bar{S})| : |S| = k\} \quad \text{and} \quad \Phi_V(G, k) = \min_{S \subset V} \{|\Gamma(S) \setminus S| : |S| = k\}.$$

We ask, given  $d, k$  and  $n$  how large these parameters can be in a  $(n, d)$ -graph.

- How large can the spectral gap be in an  $(n, d)$  graph?

In the study of an extremal problem it is often beneficial to try and identify the problem's extremal instances. This is difficult for the present questions, but if we are willing to consider infinite graphs as well, then the extremal case is clearly the  $d$ -regular infinite tree  $\mathbf{T}_d$  - the ultimate expander. The question is how close one can get to this level of expansion with *finite*  $d$ -regular graphs.

In this light we start our discussion with an analysis of the  $d$ -regular tree. We observe that its edge expansion is  $d - 2$  and show that its spectrum is the interval  $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ . This sheds both some light and some shade (it's a tree after all...) on the analogous questions in finite graphs: (i) If  $G$  is an  $(n, d)$  graph, then even if  $k \ll n$ , the

relative expansion of sets of  $k$  vertices cannot exceed  $d - 2 + o_k(1)$ . Namely,  $\Phi_E(G, k) \leq k(d - 2 + \frac{2}{k})$ , and (ii) As mentioned already, the Alon-Boppana theorem says that the second largest eigenvalue is at least  $2\sqrt{d-1} - o(1)$ .

## 5.1 The $d$ -regular tree

### 5.1.1 The expansion of $\mathbf{T}_d$

Consider the edge expansion function  $\Phi_E(\mathbf{T}_d, k)$  of  $\mathbf{T}_d$ . The minimizing set  $S$  must clearly be connected, i.e. a subtree. Therefore  $|E(S)| = |S| - 1$ , and (recalling that we are counting directed edges)  $\Phi_E(\mathbf{T}_d, k) = kd - 2(k-1) = k(d-2) + 2$ . Consequently, the expansion rate of  $\mathbf{T}_d$  is

$$h(\mathbf{T}_d) = \inf_{\text{finite } S \subseteq V} |E(S, \bar{S})|/|S| = d - 2.$$

The above argument implies that  $\Phi_E(G, k) \leq k(d-2) + 2$  for every  $(n, d)$ -graph  $G$  and every  $k$ . However,  $h(G)$  is necessarily smaller than  $h(\mathbf{T}_d) = d - 2$ . To see this, consider a random subset  $S \subseteq V(G)$  of size  $n/2$ . The expected value of  $|E(S, \bar{S})|/|S|$  is  $d/2 + o(1)$ , since every edge from  $G$  belongs to the random cut  $E(S, \bar{S})$  with probability  $1/2 + o(1)$ . Therefore, there exists some set  $S$  of cardinality  $n/2$  with  $|E(S, \bar{S})|/|S| \leq d/2 + o(1)$ , whence  $h(G) \leq d/2 + o(1)$ . A more refined analysis (see Alon [Alo97]), yields that  $h(G) \leq d/2 - c\sqrt{d}$  for every  $d \geq 3$  and sufficiently large  $n$ . Here  $c > 0$  is an absolute constant. This result is tight, up to the value of  $c$ . As we shall see later in Theorems 5.11, 7.10, there are  $(n, d, \alpha)$  graphs with  $\alpha = O(d^{-1/2})$ . The lower bound in Theorem 4.11 yields  $h(G) \geq d/2 - c\sqrt{d}$  in this case.

**Open problem 5.1.** *What is the smallest  $f = f(d, k, n)$  such that every  $(n, d)$ -graph has a set  $S$  of  $k$  vertices and  $|E(S, \bar{S})| \leq f$ .*

In full generality this question is exceedingly difficult and includes e.g., the girth problem. This question asks, for given integers  $d, g \geq 3$ , for the smallest  $n$  such that there is an  $(n, d)$  graph with no cycles of length shorter than  $g$  see e.g., [Hoo02]. Many other instances of this question are interesting and may not be as difficult as the full problem.

### 5.1.2 The spectrum of $\mathbf{T}_d$

Let  $A_T$  be the (infinite) adjacency matrix of  $\mathbf{T}_d$ . We consider it as a linear operator on  $l_2(V(\mathbf{T}_d))$ , the space of real square summable functions on the vertices of the tree, and seek its spectrum. We recall some of the basic definitions from operator theory and refer the reader to Rudin's book [Rud91] for a comprehensive discussion of spectra of general linear operators. As usual, we define the spectrum of  $A_T$  via  $\text{spec}(A_T) = \{\lambda : (A_T - \lambda I) \text{ is non-invertible}\}$ . That is,  $\lambda$  is in the spectrum if  $(A_T - \lambda I)$  has a non-trivial kernel or is not onto. For finite matrices the two conditions are obviously equivalent and we determine whether  $\lambda$  is in the spectrum by seeking a nonzero eigenvector  $u$  satisfying  $(A - \lambda I)u = 0$ . In contrast, the matrix  $A_T$  has no eigenvectors and its entire spectrum follows from the second reason.

One can go a step further, and compute the **spectral measure** of  $A_T$  that corresponds to the eigenvalue distribution for finite graphs. We will say more about it in Chapter 7. But, for now, we return to the problem of computing the spectrum of  $\mathbf{T}_d$ .

**Theorem 5.2 (Cartier [Car72]).**

$$\text{spec}(A_T) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$$

*Proof.* (Friedman [Fri91])

Fix some vertex  $v \in V(\mathbf{T}_d)$  as the root of the tree. Then

$$\lambda \in \text{spec}(A) \iff \delta_v \notin \text{Range}(\lambda I - A),$$

where  $\delta_v$  is the characteristic function of  $v$ ,

$$\delta_v(u) = \begin{cases} 1 & u = v \\ 0 & u \neq v \end{cases}.$$

The necessity of this condition is obvious. Sufficiency is not hard either but will not be proved here. We wish to find out for which values of  $\lambda$  there is a function  $f \in l_2$  satisfying

$$\delta_v = (\lambda I - A)f \tag{5.1}$$

We say that a function  $f$  on  $V(\mathbf{T}_d)$  is *spherical around the vertex  $v$* , if  $f(u)$  depends only on the distance (in the graph metric of  $\mathbf{T}_d$ ) from  $u$  to  $v$ . The *spherical symmetrization* of  $g \in l_2$  around  $v$  is a spherical function  $f$  such that  $\sum_{\text{dist}(u,v)=r} f(u) = \sum_{\text{dist}(u,v)=r} g(u)$  for every  $r \geq 0$ . It is easy to observe that if  $g \in l_2$  is a solution to (5.1) then  $g$ 's spherical symmetrization  $f$  is in  $l_2$  and solves the same equation. We may therefore assume without loss of generality that  $f$  is spherical. A spherical  $f$  is determined by a sequence of numbers  $\{x_0, x_1, \dots\}$  such that  $f(u) = x_i$  whenever  $d_T(u, v) = i$ . This reduces (5.1) to the following recurrence:

$$\begin{aligned} \lambda x_0 &= dx_1 + 1 \\ \lambda x_i &= x_{i-1} + (d-1)x_{i+1} \quad \text{for } i \geq 1. \end{aligned} \tag{5.2}$$

The solution to such a recurrence is  $x_i = \alpha \rho_1^i + \beta \rho_2^i$ , where  $\rho_{1,2} = \frac{\lambda \pm \sqrt{\lambda^2 - 4(d-1)}}{2(d-1)}$  are the roots of the quadratic  $\lambda \rho = 1 + (d-1)\rho^2$ .

If  $|\lambda| < 2\sqrt{d-1}$  the roots are complex with absolute value  $1/\sqrt{d-1}$ . In this case,  $\lambda \in \text{spec}(A)$  since  $f$  is not in  $l_2$ . Indeed,  $|x_i| = \Theta((d-1)^{-i/2})$ , and since there are  $\Theta((d-1)^i)$  vertices at distance  $i$  from  $v$ , it follows that  $\|f\|_2 = \infty$ .

We claim that when  $|\lambda| > 2\sqrt{d-1}$  Equation (5.2) has a solution in  $l_2$ , implying that  $\lambda$  is not in the spectrum of  $A$ . To see that, observe that  $|\rho_1| < 1/\sqrt{d-1}$  and so if  $x_i = \alpha \rho_1^i$ , then the resulting  $f$  belongs to  $l_2$  and satisfies Equation (5.2) for all  $i > 0$ . It is left to verify that there is a value of  $\alpha$  for which Equation (5.2) holds for  $i = 0$  as well. This additional condition reads  $\lambda \alpha = d\alpha \rho_1 + 1$ , which is possible iff  $\lambda \neq d\rho_1$ . Indeed,  $|\rho_1| < |\lambda|/2(d-1) \leq |\lambda|/d$  for all  $d \geq 2$ .  $\square$

## 5.2 The Alon-Boppana lower bound

### 5.2.1 Statement of the theorem

In this section we return to the question how small  $\lambda_2$  can be in a large  $d$ -regular graph. A weak bound was given in Section 2.5, and here we prove the Alon-Boppana lower bound  $\lambda_2 \geq 2\sqrt{d-1} - o_n(1)$ .

**Theorem 5.3 (Alon-Boppana, A. Nilli [Nil91], Friedman [Fri93]).** *There exists a constant  $c$  such that for every  $(n, d)$ -graph  $G$  of diameter  $\Delta$ :*

$$\lambda_2(G) \geq 2\sqrt{d-1} \cdot (1 - c/\Delta^2).$$

Since the diameter of a  $(n, d)$ -graph is  $\Omega(\log_{d-1} n)$ , it follows that:

**Corollary 5.4.** *For every  $(n, d)$ -graph*

$$\lambda_2 \geq 2\sqrt{d-1} \cdot (1 - O(1/\log^2 n))$$

We present two proofs of this theorem. The first proof illustrates the *moment method* which we will encounter later in Chapter 7. We first define the **tree number**  $t_{2k}$  as the number of closed walks of length  $2k$  that start and end at some given vertex in  $\mathbf{T}_d$ . The key fact of the proof is that the analogous quantity in any regular graph is  $\geq t_{2k}$ . We also use the fact that *the spectral radius*  $\rho(\mathbf{T}_d) = \max\{|\lambda| \mid \lambda \in \text{spec}(\mathbf{T}_d)\} = 2\sqrt{d-1}$ . This proof achieves a little less than we desire: (i) It only bounds  $\lambda(A) = \max_{i \geq 2} |\lambda_i(A)|$  and (ii) The resulting error term is slightly weaker than in Theorem 5.3. The second proof yields tighter estimates by computing the Rayleigh quotient of a certain function. This function is an eigenfunction of a truncation of the tree.

### 5.2.2 Proof I: Counting closed walks in $\mathbf{T}_d$

Let  $A$  be the adjacency matrix of  $G$ . Clearly,  $\lambda(A^{2k}) = (\lambda(A))^{2k}$  for every integer  $k$ . We give a lower bound on  $\lambda(A^{2k})$  by estimating the Rayleigh quotient of the function  $f = \delta_s - \delta_t$ , where  $s, t$  are two vertices at distance  $\Delta$  in  $G$ . That is  $f(s) = 1, f(t) = -1$ , and  $f(u) = 0$  for any other vertex  $u$ . Since  $f$  is orthogonal to the first eigenfunction, the constant function,

$$\lambda^{2k} \geq \frac{fA^{2k}f^T}{\|f\|^2} = \frac{(A^{2k})_{ss} + (A^{2k})_{tt} - 2(A^{2k})_{st}}{2}.$$

We choose  $k = \lfloor \frac{\Delta-1}{2} \rfloor$ , so the negative term in the numerator vanishes. The positive terms in the numerator count closed walks of length  $2k$  that start and end at  $s$  and  $t$  respectively and are therefore  $\geq t_{2k}$ . Consequently,

$$\lambda^{2k} \geq t_{2k}.$$

The tree numbers  $t_{2k}$  have been studied in great detail, see [McK81], and [Lub94]. Good estimates, a recursion, and their generating function are known, but all we need here is a rough estimate. (Slightly more will be needed below, Lemma 7.3.) Associated with every walk that starts and ends at the same vertex  $v$  in  $\mathbf{T}_d$  is a **sign pattern**. Each step is associated a  $+1$  or  $-1$  according to its being directed either away or toward  $v$ . Clearly such a sign pattern is characterized by two conditions: (i) It sums up to zero and (ii) The sum of each prefix is nonnegative. It is well known that the number of such sequences of length  $2k$  is the  $k$ -th Catalan number  $C_k = \binom{2k}{k}/(k+1)$  (see [vLW01]). Corresponding to every such sign pattern are at least  $(d-1)^k$  walks, since there are exactly  $k$  occurrences of  $+1$  in the sequence, and at least  $d-1$  choices for moving away from  $v$  regardless of the current position of the walk. (For an accurate estimates of  $t_{2k}$  note that a prefix that sums to zero corresponds to a time when the walk reaches the vertex  $v$  at which time the number of choices for the next step is  $d$ , not  $d-1$ ). Therefore

$$\lambda^{2k} \geq t_{2k} \geq C_k \cdot (d-1)^k = \Theta((2\sqrt{d-1})^{2k} \cdot k^{-3/2}).$$

Taking the  $2k$ -th root, and recalling that  $k = \lfloor (\Delta-1)/2 \rfloor$  yields:

$$\lambda(A) \geq 2\sqrt{d-1} \cdot (1 - O(\log \Delta / \Delta)).$$

### 5.2.3 Proof II: Using spherical functions

This argument follows Friedman [Fri93]. Here we derive a lower bound on  $\lambda_2 = \max_{f \perp \mathbf{1}} fAf^T / \|f\|^2$  through a proper choice of a *test function*  $f$ . The function we use is an adaptation of an eigenfunction for a truncation of  $\mathbf{T}_d$ . Given two vertices  $s, t$  at distance  $\Delta$ , we construct a function  $f$  that is positive on vertices at distance  $\leq k = \lfloor \frac{\Delta}{2} \rfloor - 1$  from  $s$ , negative on vertices at distance  $\leq k$  from  $t$ , and zero elsewhere. The values of  $f$  are derived from those of the eigenfunction  $g$  with maximal eigenvalue  $\mu$  for the  $d$ -regular tree of height  $k$ . We view  $s$  and  $t$  as roots of (separate)  $k$ -tall trees. We show that  $f^+$ , the positive part of  $f$  satisfies  $Af^+ \geq \mu f^+$ , and likewise for the negative part  $Af^- \leq -\mu f^-$ , so that  $fAf^T \geq \mu \|f\|^2$ , and  $fAf^T / \|f\|^2 \geq \mu$ . Finally, the positive and negative parts of  $f$  are normalized so that  $\sum f(x) = 0$ , so we can conclude that  $\lambda_2 \geq \mu$ .

We now get down to business. Let  $k = \lfloor \frac{\Delta}{2} \rfloor - 1$  and select two vertices  $s, t$  in  $G$  at distance  $\Delta$ . We classify the vertices according to their distance from  $s$  or  $t$ , along with a no-man's-land of the vertices that are far from both vertices.

$$\begin{aligned} S_i &= \{v : d(s, v) = i\} && \text{for } i = 0, \dots, k, \\ T_i &= \{v : d(t, v) = i\} && \text{for } i = 0, \dots, k, \\ Q &= V(G) \setminus \bigcup_{0 \leq i \leq k} (S_i \cup T_i). \end{aligned}$$

There are, of course, no edges between any  $S_i$  and  $T_j$ .

Let  $\mathbf{T}_{d,k}$  denote the  $d$ -regular tree of height  $k$ , and let  $A_{T_k}$  be its adjacency matrix.

**Claim 5.5.** *Let  $\mu$  be the largest eigenvalue of  $A_{T_k}$ . There is a unique function  $g : V(\mathbf{T}_{d,k}) \rightarrow \mathbb{R}$  satisfying  $A_{T_k}g = \mu g$ . The function  $g$  is nonnegative and spherically symmetric.*

*Proof.* This can either be verified directly or by appealing to the Perron Frobenius theorem. The spherical symmetry can be verified as in the proof of Theorem 5.2.  $\square$

Let  $g_i$  be the value that  $g$  takes on vertices at the  $i$ -th level. These numbers clearly satisfy the following recursion and boundary conditions:

$$\begin{aligned} \mu g_0 &= dg_1, \\ \mu g_i &= g_{i-1} + (d-1)g_{i+1} \quad \text{for } i = 1, \dots, k, \\ g_{k+1} &= 0. \end{aligned} \tag{5.3}$$

We define  $f : V(G) \rightarrow \mathbb{R}$  as follows:

$$f(v) = \begin{cases} c_1 g_i & v \in S_i \\ -c_2 g_i & v \in T_i \\ 0 & \text{otherwise} \end{cases},$$

where  $c_1, c_2$  are non-negative constants to be determined later. We next prove that  $f$  gives us the desired properties:

**Lemma 5.6.** *If  $g$  is non-increasing (as indeed will be shown below), then:*

$$\begin{aligned} (Af)_v &\geq \mu f_v \quad \text{for } v \in \cup_i S_i, \\ (Af)_v &\leq \mu f_v \quad \text{for } v \in \cup_i T_i. \end{aligned}$$

*Proof.* Let  $v \in S_i$  for some  $i > 0$ . Then of its  $d$  neighbors  $p \geq 1$  belong to  $S_{i-1}$ ,  $q$  neighbors to  $S_i$ , and  $d - p - q$  to  $S_{i+1}$ . Therefore,

$$(Af)_v = p \cdot c_1 g_{i-1} + q \cdot c_1 g_i + (d - p - q) \cdot c_1 g_{i+1}.$$

Comparing with (5.3) and using the fact that  $g$  is non-negative and non-increasing we obtain:

$$\begin{aligned} (Af)_v &= c_1 \cdot (pg_{i-1} + qg_i + (d - p - q)g_{i+1}) \\ &\geq c_1 \cdot (g_{i-1} + (d-1)g_{i+1}) \\ &= c_1 \cdot (A_{T_k}g)_i = c_1 \mu g_i = \mu f_v. \end{aligned}$$

A similar argument works for  $v = s$  and for  $v \in \cup_i T_i$ , to yield the required result.  $\square$

**Corollary 5.7.**  $\lambda_2 \geq \mu$ .

*Proof.* First observe that the previous lemma implies that

$$\begin{aligned} fAf^T &= \sum_{v \in V(G)} f_v (Af)_v \\ &= \sum_{v \in \cup_i S_i} f_v (Af)_v + \sum_{v \in \cup_i T_i} f_v (Af)_v + \sum_{v \in Q} f_v (Af)_v \\ &\geq \sum_{v \in \cup_i S_i} f_v \mu f_v + \sum_{v \in \cup_i T_i} f_v \mu f_v = \mu f f^T. \end{aligned}$$

Also, a proper choice of  $c_1, c_2$  gives  $\sum_{v \in \cup_i S_i} f_v = -\sum_{v \in \cup_i T_i} f_v$  and hence  $f \perp \vec{1}$ . Therefore,  $\lambda_2 \geq fAf^T / \|f\|^2 \geq \mu$  as claimed.  $\square$

We still need to show that  $g$  is non-increasing, and prove a lower bound on  $\mu$ . We do that by giving an explicit solution  $h : \{0, \dots, k+1\} \rightarrow \mathbb{R}$  to the recursion (5.3). By Claim 5.5  $h$  must coincide with  $g$ . We then show that  $h$  is non-increasing, and derive the required lower bound on  $\mu$ . Let

$$h_i = (d-1)^{-i/2} \cdot \sin((k+1-i)\theta)$$

where the parameter  $\theta$  will be determined below. It is easy to check that  $h_{k+1} = 0$  and that  $h$  is non-negative and decreasing provided that  $0 < \theta < \pi/(k+1)$ . We claim that it satisfies the recursion (5.3), with  $\mu = 2\sqrt{d-1} \cos \theta$ . For  $0 < i \leq k$  we have:

$$\begin{aligned} h_{i-1} + (d-1)h_{i+1} &= (d-1)^{-(i-1)/2} \cdot [\sin((k+1-(i-1))\theta) + \sin((k+1-(i+1))\theta)] \\ &= \sqrt{d-1} \cdot (d-1)^{-i/2} \cdot [\sin((k+2-i)\theta) + \sin((k-i)\theta)] \\ &= 2\sqrt{d-1} \cdot (d-1)^{-i/2} \sin((k+1-i)\theta) \cos \theta = \mu \cdot h_i. \end{aligned}$$

The condition for  $i = 0$  is that  $\mu h_0 = dh_1$ , or equivalently:

$$(2d-2) \cdot \cos \theta \cdot \sin((k+1)\theta) = d \cdot \sin(k\theta). \quad (5.4)$$

The smallest positive root of this equation,  $\theta_0$  satisfies  $0 < \theta_0 < \pi/(k+1)$ . This is because the difference between the two sides of (5.4) changes sign between 0 and  $\pi/(k+1)$ . We set  $\theta = \theta_0$ , and so the recursion (5.3) is satisfied and  $h \geq 0$ . Now  $\theta_0 < \pi/(k+1) \approx 2\pi/\Delta$ , since  $k = \lfloor \Delta/2 \rfloor - 1$ . By Taylor expansion of the cosine,

$$\cos(\theta_0) > 1 - c/\Delta^2,$$

yielding the required lower bound on  $\mu$  with constant  $c \approx 2\pi^2$ .

## 5.2.4 Extensions of the Alon-Boppana theorem

A quantitative variation of the Alon-Boppana theorem states that a constant fraction of the  $n$  eigenvalues must exceed  $2\sqrt{d-1} - \epsilon$  for any fixed  $\epsilon > 0$ .

**Theorem 5.8 (Serre).** *For every integer  $d$  and  $\epsilon > 0$  there is a constant  $c = c(\epsilon, d)$ , such that every  $(n, d)$  graph  $G$  has at least  $c \cdot n$  eigenvalues greater than  $2\sqrt{d-1} - \epsilon$ .*

There are several available proofs of this theorem e.g., [DSV03], [Fri93], [Nil04]. These papers also give the proper credit to the theorem's originator J. P. Serre. The simple proof we present is due to Cioabă [Cio05].

*Proof.* Let  $A$  be the adjacency matrix of  $G$ . We seek a lower bound on  $n_\epsilon$ , the number of eigenvalues larger than  $2\sqrt{d-1} - \epsilon$ . Consider the matrix  $(A + dI)^k$ , where the positive integer  $k$  will be specified below. On one hand

$$\begin{aligned} \text{trace}(A + dI)^k &= \sum_{i=1}^n (\lambda_i + d)^k \\ &\leq (2d)^k \cdot n_\epsilon + (d + 2\sqrt{d-1} - \epsilon)^k \cdot n. \end{aligned} \quad (5.5)$$

On the other hand,

$$\text{trace}(A + dI)^k = \sum_{j=0}^k \binom{k}{j} \cdot \text{trace}(A^j) \cdot d^{k-j} \geq \sum_{l=0}^{\lfloor k/2 \rfloor} \binom{k}{2l} n \cdot t_{2l} \cdot d^{k-2l}.$$

Here we have eliminated the (positive) terms for odd  $j$ . We can use the estimates of the tree numbers  $t_j$  from section 5.2.2 to conclude

$$\begin{aligned} \text{trace}(A + dI)^k &\geq (c'/k^{3/2}) \cdot \sum_{l=0}^{\lfloor k/2 \rfloor} \binom{k}{2l} n \cdot (2\sqrt{d-1})^{2l} \cdot d^{k-2l} \\ &= (c'/2k^{3/2}) \cdot n \cdot [(d + 2\sqrt{d-1})^k + (d - 2\sqrt{d-1})^k] \geq (c'/2k^{3/2}) \cdot n \cdot (d + 2\sqrt{d-1})^k \end{aligned}$$

for some absolute constant  $c' > 0$ . In combination with (5.5), this yields

$$\frac{n_\epsilon}{n} \geq \frac{(c'/2k^{3/2}) \cdot (d + 2\sqrt{d-1})^k - (d + 2\sqrt{d-1} - \epsilon)^k}{(2d)^k}.$$

This expression is positive for  $k \geq \Omega(\frac{d}{\epsilon} \log(\frac{d}{\epsilon}))$ , and the theorem follows.  $\square$

Much less is known about the spectrum of irregular graphs. For example, is the second largest eigenvalue  $\geq 2\sqrt{d-1} - o(1)$ , where  $d$  is the *average degree* of the graph? Not necessarily. The *lollipop graph*  $L_n$  has  $2n$  vertices, and is obtained from an  $n$ -clique  $K_n$  and a path  $P_{n+1}$  on  $n+1$  vertices by identifying a vertex in  $K_n$  with end vertex in  $P_{n+1}$ . It is not hard to check that unlike a regular graph, the diameter and average degree of  $L_n$  are  $\Theta(n)$ , but  $\lambda(L_n) \leq 2$ . The following theorem shows that a simple additional condition yields the Alon-Boppana bound for irregular graphs.

**Theorem 5.9 (Hoory [Hoo05]).** *Let  $d, r \geq 2$  be integers. Suppose that the average degree in the graph  $G$  is  $\geq d$  whenever a ball of radius  $r$  is deleted from the graph. Then  $\lambda(G) \geq 2\sqrt{d-1} \cdot (1 - c \cdot \log r/r)$ , for some absolute constant  $c > 0$ .*

### 5.3 Ramanujan graphs

In light of the Alon Boppana bound (Theorem 5.3) we define:

**Definition 5.10.** A  $d$ -regular graph  $G$  is **Ramanujan** if  $\lambda(G) \leq 2\sqrt{d-1}$ .

It is a major result discovered by Lubotzky-Phillips-Sarnak [LPS88] (who also coined this term) and independently by Margulis [Mar88] that arbitrarily large  $d$ -regular Ramanujan graphs exist when  $d-1$  is prime, and moreover can be explicitly constructed. Morgenstern [Mor94] extended this to the case when  $d-1$  is a prime power. Here we only state the result and describe the construction. The book by Davidoff, Sarnak, and Valette [DSV03] offers a self contained description of the beautiful mathematics around it. Lubotzky's book [Lub94] should be consulted as well.

**Theorem 5.11 (Lubotzky-Phillips-Sarnak [LPS88] Margulis [Mar88] Morgenstern [Mor94]).** *For every prime  $p$  and every positive integer  $k$  there exist infinitely many  $d$ -regular Ramanujan graphs with  $d = p^k + 1$ .*

The following suggests itself:

**Conjecture 5.12.** *For every integer  $d \geq 3$  there exist arbitrarily large  $d$ -regular Ramanujan Graphs.*

We will review in the following chapters, some recent attempts at solving this problem using combinatorial and probabilistic methods.

We now review the construction of the Ramanujan graphs  $X^{p,q}$ , as described in [LPS88]. Let  $p, q$  be distinct primes that are congruent to 1 mod 4. Then  $X^{p,q}$  is a  $p+1$ -regular graph of size  $\Theta(q^3)$ . Their explicitness level depends on the complexity of finding large primes.

Let us recall the definition of a *Cayley graph*. Let  $G$  be a group and  $S \subset G$ , a subset that is closed under inversion. The corresponding Cayley graph  $C(G, S)$  is a graph with vertex set  $G$  and edge set  $\{(x, xs) : x \in G \text{ and } s \in S\}$ . In the present case,  $G = \text{PGL}(2, q)$ , the group of 2 by 2 non-singular matrices over  $\mathbb{F}_q$ , where we identify two matrices that are proportionate to each other. Fix some integer  $i$  with  $i^2 \equiv -1 \pmod{q}$ . We define  $S$  as:

$$S = \left\{ \begin{pmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{pmatrix} : a_0^2 + a_1^2 + a_2^2 + a_3^2 = p, \text{ with odd } a_0 > 0 \text{ and even } a_1, a_2, a_3 \right\}.$$

By a theorem of Jacobi, there are exactly  $p+1$  such solutions (over the integers!), so  $|S| = p+1$ . It can be verified that  $S$  is closed under inversion, as needed. The graph  $X^{p,q}$  is obtained by taking the connected component of the identity of  $C(G, S)$ . (It can be shown that  $C(G, S)$  is either connected or has exactly two equal connected components, depending on the quadratic residue symbol  $(\frac{q}{p})$ ).



## Chapter 6

# Spectrum and Expansion in Lifts of Graphs

### 6.1 Covering maps and lifts

From a topological perspective graphs are one-dimensional simplicial complexes. The notion of covering maps from topology turns out to be quite useful for our subject. For one thing,  $\mathbf{T}_d$  is the universal covering space of any  $d$ -regular graph. This explains the role of  $\mathbf{T}_d$  in Chapter 5. Also if there is a covering map from a graph  $H$  onto a graph  $G$ , then every eigenvalue of  $G$  is also an eigenvalue of  $H$ . We also show how lifts allow us to construct regular graphs with near-optimal spectral gaps. We do not need anything substantial from topology, but we do borrow some of the terminology and several of our observations are special cases of more general phenomena in the theory of covering maps in topology.

We start by defining the notion of coverings and lifts.

**Definition 6.1.** Let  $G$  and  $H$  be two graphs. We say that a function  $f : V(H) \rightarrow V(G)$  is a **covering map**, if for every  $v \in V(H)$ ,  $f$  maps  $v$ 's neighbor set  $\Gamma_H(v)$  one to one and onto  $\Gamma_G(f(v))$ . If there exists a covering function from  $H$  to  $G$ , we say that  $H$  is a **lift** of  $G$  or that  $G$  is a **quotient** of  $H$ .

This definition is appropriate for simple graphs (with no parallel edges and self loops) but can be easily extended to all graphs.

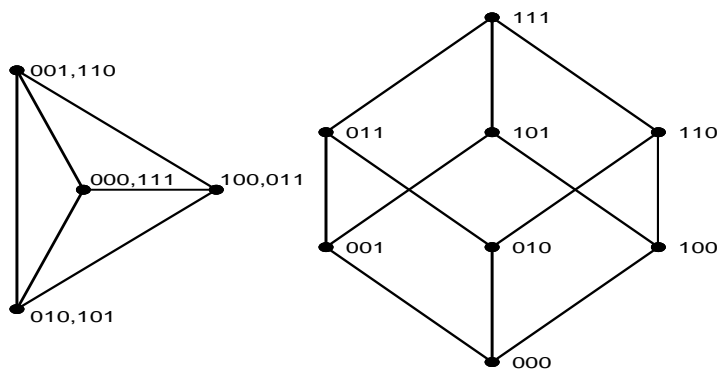


Figure 6.1: The three dimensional cube is a 2-lift of the clique on four vertices. The cover map identifies antipodal vertices in the cube.

If  $f$  is a covering map onto  $G$  and  $v$  is a vertex in  $G$ , we call the set  $f^{-1}(v)$  the **fiber** of  $v$ . Similarly, if  $e \in E(G)$ , we say that  $f^{-1}(e)$  is the fiber of  $e$ . Also, it is easily verified that if  $G$  is connected, then every covering map onto  $G$  has a well defined **covering number**  $n$ , such that all vertex and edge fibers have fixed size  $n$ .

Let  $G$  be a connected graph. We denote by  $L_n(G)$  the set of all lifts of  $G$  with covering number  $n$ , which we call  $n$ -lifts of  $G$ . There is a simple and convenient description to the members of  $L_n(G)$ . If  $H \in L_n(G)$ , then

$V(H) = V(G) \times [n]$ . That is, the fiber of every  $v \in V(G)$ , consists of the vertices  $(v, 1), \dots, (v, n)$  in  $V(H)$ . To define the edges of  $H$ , we associate with every edge  $e = (v, u) \in E(G)$  a permutation  $\pi_e$  from  $S_n$ . The fiber of the edge  $(u, v)$  consists of the edges  $((u, i), (v, \pi_e(i)))$  for all  $i \in [n]$ . (Note that here we consider  $e = (u, v)$  as a *directed* edge, and the permutation corresponding to  $(u, v)$  is the inverse  $\pi_e^{-1}$ .) Thus every choice of permutations in  $S_n$ , one for each edge in  $G$  defines a member in  $L_n(G)$ . This also gives us a natural way to sample a *random*  $n$ -lift of  $G$  as we elaborate in Section 7.3.3.

## 6.2 Eigenvalues - old and new

What can we say about the eigenvalues of lifts of  $G$ ? The eigenvalues of  $G$  are eigenvalues of its lifts as well. To see this, let  $h : V(G) \rightarrow \mathbb{R}$  be an eigenfunction of  $G$  with eigenvalue  $\lambda$ , and let  $H$  cover  $G$  via the map  $f : V(H) \rightarrow V(G)$ , then  $h \circ f$  is an eigenfunction of  $H$  with the same eigenvalue. Such an eigenfunction and the corresponding eigenvalue are considered *old*. Thus if  $H$  is a lift of  $G$ , we talk about its *old* eigenvalues and eigenfunctions which are inherited from  $G$ , and the rest of the eigenvalues and eigenfunctions which are considered *new*.

**Example 6.2.** In the above example of the 3-dimensional cube and  $K_4$ , the spectrum of the cube is composed of: The old spectrum  $\{3, -1, -1, -1\}$  inherited from  $K_4$ , and the new spectrum  $\{1, 1, 1, -3\}$ .

This leads us to two simple but useful observations from [BL]. Since the eigenfunctions of  $G$  span the space of real functions on  $V(G)$  and since distinct eigenfunctions can always be chosen to be mutually orthogonal, we conclude:

**Proposition 6.3.** *Let  $f : V(H) \rightarrow V(G)$  be a covering map and let  $\psi$  be a new eigenfunction of  $H$ . Then  $\sum_{f(x)=v} \psi(x) = 0$  for every  $v \in V(G)$ . In words,  $\psi$  sums to zero on every fiber.*

This leads to a particularly pleasing description of the spectrum for 2-lifts of  $G$ . Let  $A = A_G$  be the adjacency matrix of  $G$ . A *signing*  $\tilde{A}$  of  $A$  is a symmetric matrix that is obtained by replacing some of the 1-entries in  $A$  by  $-1$ . There is a natural  $1 : 1$  correspondence between 2-lifts of  $G$  and signings of  $A_G$ . Namely, For every edge  $e = (u, v) \in E(G)$  we define  $\tilde{A}(u, v)$  to be  $\pm 1$  according to the permutation  $\pi_e$  being the identity or  $(2, 1) \in S_2$ . It follows that there is a  $1 : 1$  correspondence between the eigenfunctions  $\phi$  of  $\tilde{A}$  and new eigenfunctions  $\psi$  of the two lift. The correspondence is given by  $\psi(v, 1) = -\psi(v, 2) = \phi(v)$  for every  $v \in V(G)$ .

In particular:

**Proposition 6.4.** *Let  $H$  be a 2-lift of  $G$  that's encoded by the matrix  $\tilde{A}$ . The new eigenvalues of  $H$  are the eigenvalues of  $\tilde{A}$ .*

These observation are used below (Section 6.4) to construct graphs with a near-optimal spectral gap.

## 6.3 The universal covering tree

Associated with every connected graph  $G$  is its *universal covering space*  $\hat{G}$ . This is an infinite tree that is uniquely defined through the condition that every connected lift of  $G$  is a quotient of  $\hat{G}$ . It is not hard to explicitly construct  $\hat{G}$ . Fix some vertex  $v_0$  in  $G$ . The vertices of  $\hat{G}$  are in  $1 : 1$  correspondence with all non-backtracking walks in  $G$  starting at  $v_0$ . That is, all finite sequences  $v_0, v_1, \dots$ , such that  $v_i$  is adjacent to  $v_{i+1}$  and  $v_i \neq v_{i+2}$  for all  $i$ . Two vertices of  $\hat{G}$  are adjacent if one walk is a single-step extension of the other. It is easy to verify that:

**Example 6.5.**  $T_d$  is the universal covering space of every  $d$ -regular graph.

### 6.3.1 Irregular Ramanujan graphs?

As we have already mentioned, the counterpart of the present theory for irregular graphs is still quite poorly understood. It is not even clear how to define an irregular Ramanujan Graph. In particular, is there a natural lower bound on  $\lambda(G)$  for an irregular graph? We recall that the spectral radius of a (finite or infinite) graph  $H$  is defined via  $\rho(H) = \sup\{|\lambda| \mid \lambda \in \text{spec}(H)\}$ . The following generalization for the Alon-Boppana lower bound is due to Greenberg and Lubotzky [Gre95].

**Theorem 6.6 (Greenberg and Lubotzky [Gre95]).** *Let  $\{G_i\}$  be a family of graphs covered by the same universal cover  $T$ . Then  $\lambda(G_i) \geq \rho(T) - o(1)$ .*

The proof is similar to the first proof we gave in the regular case, Section 5.2.2. One considers  $fA^{2k}f^T/\|f\|^2$ , where  $f$  is  $\pm 1$  on two faraway vertices, and zero elsewhere. However, the numbers  $t_{2k}$  of the  $d$ -regular tree, are replaced by the number of closed walks of length  $2k$  from  $v$  to  $v$  in the universal cover  $T$ , which is at least  $(\rho - o(1))^{2k}$ .

This naturally suggests a definition of (not necessarily regular) Ramanujan Graphs.

**Definition 6.7.** A graph  $G$  is Ramanujan if  $\lambda(G) \leq \rho(\hat{G})$ . In other words, if the absolute value of all of  $G$ 's eigenvalues excluding  $\lambda_1$  are bounded by the spectral radius of its universal cover.

If  $G$  is  $d$ -regular, then  $\hat{G} = \mathbf{T}_d$ , so that  $\rho(\hat{G}) = 2\sqrt{d-1}$ , and this definition coincides with the definition of  $d$ -regular Ramanujan graphs.

A possible restatement of Theorem 5.11 is that if  $d-1$  is a prime power, then infinitely many quotients of  $\mathbf{T}_d$  are Ramanujan graphs. Conjecture 5.12 posits that the same holds for every  $d \geq 3$ . Might it be that a similar statement holds for every infinite tree  $T$  with infinitely many finite quotients? This is not so, as pointed out by Lubotzky and Nagnibeda [LN98]. They constructed such a tree  $T$  such that there is a single graph  $G^*$  that is covered by every finite quotient of  $T$ . Moreover, the second eigenvalue of  $G^*$  is larger than  $\rho(T)$ . Since this large eigenvalue is inherited by all graphs covering  $G^*$ , none of them can be Ramanujan.

## 6.4 Nearly-Ramanujan graphs by way of 2-lifts

It seems plausible that unlike the currently known proof of Theorem 5.11, a proof of Conjecture 5.12 would have to resort to methods outside of number theory. For this and for several other reasons, it is a major challenge to develop combinatorial and probabilistic arguments that can establish such results. Such an attempt was recently made by Bilu and Linial [BL] as we now describe. We start from a small  $d$ -regular Ramanujan graph, such as the complete graph  $K_{d+1}$ , and attempt to construct large  $d$ -regular Ramanujan graphs, by applying a series of 2-lifts. In view of Proposition 6.4, the following conjecture would guarantee the success of this plan, and in particular implies the correctness of Conjecture 5.12.

**Conjecture 6.8.** *Every  $d$ -regular Ramanujan graph  $G$  has a 2-lift such that all new eigenvalues are in the range  $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ .*

Extensive computer experiments tend to suggest an even more daring conjecture.

**Conjecture 6.9.** *Every  $d$ -regular graph  $G$  has a 2-lift such that all new eigenvalues are in the range  $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ .*

Equivalently,

**Conjecture 6.10.** *Let  $A$  be the adjacency matrix of some  $d$ -regular graph. Then  $A$  has a signing  $\tilde{A}$  with spectral radius  $\rho(\tilde{A}) \leq 2\sqrt{d-1}$ .*

A somewhat weaker statement can be proved. It uses the Lovasz Local Lemma, a standard tool in the probabilistic method (see [AS00]) and an enumerative argument to show:

**Lemma 6.11.** *Let  $A$  be the adjacency matrix of some  $d$ -regular graph. Then  $A$  has a signing  $\tilde{A}$  such that*

$$x\tilde{A}x \leq O\left(\sqrt{d} \log d \cdot \|x\|^2\right)$$

for every vector  $x$  in which each coordinate is  $-1, 0$  or  $1$ .

Using Lemma 2.6 we reach the main result of that paper:

**Theorem 6.12 (Bilu-Linial [BL]).** *For every  $d \geq 3$  there is a mildly explicitly constructed family family of  $(n, d)$  graphs, the second eigenvalue of each is  $O(\sqrt{d \log^3 d})$ .*

In fact, the algorithmic aspects of this construction need a little additional elaboration, but we refer the interested reader to the original paper.



## Chapter 7

# The Spectrum of Random Graphs

The *probabilistic method* is one of the most fruitful ideas in modern Discrete Mathematics. The basic tenet of this method can be phrased as follows: It is in general essentially impossible for a human to investigate a large graph (or other discrete mathematical objects). However, it is possible and very beneficial to craft certain *ensembles* of large graphs and investigate their *typical* properties. Specifically, it is possible to introduce probability spaces whose elements are graphs. Any graph parameter then becomes a random variable which can be investigated using tools from probability theory. These probability spaces and these random variables are being investigated in their own right (see e.g. the book [JŁR00]) or as means for solving other problems (here [AS00] is the standard reference). This important idea was first crystallized nearly a half century ago in a series of ground-breaking papers by Erdős and Rényi, starting with [ER59]. These authors have introduced  $G(n, p)$  which is a basic model of random graphs<sup>1</sup>. To sample a graph from this distribution, start with  $n$  vertices. Independently, for every pair of vertices, introduce an edge with probability  $1 > p > 0$ . This important model is, however, not very useful for our purposes, since we are interested mostly in regular graphs. Regular graphs have only a tiny probability in the  $G(n, p)$  model, and different random models are needed. Indeed it took about two decades of research in random graphs until it was discovered how to sample uniformly  $(n, d)$  graphs (see [JŁR00] Chapter 9, and [Wor99]).

In this chapter we study what the typical eigenvalues of random  $(n, d)$  graphs look like. The first section deals with the bulk of the spectrum (where most eigenvalues tend to be), and the second with the extreme eigenvalues (those defining expansion). In both cases, this study parallels earlier investigations of the spectrum of random matrices, as we mention. A central tool in both parts is the trace method (or moment method) which we have already seen. This method is particularly appropriate for the random setting. In the third section we consider some variations of this study of random sparse graphs.

### 7.1 The bulk of the spectrum

Our focus has so far been the extreme eigenvalues of graphs, namely  $\lambda_2$  and  $\lambda_n$ , the study of which is rather challenging, as we saw. It turns out that the bulk of the spectrum is more amenable to analysis. Here is some of what we know about these parts of the spectrum. The adjacency matrix of a graph is a real symmetric matrix, and a large body of work exists about the typical spectra of such matrices. The grandfather of this whole area is the “Wigner’s semicircle law”, which states that, under some conditions, the eigenvalues of a large random symmetric matrix with independent entries are distributed close to a semicircle.

**Theorem 7.1 (Wigner [Wig58]).** *Let  $A_n$  be an  $n \times n$  real symmetric matrix, where off-diagonal entries are sampled independently from the distribution  $F$ , and the diagonal entries from the distribution  $G$ . Furthermore, assume that  $\text{var}(F) = \text{var}(G) = \sigma^2$ , and that  $F$  and  $G$  have finite moments, i.e.  $\int |x|^k dF(x)$  and  $\int |x|^k dG(x)$  are finite for all  $k$ .*

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<sup>1</sup>This model has many important connections to statistical physics, some of which are related to expansion in both the random graphs thus generated, as well as in certain natural dynamics on them, but these are beyond the scope of this book.

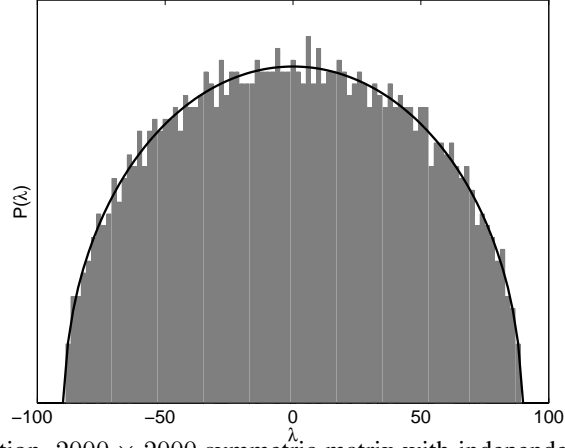


Figure 7.1: Eigenvalue distribution,  $2000 \times 2000$  symmetric matrix with independent standard normal entries. The gray area is a 100 bin histogram, where the height is the portion of eigenvalues in the respective bin. The black line, is the prediction of Theorem 7.1.

Also, define the empirical eigenvalue distribution as

$$W_n(x) = \frac{1}{n} |\{i : \lambda_i(A_n) \leq x\}|,$$

where  $\lambda_1(A_n) \geq \dots \geq \lambda_n(A_n)$  are the eigenvalues of  $A_n$ . Then for every  $x$ ,

$$W(x) = \lim_{n \rightarrow \infty} W_n(2x\sigma\sqrt{n}) = \frac{2}{\pi} \int_{-1}^x \sqrt{1-z^2} dz.$$

What about the analogous questions when we deal with the adjacency matrices of random  $d$ -regular graphs? As can be seen in the following figure, the eigenvalue distribution no longer resembles a semicircle. Nevertheless, for large  $d$ , the distribution does approach a semicircle. This behavior is explained by the following theorem of McKay. In fact, this theorem applies not only to random graphs. It only assumes that the graph in question has few short cycles. Specifically, let  $C_k(G)$  be the number of cycles in  $G$  of length  $k$ . The theorem applies whenever  $C_k(G) = o(|V(G)|)$  for every fixed  $k \geq 3$ , a property that holds almost surely for random  $d$ -regular graphs.

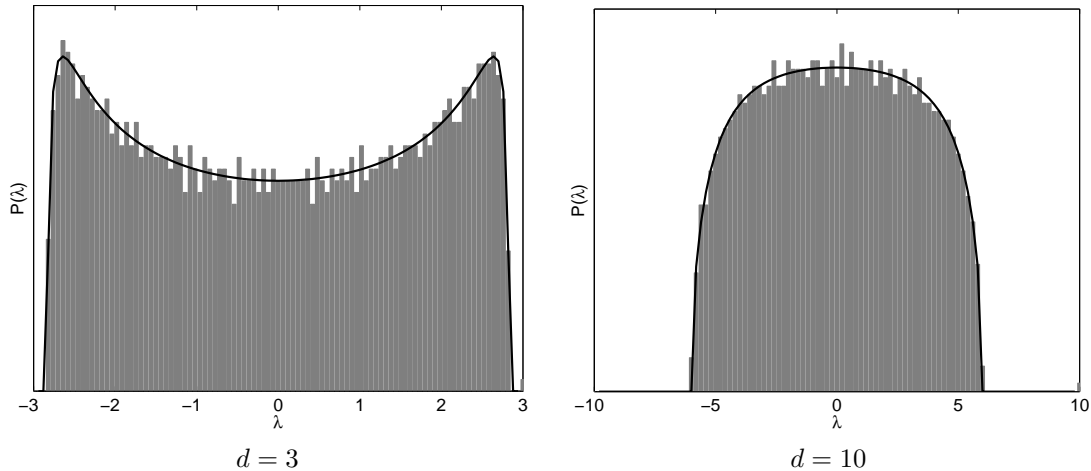


Figure 7.2: Eigenvalue distribution,  $d$ -regular graph with 2000 vertices. The gray area is a 100 bin histogram, where the height is the portion of eigenvalues in the respective bin. The black line, is the prediction of Theorem 7.2.

**Theorem 7.2 (McKay [McK81]).** Let  $G_n$  be an infinite sequence of  $d$ -regular graphs, such that,  $C_k(G_n) = o(|V(G_n)|)$  for all  $k \geq 3$ , where  $C_k(G_n)$  is the number of length  $k$  cycles in  $G_n$ . Define the empirical eigenvalue distribution as

$$F(G_n, x) = \frac{1}{|V(G_n)|} |\{i : \lambda_i(G_n) \leq x\}|.$$

Then for every  $x$ ,

$$F(x) = \lim_{n \rightarrow \infty} F(G_n, x) = \int_{-2\sqrt{d-1}}^x \frac{d\sqrt{4(d-1)-z^2}}{2\pi(d^2-z^2)} dz.$$

Note that the limit distribution  $F(x)$  is supported on  $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ , which is the spectrum of  $\mathbf{T}_d$  (Theorem 5.2). Here is the main idea: Since there are only few short cycles, the neighborhood of most vertices is (nearly) a tree. Consequently, for most vertices  $v$ , the number of walks of length  $k$  that start and end at  $v$  roughly equals  $t_k$ , the analogous quantity for  $\mathbf{T}_d$ . To proceed, we need a good estimate for  $t_k$  (Compare with the lower bound in Section 5.2.2).

**Lemma 7.3.**

$$t_{2s+1} = 0, \quad t_{2s} = \sum_{j=1}^s \binom{2s-j}{s} \frac{j}{2s-j} d^j (d-1)^{s-j}.$$

*Proof.* The first claim follows from the fact that a tree is a bipartite graph and contains no odd length cycles. To determine  $t_{2s}$ , consider any path of length  $2s$  from  $v$  to  $v$ . Associate with it a sequence  $0 = \delta_0, \delta_1 \cdots \delta_{2s} = 0$ , where  $\delta_i$  is our distance from  $v$  at time  $i$ . Clearly,  $\delta_i$  are non-negative integers and  $|\delta_i - \delta_{i-1}| = 1$  for all  $i$ . The number of such sequences in which exactly  $j$  of the terms  $\delta_0 \cdots \delta_{2s-1}$  are 0 is

$$\binom{2s-j}{s} \frac{j}{2s-j}.$$

This is a simple generalization of Catalan numbers (see [Fel68]). How many paths correspond to such a sequence? Each such path takes  $s$  steps away from  $v$  and  $s$  steps towards  $v$ . Namely there are  $s$  indices each with  $\delta_{i+1} - \delta_i = 1$  or  $-1$  respectively. In a step towards  $v$ , the next vertex is uniquely determined. On steps away from  $v$  we have  $d$  choices whenever  $\delta_i = 0$  (and the path resides at  $v$ ), and  $d-1$  choices at each of the remaining  $s-j$  times. The conclusion follows.  $\square$

If  $C_k(G_n) = o(|V(G_n)|)$  for every  $k \geq 3$ , then for every constant  $r$ , almost every vertex has a cycle-free  $r$ -neighborhood. Therefore there are  $(1 + o_n(1))|V(G_n)| \cdot t_r$  closed paths of length  $r$  in  $G_n$ . It follows that the function  $F(x)$  satisfies  $\int x^r dF(x) = t_r$  for all  $r$ . In order to finish the proof of the theorem we need to explicitly recover  $F(x)$  from its moments. This is accomplished by expanding  $F$  in the basis of Chebyshev polynomials. The details are omitted.

## 7.2 The extreme eigenvalues

Quite a lot is known about the extreme eigenvalues of random symmetric matrices as well.

**Theorem 7.4 (Füredi-Komlós [FK81], Vu [Vu05]).** Let  $A = A_n$  be an  $n \times n$  real symmetric matrix with independent entries from a distribution  $F$ , that has zero expectation, variance  $\sigma^2$ , and is supported on  $[-K, K]$  for some constant  $K$ . Then with probability  $1 - o_n(1)$ , all eigenvalues of  $A$  satisfy  $|\lambda_i| < 2\sigma\sqrt{n} + O(n^{\frac{1}{3}} \log n)$ .<sup>2</sup>

The proof of this theorem is based on the **trace method** (a.k.a. the moment method) which we have already encountered in Section 5.2.2. This old work horse of the theory stands as well behind Friedman's Theorem 7.10 and many other results. One estimates first, by combinatorial reasoning the number  $\text{trace}(A^{2k})$  which is a count of closed

<sup>2</sup>There is a version where  $F$  has expectation  $\mu > 0$ . In this case, the bound of the theorem still holds for  $i \geq 2$ . The remaining eigenvalue  $\lambda_1$ , asymptotically, has normal distribution with expectation  $n\mu + \sigma^2/\mu$  and variance  $2\sigma^2$ .

paths of length  $2k$  in a certain (possibly edge-weighted) graph. We subtract from it  $\lambda_1^{2k}$  and claim that if  $k$  is large enough, then the difference is dominated by  $\lambda_2^{2k}$ , whence we can estimate  $|\lambda_2|$ . On the one hand, the larger  $k$  is, the more dominant the contribution of  $\lambda_2^{2k}$ , and the better our estimates become. However, as  $k$  grows, the enumeration of closed walks becomes hairy, so one needs to strike a balance between these two conflicting needs.

Before we proceed to exemplify this method, let us mention an alternative approach to the same problem due to Kahn and Szemerédi [FKS89]. They show that if  $A$  is the adjacency matrix of a random  $d$ -regular graph, then  $A$ 's Rayleigh quotient is likely to be small on all points of an appropriate  $\epsilon$ -net on the unit sphere. Since the Rayleigh quotient of a matrix is a Lipschitz function on unit vectors, the eigenvalues of  $A$  can be bounded. Lemma 2.6 suggests some ways in which this method can be further advanced.

## 7.2.1 An illustration of the trace method

We now illustrate the power of the trace method by proving the following upper bound on  $\lambda(G)$  for a random  $(n, d)$ -graph. A much stronger (possibly tight) result (Theorem 7.10) is now known, but its proof is much too long and complex to be reviewed here.

**Theorem 7.5 (Broder-Shamir [BS87]).** *The largest nontrivial eigenvalue of almost every  $2d$ -regular graph  $G$  satisfies  $\lambda(G) = O(d^{3/4})$ .*

There is a technicality that has to be addressed here on how to sample from the space of  $2d$ -regular graphs. In the proof below we sample such a graph using the *permutation model*. A  $2d$ -regular graph on  $n$  vertices in the permutation model is constructed by independently choosing at random  $d$  permutations  $\pi_1, \dots, \pi_d$  in the symmetric group  $S_n$ , and introducing an edge  $(v, \pi_i(v))$  for every  $v \in [n]$  and  $i \in [d]$ .

It turns out that this does *not* yield a uniform distribution on all  $(n, 2d)$ -graphs. However the theorem is valid as stated, since it is known (see e.g. [JLR00] Chapter 9, and Wormald [Wor99]) that the distribution induced by the permutation model is *contiguous* with the uniform distribution. Namely a family of events have probability  $1 - o(1)$  in one distribution iff they have probability  $1 - o(1)$  in the other. Put differently, both distributions agree on the notion of ‘‘asymptotically almost sure events’’. It is possible to sample uniformly from among the  $(n, 2d)$  graphs using the so-called *configuration model*, but that would complicate matters significantly. By contiguity this additional complication is unnecessary.

*Proof.* Let  $G$  be a random  $2d$ -regular graph on  $n$  vertices in the permutation model. Let  $P$  be the transition matrix of the random walk on  $G$ , i.e.  $G$ 's adjacency matrix divided by  $2d$ . Let  $1 = \mu_1 \geq \mu_2 \geq \dots \geq \mu_n$  be the eigenvalues of  $P$ , and  $\rho = \max\{|\mu_2|, |\mu_n|\}$ . Since the eigenvalues of  $P^k$  are  $\{\mu_i^k\}_{i=1}^n$ , we have  $\rho^{2k} \leq \text{trace}(P^{2k}) - 1$  for any  $k$ . Therefore, by Jansen's inequality,

$$\mathbb{E}[\rho] \leq (\mathbb{E}[\rho^{2k}])^{1/2k} \leq (\mathbb{E}[\text{trace}(P^{2k})] - 1)^{1/2k}. \quad (7.1)$$

This basic inequality is at the heart of the trace method. It bounds the eigenvalues, by estimating the trace of powers of the matrix. These traces are combinatorial quantities, counting the number of close paths of length  $2k$  in our (random) graph.

Observe that the paths in  $G$  starting at vertex 1 are in one to one correspondence with words over the alphabet  $\Sigma = \{\pi_1, \pi_1^{-1}, \dots, \pi_d, \pi_d^{-1}\}$ . Just think of the directed edge  $(v, \pi_i(v))$  (resp.  $(\pi_i(v), v)$ ) as being labeled by  $\pi_i$  (resp.  $\pi_i^{-1}$ ). Now interpret a word as a sequence of directed edge labels to follow. Therefore

$$\mathbb{E}[\text{trace}(P^{2k})] = \mathbb{E}[\text{f.p.}(\omega)] = n \cdot \Pr[\omega(1) = 1] \quad (7.2)$$

where  $\omega$  is a uniformly chosen word in  $\Sigma^{2k}$  and  $\text{f.p.}(\omega)$  is the number of fixed points of the permutation  $\omega$  i.e., the number of  $i$  with  $\omega(i) = i$ .

Our analysis is in two parts. We first consider the structure of the word  $\omega$  as an element of the free group in  $d$  generators. We then investigate the actual walk corresponding to  $\omega$  when these  $d$  generators take on particular (random) permutations from  $S_n$ . For the first part it is natural to *reduce* this word, namely repeatedly remove every two consecutive letters one of which is  $\pi_j$  and the other  $\pi_j^{-1}$ . Let  $\omega' = \text{red}(\omega)$  be the reduction of  $\omega$ . Clearly,

$\Pr[\omega(1) = 1] = \Pr[\omega'(1) = 1]$ . We show that such a random reduced word is very unlikely to exhibit a nontrivial periodicity of a type specified below. A word that is periodic in this sense is called a *bad* reduced word.

For the second part we examine a fixed reduced *good* word  $\omega'$  and derive an upper bound on  $\Pr[\omega'(1) = 1]$ . As mentioned,  $\omega'$  is fixed and the probability is considered under the random choice of the  $d$  permutations  $\pi_1, \dots, \pi_d \in S_n$ . This is relatively simple when  $\omega'$  is good and reduced.

A reduced word  $\omega'$  is **bad** if it has the form  $\omega' = \omega_a \omega_b^j (\omega_a)^{-1}$  for some words  $\omega_a, \omega_b$  and some  $j \geq 2$ . Note that the empty word is bad. We estimate two probabilities, whose sum is an upper bound on  $\Pr[\omega(1) = 1]$ :

1. The probability that  $\omega'$  is a bad word.
2. The probability  $\Pr[\omega'(1) = 1]$  for some arbitrary **fixed** good reduced word  $\omega'$ .

The first probability is estimated in the next lemma.

**Lemma 7.6.** *The probability that  $\omega'$  is bad, for a random uniformly drawn  $\omega \in \Sigma^{2k}$ , is at most  $O((2/d)^k)$ .*

*Proof.* The idea is that for  $\omega$  to have a bad reduction, at least half of its letters are determined by the rest (either to get the cancelations, or to guarantee periodicity). Implementing this idea takes a bit of care. First, we upper bound the probability that  $\omega$  reduces to a word of length  $2l$  for some  $l \leq k$ . Then we bound the probability that a reduced word has the structure of a bad word.

We observe that all words  $\omega$  of length  $2k$  which reduce to a word of length  $2l$  can be generated as follows. Start with an sequence of  $k - l$  left brackets and  $k - l$  right brackets where each initial segment contains at least as many left brackets as right brackets. A *breakpoint* in this sequence is a place where the initial segment contains an equal number of left and right brackets. (This includes the positions to the left and to the right of the whole sequence). In every breakpoint we place an arbitrary number of  $\star$  symbols with a total of  $2l$   $\star$ 's. The brackets in such a sequence can be paired up in a unique allowable way. Now associate letters from our alphabet  $\Sigma$  to the brackets where a pair of brackets that are matched are assigned a letter and its inverse. The  $\star$ 's are relaced by symbols from  $\Sigma$  subject to the following condition: The assignment of symbols to the brackets specifies the cancellations steps that reduce  $\omega$  to a word of length  $2l$ . The other letters of  $\omega$  are to be chosen so that no additional cancellations should take place. A small adaptation of the expression in the proof of Lemma 7.3 yields that there are  $\binom{2k-2l-j}{k-l} \frac{j}{2k-2l-j}$  sequences of brackets with  $j + 1$  breakpoints. There are  $\binom{2l+j}{j}$  ways to place the  $2l$  stars in these  $j + 1$  breakpoints. It follows that the probability that the reduced word has length  $2l$  is at most

$$(2d)^{-k+l} \sum_j \binom{2l+j}{j} \binom{2k-2l-j}{k-l} \frac{j}{2k-2l-j}.$$

(This is not an equality since we ignored the condition that the  $\star$ -word is irreducible). This bound is easily seen to be at most  $\binom{2k}{k-l} (2d)^{-k+l}$ .

Now it is easy to see that the probability that a reduced word of length  $2l$  has the form  $\omega' = \omega_a \omega_b^j (\omega_a)^{-1}$  for some words  $\omega_a, \omega_b$  and some  $j \geq 2$  is at most  $(2d)^{-l}$ . This is because the words  $\omega_a, \omega_b$  uniquely determine the rest of the letters in  $\omega'$  of which there are at least  $l$ .

Taken together, these estimates imply that the probability that  $\omega'$  is bad does not exceed

$$\sum_l (2d)^{-l} \cdot \binom{2k}{k-l} (2d)^{-k+l} \leq (2/d)^k,$$

as claimed. □

Now we fix a good reduced word  $\omega'$  of length  $s \leq 2k$ , and turn to estimate  $\Pr[\omega'(1) = 1]$ . The main conceptual idea of Broder and Shamir is that we expose the path that we follow only "as we go". It is here that the value of the permutation model in the analysis becomes apparent. We gradually reveal the relevant information about the random permutations  $\pi_1, \dots, \pi_d$  and their inverses, as follows. We start with the vertex  $v_0 = 1$ , and compute the path  $v_0, v_1, v_2, \dots, v_s$  one step at a time. For  $i \geq 1$  we compute  $v_i$  as follows. If  $\sigma \in \Sigma$  is the  $i$ -th letter in  $\omega'$ , then we should set  $v_i = \sigma(v_{i-1})$ . Call step  $i$  *free* if the value of  $\sigma(v_{i-1})$  was not "revealed" in a previous step and is still

undetermined. If this is the case and if  $t$  values of  $\sigma$  were previously revealed, then  $v_i$  is selected from among the  $n - t$  vertices not yet assigned to  $\sigma$ 's range. A step that is not free is called *forced*.

Call step  $i$  a *coincidence* if it is free, and moreover (the randomly selected vertex)  $v_i$  coincides with a previous vertex on the path. Namely  $v_i \in \{v_0, \dots, v_{i-1}\}$ . Let  $C_i$  denote the event of a coincidence at step  $i$ . By what we said above  $\Pr[C_i | v_0 = u_0, \dots, v_{i-1} = u_{i-1}] \leq s/(n - s)$  for every  $u_0 = 1, u_1, \dots, u_{i-1}$ . We see that the probability of a coincidence at step  $i$  is at most  $s/(n - s) \leq 2k/(n - 2k)$ , regardless of preceding history.

Clearly for the event that  $\omega'(1) = 1$  to hold, at least one coincidence must occur, so we bound the probability of this event by the sum of the probabilities of two events:

1. At least two coincidences have occurred along the path.
2. Exactly one coincidence has occurred and  $v_s = 1$ .

The first probability is easily bounded as follows. Given the positions  $i, j$  of the first two coincidences, as mentioned above, the probability that these coincidences occur is at most  $\Pr[C_i] \cdot \Pr[C_j | C_i] \leq (s/(n - s))^2$ . There are at most  $s^2$  such positions, and so the total probability is at most  $s^4/(n - s)^2 = O(k^4/n^2)$ . The last inequality holds, since our choice will be that  $k = o(n)$ .

The second probability is bounded in the following lemma.

**Lemma 7.7.** *Let  $\omega'$  be a good reduced word of length  $s \leq 2k$ , and consider the path starting at 1 and following  $\omega'$ . Then the probability of having exactly one coincidence, and ending at vertex 1 is at most  $\frac{1}{n-s+1} \leq \frac{1}{n-2k}$ .*

*Proof.* A realization of  $\omega'$  is any walk corresponding to  $\omega'$  that is determined by a choice of the permutations in  $\Sigma$ . Since  $\omega'$  is reduced, any initial segment of such a realization preceding the (single) coincidence is a simple path. The step at which the coincidence occurs turns this path into a cycle with a (possibly empty) tail. Since no more coincidences take place and since  $\omega'$  is reduced, no additional edges will be visited, and all future steps are forced. In order to eventually reach vertex 1, the walk could proceed in one of two ways: (i) Revolve around the cycle any number of times and then proceed to vertex 1; Or (ii) Turn immediately to the end of the tail and terminate the walk at vertex 1 there. The former possibility is ruled out since the word  $\omega'$  is good.

It follows that  $\omega' = \omega_a \omega_b (\omega_a)^{-1}$ . Here  $\omega_a$  may be empty and corresponds to the walk along the tail and  $(\omega_a)^{-1}$  - to the way back. The word  $\omega_b$  is nonempty and corresponds to the walk around the cycle. Let  $r = |\omega_a| + |\omega_b|$ . It follows that the event  $\omega'(1) = 1$  is included in the event that the  $(r - 1)$ -th step is a coincidence. The probability of the latter is  $\leq \frac{1}{n-r} \leq \frac{1}{n-s+1}$ , and the claim follows. □

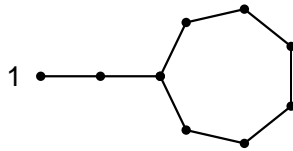


Figure 7.3: A cycle with a tail

Adding the two bounds, the proof of the lemma follows. □

We now put the bounds of the two lemmas together, obtaining

$$\Pr[\omega(1) = 1] \leq 1/(n - 2k) + O((2/d)^k) + O(k^4/n^2)$$

We choose  $k = 2 \log_{d/2} n$  to minimize the upper bound. Substituting it back into (7.1) and (7.2), yields  $\mathbb{E}[\rho] \leq (2/d)^{1/4} \cdot (1 + o(1))$ . We finish up the proof by Markov's inequality, stating that a random variable is not likely to take on values much larger than its expectation.

Broder and Shamir [BS87] have also observed that the random variable  $\lambda(G)$  considered for the probability space of  $d$ -regular graphs  $G$ , is highly concentrated. Namely, for a large  $d$ -regular graph  $G$ , almost surely,  $\lambda(G)$  is within  $O(\sqrt{d})$  of its expected value. This is done using martingales and Azuma's inequality. For more on this very useful technique see e.g., [AS00, JLR00].

## 7.3 Variations on a theme

### 7.3.1 Back to the irregular case

Let us come back to Definition 6.7 which extends the concept of Ramanujan Graphs to the irregular case. It is tempting to consider problems such as Conjecture 5.12 in this more general context. Is it true that for every graph  $G$ , almost all sufficiently high lifts are Ramanujan?

To address this problem, Friedman [Fri03], rephrased the conjecture:

**Conjecture 7.8.** *For almost all sufficiently high lifts of any graph  $G$ , the absolute value of all new eigenvalues is bounded by  $\rho + o(1)$ , where  $\rho$  is the spectral radius of the universal cover of  $G$ .*

To support his conjecture, Friedman proved the following theorem by generalizing the Broder Shamir argument of Theorem 7.5.

**Theorem 7.9 (Friedman [Fri03]).** *Let  $G$  be a graph with a largest eigenvalue  $\lambda_1$ , and let  $\rho$  be the spectral radius of its universal cover. Then for almost all sufficiently high lifts of  $G$ , the absolute value of all new eigenvalues is bounded by  $\sqrt{\lambda_1 \rho} + o(1)$ .*

Indeed, this generalizes Theorem 7.5, since for  $2d$ -regular graphs, the permutations model coincides with random  $n$ -lifts of a  $2d$ -regular graph consisting of one vertex with  $d$  loops. In this case, the old spectrum is the single eigenvalue  $\lambda_1 = 2d$  and  $\rho = 2\sqrt{d-1} = O(\sqrt{d})$ .

As mentioned before, even in the  $d$ -regular case, Conjecture 7.8 is not settled yet.

### 7.3.2 Are most regular graphs Ramanujan?

A major result due to Friedman, is a relaxed version of Conjecture 5.12, and the regular case of Conjecture 7.8:

**Theorem 7.10 (Friedman [Fri]).** *For every  $\epsilon > 0$ ,*

$$Pr\left(\lambda(G) \leq 2\sqrt{d-1} + \epsilon\right) = 1 - o_n(1)$$

where  $G$  is a random  $(n, d)$  graph.

To arrive at an educated guess, whether most  $d$ -regular graphs are Ramanujan, the most natural approach it is to carry out some computer experiments. Such experiments were conducted by Novikoff [Nov02] (see also [Sar04]) and by Hoory. We show here the results obtained by taking 1000 random 4-regular graphs in the permutation model, for varying graph sizes up to  $n = 400000$ .<sup>3</sup>

$n$	100	400	1000	4000	10000	40000	100000	400000
$\Pr(\lambda(G) < 2\sqrt{3})$	0.62	0.64	0.63	0.66	0.68	0.67	0.67	0.68
$2\sqrt{3} - \text{median}(\lambda(G))$	0.01519	0.00593	0.00361	0.00157	0.00100	0.00036	0.00019	0.00007
$2\sqrt{3} - \text{mean}(\lambda(G))$	0.01026	0.00487	0.00283	0.00129	0.00086	0.00029	0.00017	0.00007
$\text{std}(\lambda(G))$	0.06376	0.01872	0.01012	0.00372	0.00190	0.00079	0.00042	0.00017

It is evident, that the median and expected value of  $\lambda$  tend to  $2\sqrt{d-1}$  from below, and that its standard deviation tends to zero with growing graph size. In [Nov02], it is conjectured that, after normalization, the distribution of  $\lambda$  tends to a Tracy-Widom distribution. Furthermore, they conjecture that the mean approaches faster than the standard deviation, and that the probability of being Ramanujan tends to a constant which is *different* than zero and one.

<sup>3</sup> $\lambda(G)$  was calculated using the matlab function `eigs`. The 3 smallest and largest values were excluded in the calculation of the mean and the standard deviation.

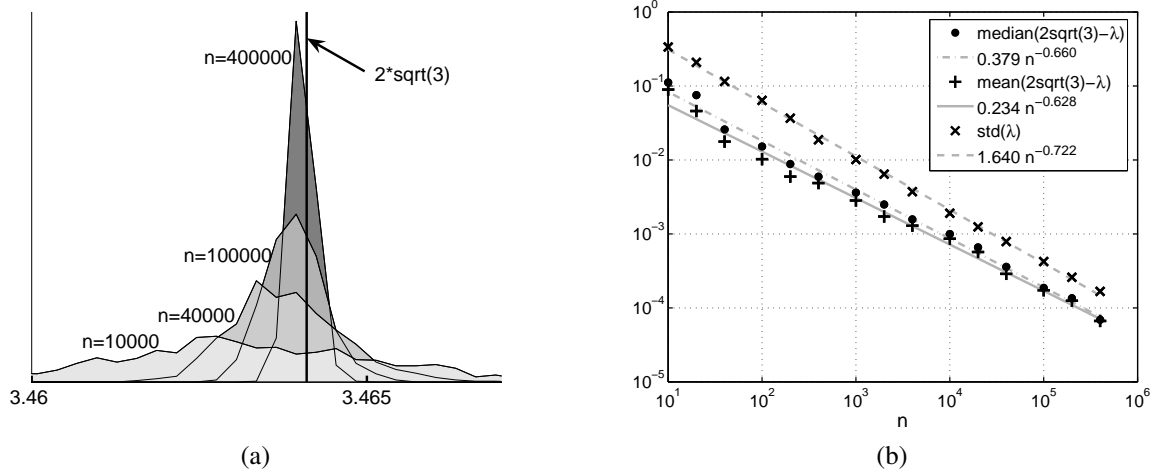


Figure 7.4: (a) Distribution of  $\lambda(G)$  for 1000 random 4-regular graphs in the permutation model. Four 40 bin histograms of  $\lambda(G)$  for graph sizes 10000, 40000, 100000, 400000. (b) median, mean and standard deviation of  $2\sqrt{d-1} - \lambda(G)$  as a function of the graph size  $n$ . A log-log graph, along with the best linear interpolations.

### 7.3.3 More on random lifts

As pointed out in 6.1,  $n$ -lifts of a given connected graph can be generated at random. This is an entirely different source of random graphs whose vertex degrees can be completely controlled. A recent application for these “crafted” graph distributions, is the random generation of graphs, where belief propagation decoding does better than graph generated by the standard configuration model, see [Tho03, RU]. We illustrate below some of what is known about combinatorial properties of random lifts. The following statements refer to graphs in  $L_n(G)$ , where  $G$  is a fixed connected graph with more edges than vertices and where  $n$  is large. In particular “almost all” means that the statement holds in the above space with probability  $1 - o_n(1)$ .

[AL] For every  $G$  there is an  $\epsilon_G > 0$ , such that almost all lifts of  $G$  have expansion ratio  $\geq \epsilon_G$ .

[AL02] If  $\delta \geq 3$  is the smallest vertex degree in  $G$ , then no graph in  $L_n(G)$  has connectivity  $> \delta$  and almost all graphs there are  $\delta$ -connected.

[LR05] A zero/one law about the existence of perfect matching. Here  $n$  is even and large. All graphs  $G$  fall into two categories. One where almost all  $n$ -lifts of  $G$  have a perfect matching, and the other where almost none of  $G$ 's  $n$ -lifts have a perfect matching.

There are also some results about chromatic numbers [ALM02], graph minors [DL], and Hamiltonian circuits in lifts [BCCF05].

In analogy with Conjecture 5.12, it seems natural to ask questions such as:

**Open problem 7.11.** Fix a  $d$ -regular base graph, e.g.,  $G = K_{d+1}$ . How likely is it that all new eigenvalues of an  $n$ -lift of  $G$  fall in the range  $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ ?

### 7.3.4 The eigenvectors

There is very little we have to say here, except to point out what seems at present like a total mystery that's worth investigating. Can anything be said about the distribution of the coordinates in eigenvectors for any of the above classes of random matrices? In particular:

**Open problem 7.12.** Fix an integer  $d \geq 3$  and consider large random  $(n, d)$ -graphs. Clearly, all the coordinates of the first normalized eigenvector  $v_1$  equal  $1/\sqrt{n}$ . What can be said about the distribution of the coordinates of  $v_2$ ?

*Specifically, does  $v_2$  tend to be uniformly distributed on the unit sphere (in which the distribution of these coordinates is nearly normal)?*

We hesitate to say here much, but some preliminary numerical experiments carried out by N.L.'s students suggest that the answer is negative. What's perhaps even more intriguing is that there seems to be another (nonuniform) limit distribution involved. At present we do not know much about these fascinating issues.



## Chapter 8

# The Margulis Construction

In this chapter we return to the oldest explicit construction of a family of expander graphs. This construction is still among the most elegant, and most easy to generate of all known constructions. Nevertheless, the simplest known proof that it is an expander (only a few pages of basic linear algebra), which we present here, is still subtle and mysterious. We recall an elementary approach to these graphs that is mentioned in Section 4.3. The concrete conjecture stated there suggests how a more intuitive proof may look like.

We now turn to describe the construction. Again we recall from Section 4.3 an infinite analog of this construction. There we start with the action of two linear transformations on the unit torus. Here, in the finite setting, we take essentially the same linear transformations, together with their affine shifts (which serve as a discrete substitute for continuity). These act on the finite torus  $(\mathbb{Z}_n)^2$ . In Chapter 12 we will see that the linear transformations themselves suffice for expansion, at least when  $n$  is a prime. It will be viewed as the action of the group  $SL_2(n)$  of  $2 \times 2$  matrices of determinant 1, which is generated by these two linear transformations. That proof of expansion is, however, far from elementary.

**Construction 8.1.** Define the following 8-regular graph  $G_n = G = (V, E)$  on the vertex set  $V = \mathbb{Z}_n \times \mathbb{Z}_n$ . Let

$$T_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Each vertex  $v = (x, y)$  is adjacent to the four vertices  $T_1v, T_2v, T_1v + e_1, T_2v + e_2$ , and the other four neighbors of  $v$  obtained by the four inverse transformations. Note that all calculations are mod  $n$ , and that this is an undirected 8-regular graph (that may have multiple edges and self loops).

**Theorem 8.2.** [GG81]  $\lambda(G) \leq 5\sqrt{2} < 8$

We will, in fact, prove in full a slightly weaker bound on  $\lambda$  that is still smaller than 8. As we saw in Section 4.5, this already implies the graphs  $G_n$  form a family of expanding graphs.

Margulis [Mar73] proved the same conclusion for a closely related family of graphs in 1973. However, his method was inherently existential (using the machinery of Kazhdan's property (T)) and couldn't give an explicit lower bound on the spectral gap. In 1981 Gabber and Galil [GG81] were able to derive a lower bound on the gap. Their argument uses classical harmonic analysis. In 1987 Jimbo and Marouka [JM87] improved it further using Fourier analysis on the group  $G = \mathbb{Z}_n^2$ . We present here a slight simplification of their proof, due to Boppana.

By the variational formula for the eigenvalues (Section 4.5) we are trying to prove that if  $f : \mathbb{Z}_n^2 \rightarrow \mathbb{R}$  satisfies  $\sum_x f(x) = 0$ , then

$$\sum_{(x,y) \in E} f(x)f(y) \leq \frac{5\sqrt{2}}{2} \sum f^2(x).$$

Taking into account the definition of the graph  $G$ , Theorem 8.2 can therefore be restated as:

**Theorem 8.3.** For any  $f : \mathbb{Z}_n^2 \rightarrow \mathbb{R}$  satisfying  $\sum_z f(z) = 0$ , the following inequality holds:

$$\sum_{z \in \mathbb{Z}_n^2} f(z) \cdot [f(T_1 z) + f(T_1 z + e_1) + f(T_2 z) + f(T_2 z + e_2)] \leq \frac{5\sqrt{2}}{2} \sum f^2(x) \quad (8.1)$$

## 8.1 A detour into harmonic analysis

The intensifying connections between harmonic analysis and discrete mathematics are among the most wonderful recent developments of both fields. We are unable to do here any justice to this domain of research. The reader can learn the fundamentals of harmonic analysis from excellent texts such as Körner [Kör89] and Terras [Ter99]. Certain parts of this area are covered by lecture notes available at [www.cs.huji.ac.il/~nati/PAPERS/uw/](http://www.cs.huji.ac.il/~nati/PAPERS/uw/).

Below we collect some of the basic tools that we need to proceed with our analysis. None of this is hard and the proofs can be found in the above references (or be carried out on the reader's own). We will consider similar issues in a more general context in Chapter 12. Namely, Fourier analysis on general groups, a.k.a representation theory. Here we confine ourselves to Abelian groups. It is remarkable, though, that we are able to investigate the action of the (non-Abelian) group of  $2 \times 2$  matrices using “only” harmonic analysis.

### 8.1.1 Characters

**Definition 8.4.** A *character* of a group  $H$  is a homomorphism  $\chi : H \rightarrow \mathbb{C}$  (i.e.  $\chi(gh) = \chi(g) \cdot \chi(h)$  for all  $h, g \in H$ ).

Note that this definition implies that the range of any character is actually contained in the unit circle, namely complex numbers of absolute value 1. Also, when  $H$  is Abelian (as is the case in the present chapter), we denote, as customary, its group operation by  $+$ .

Here are some examples of characters.

- The *trivial character* maps all elements of  $H$  to 1.
- For  $H$  the cyclic group  $\mathbb{Z}_n$ , the characters are  $\chi_k(h) = e^{2\pi i k h / n}$ . (The trivial character corresponds to  $k = 0$ ).
- For  $H = (\mathbb{Z}_2)^d$ : For some  $a = (a_1, \dots, a_d) \in (\mathbb{Z}_2)^d$  the characters are  $\chi_a(x) = (-1)^{\langle a, x \rangle}$  where  $\langle x, y \rangle = \sum_i x_i y_i$  is the inner product. (The trivial character corresponds to  $a = 0^d$ ).
- The group that is most relevant for the present chapter is  $H = \mathbb{Z}_n^2$ . It has a character  $\chi_b$  for each  $b = (b_1, b_2) \in \mathbb{Z}_n^2$ , where  $\chi_b(a_1, a_2) = \omega^{a_1 b_1 + a_2 b_2}$ . Here  $\omega = e^{2\pi i / n}$  is the primitive  $n$ -th root of unity.

The collection  $\mathcal{F}$  of all complex functions on  $H$  is a linear space with inner product  $\langle f, g \rangle = \sum_{x \in \Gamma} \overline{f(x)} g(x)$ . Fourier analysis on  $H$  entails expanding functions in  $\mathcal{F}$  as linear combinations of characters.

**Proposition 8.5.** Every finite Abelian group  $H$  has  $|H|$  distinct characters which can be naturally indexed as  $\{\chi_x\}_{x \in H}$ . They form an orthonormal basis of  $\mathcal{F}$ . Thus every  $f : H \rightarrow \mathbb{C}$  can be uniquely expressed as  $f = \sum_x \widehat{f}(x) \chi_x$ , where  $\widehat{f} : H \rightarrow \mathbb{C}$  is the discrete Fourier transform of  $f$ ,

$$\widehat{f}(x) = \langle f, \chi_x \rangle = \sum_y \overline{f(y)} \cdot \chi_x(y)$$

Therefore, in the case of interest here,  $H = \mathbb{Z}_n^2$ , the discrete Fourier transform of  $f$  takes the form

$$\widehat{f}(x) = \sum_b \overline{f(b)} \cdot \omega^{b_1 x_1 + b_2 x_2}.$$

In the following proposition we collect some of the basic properties of the Fourier transform. We state the results for the group  $\mathbb{Z}_n^2$ , though most of these claims apply just as well for any Abelian group.

**Proposition 8.6.** (a)  $\sum_a f(a) = 0 \Leftrightarrow \widehat{f}(0) = 0$ .

(b)

$$\langle f, g \rangle = \frac{1}{n^2} \langle \widehat{f}, \widehat{g} \rangle \text{ for any } f, g \in \mathcal{F}$$

(c) A special case of (b) when  $f = g$ , is Parseval's identity:

$$\sum_a |f(a)|^2 = \frac{1}{n^2} \sum_a |\widehat{f}(a)|^2.$$

(d) The inverse formula :

$$f(a) = \frac{1}{n^2} \sum_b \widehat{f}(a) \omega^{-\langle a, b \rangle}.$$

(e) The shift property<sup>1</sup> If  $A$  is a non-singular  $2 \times 2$  matrix over  $\mathbb{Z}_n$ ,  $b \in \mathbb{Z}_n^2$  and  $g(x) = f(Ax + b)$  then

$$\widehat{g}(y) = \omega^{-\langle A^{-1}b, y \rangle} \widehat{f}((A^{-1})^t y).$$

## 8.2 Back to the proof

To prove Theorem 8.3, we express inequality (8.1) using the Fourier coefficients of  $f$ . The condition  $\sum_z f(z) = 0$  is restated as  $\widehat{f}(0, 0) = 0$ . Using Parseval's identity and the behavior of the transform under composition with an affine transformation, we can rewrite Theorem 8.3 in the form below. It would suffice to show this with  $F$  the Fourier transform of  $f$ , but we prove this claim in its generality.

**Theorem 8.7.** For every  $F : \mathbb{Z}_n^2 \rightarrow \mathbb{C}$  with  $F(0, 0) = 0$ ,

$$\left| \sum_{z=(z_1, z_2) \in \mathbb{Z}_n^2} \overline{F(z)} \cdot [F(T_2^{-1}z)(1 + \omega^{-z_1}) + F(T_1^{-1}z)(1 + \omega^{-z_2})] \right| \leq \frac{5\sqrt{2}}{2} \sum_{z \in \mathbb{Z}_n^2} |F(z)|^2.$$

We define  $G : \mathbb{Z}_n^2 \rightarrow \mathbb{R}$  via  $G = |F|$ . By the triangle inequality and the identity  $|1 + \omega^{-t}| = 2|\cos(\pi t/n)|$ , it suffices to prove:

**Theorem 8.8.** For every non-negative function  $G : \mathbb{Z}_n^2 \rightarrow \mathbb{R}$  with  $G(0, 0) = 0$ :

$$\sum 2G(z) \cdot [G(T_2^{-1}z) \cdot |\cos(\pi z_1/n)| + G(T_1^{-1}z) \cdot |\cos(\pi z_2/n)|] \leq \frac{5\sqrt{2}}{2} \sum G^2(z) \quad (8.2)$$

where the summations are over  $z = (z_1, z_2) \in \mathbb{Z}_n^2$ .

We seek to bound the terms on the left by squares that match the terms on the right. We will do this by means of the elementary inequality  $2\alpha\beta \leq \gamma\alpha^2 + \gamma^{-1}\beta^2$  that holds for any non-negative  $\alpha, \beta$ , and  $\gamma$ . The key will be to choose  $\gamma$  in a clever way; it will not be constant, but rather a mapping  $\gamma : (\mathbb{Z}_n^2)^2 \rightarrow \mathbb{R}$ .

How should we choose  $\gamma$ ? First note that setting  $\gamma$  to be identically 1 is problematic for small  $z_1, z_2$  where the cosines are near 1. That would yield the inequality with a coefficient of 8, which is just what we wanted to avoid. This suggests, though, that we let  $\gamma$  take only values close to 1. The idea is to define a partial order on  $\mathbb{Z}_n^2$ , and let  $\gamma$  satisfy:

$$\gamma((z_1, z_2), (z'_1, z'_2)) = \begin{cases} \alpha & \text{if } (z_1, z_2) > (z'_1, z'_2) \\ 1/\alpha & \text{if } (z_1, z_2) < (z'_1, z'_2) \\ 1 & \text{otherwise.} \end{cases}$$

<sup>1</sup>This shows how the Fourier Transform behaves under a composition with a nonsingular affine transformation. It is this relation which makes it possible to analyze the non-Abelian group action of  $2 \times 2$  matrices based on Fourier analysis of the Abelian group  $(\mathbb{Z}_n)^2$

Here  $\alpha$  should be a constant slightly bigger than 1. Indeed we set  $\alpha = 5/4$ , so that  $\gamma$  takes only the values 1,  $\alpha = 5/4$  and  $1/\alpha = 5/4$ . This definition implies that:

$$\gamma(x, y) \cdot \gamma(y, x) = 1. \quad (8.3)$$

for every  $x, y \in \mathbb{Z}_n^2$ . We write

$$2G(x)G(y) \leq \gamma(x, y)G^2(x) + \gamma(y, x)G^2(y),$$

to derive the following upper bound on the left side of (8.2):

$$\begin{aligned} \sum_z & |\cos(\pi z_1/n)| \cdot [\gamma(z, T_2^{-1}z)G^2(z) + \gamma(T_2^{-1}z, z)G^2(T_2^{-1}z)] \\ & + |\cos(\pi z_2/n)| \cdot [\gamma(z, T_1^{-1}z)G^2(z) + \gamma(T_1^{-1}z, z)G^2(T_1^{-1}z)]. \end{aligned}$$

So far everything applies to any linear transformations  $T_1, T_2$ . We now use the fact that they are triangular. The exact choice of the matrices will come in only later. We use the fact that  $z_2$  is invariant under  $T_1$  and likewise with  $z_1$  and  $T_2$ . Inequality (8.2) would follow from:

$$\begin{aligned} & \sum_z G^2(z) \cdot (|\cos(\pi z_1/n)| \cdot [\gamma(z, T_2z) + \gamma(z, T_2^{-1}z)] + |\cos(\pi z_2/n)| \cdot [\gamma(z, T_1z) + \gamma(z, T_1^{-1}z)]) \\ & \leq \frac{5\sqrt{2}}{2} \sum_z G^2(z). \end{aligned}$$

Since this should hold for every  $G$ , let us examine the case where  $G$  is nonzero on a single point  $z \neq (0, 0)$ . If the present approach to proving Theorem 8.2 is viable, then it should work *term by term*. Thus our proof will succeed if our function  $\gamma$  satisfies

$$|\cos(\pi z_1/n)| \cdot [\gamma(z, T_2z) + \gamma(z, T_2^{-1}z)] + |\cos(\pi z_2/n)| \cdot [\gamma(z, T_1z) + \gamma(z, T_1^{-1}z)] \leq \frac{5\sqrt{2}}{2}. \quad (8.4)$$

for every  $z = (z_1, z_2) \in \mathbb{Z}_n^2 \setminus (0, 0)$ .

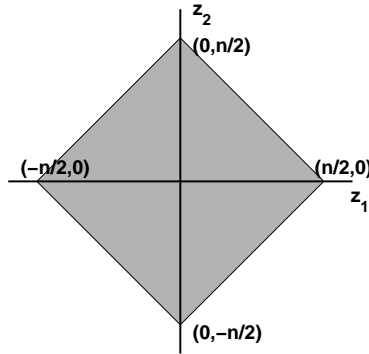


Figure 8.1: The diamond

We split our proof to two domains. **Outside the diamond** in Figure 8.1 we overestimate all the  $\gamma$  terms by  $\alpha = \frac{5}{4}$ . As we verify below, in this range  $|\cos(\pi z_1/n)| + |\cos(\pi z_2/n)| \leq \sqrt{2}$  which implies the necessary inequality. Let us assume  $z_1, z_2$  are in the first quadrant. The other cases follow similarly. Since  $\cos(\pi z_2/n)$  is decreasing, and since we are outside the diamond, this expression is maximized on the boundary of the diamond  $z_2 = n/2 - z_1$ , where  $\cos(\pi z_2/n) = \sin(\pi z_1/n)$ . Consequently,  $\cos(\pi z_1/n) + \cos(\pi z_2/n) = \cos(\pi z_1/n) + \sin(\pi z_1/n) \leq \sqrt{2}$ , as needed.

When  $(z_1, z_2)$  is **inside the diamond**, we just bound the cosines by 1, and wish to prove that

$$\gamma(z, T_1z) + \gamma(z, T_1^{-1}z) + \gamma(z, T_2z) + \gamma(z, T_2^{-1}z) \leq \frac{5\sqrt{2}}{2}. \quad (8.5)$$

That would follow from the following claim:

**Proposition 8.9.** *For every  $(z_1, z_2)$  inside the diamond, either:  
Three of the four points  $T_1z, T_2z, T_1^{-1}z$  and  $T_2^{-1}z$  are  $> z$  and one is  $< z$ .*

*or*

*Two of the four points  $T_1z, T_2z, T_1^{-1}z$  and  $T_2^{-1}z$  are  $> z$  and two are incomparable with  $z$ .*

In the first case, the l.h.s. of (8.5) is  $3/\alpha + \alpha$ , while in the second case it is  $2/\alpha + 2$ . Since  $\alpha = 5/4$ , the l.h.s. of (8.4) is bounded by 3.65. This is not as good as  $5\sqrt{2}/2 = 3.53\dots$ , but it does prove that the graphs are a family of good expanders. A different choice for  $\alpha$ , and a more careful analysis of the inequality inside the diamond yield the  $5\sqrt{2}/2$  bound.

To define our partial order on  $\mathbb{Z}_n^2$ , we use the convention that our variables take values in the interval  $[-n/2, n/2)$ . Say that  $(z_1, z_2) > (z'_1, z'_2)$  if  $|z_1| \geq |z'_1|$  and  $|z_2| \geq |z'_2|$ , and at least one of the inequalities is strong.

We only know how to verify Proposition 8.9 by a case analysis, so here is one of the cases. It is easy to verify that if  $|z_1| = |z_2|$ , then the second case of the proposition holds, so we assume w.l.o.g. that  $|z_1| > |z_2|$ . As mentioned, the second coordinate of  $T_1^{\pm 1}z$  is  $z_2$  and likewise with  $T_2$  and  $z_1$ . So in each case there is only one inequality to analyze. By symmetry we may assume that  $z_1 > z_2 \geq 0$ , and since we are inside the diamond,  $z_1 + z_2 \leq n/2$ . It follows that  $|z_1 - 2z_2| < |z_1|$ . Consequently,  $T_1^{-1}z < z$ , as in the first case of the proposition.



# Chapter 9

## The Zig-Zag Product

In this chapter we introduce a new kind of graph product called the **Zig-Zag Product**. We show that the zig-zag product of two expanders is a (not much worse) expander as well. We start by showing how this fact leads to an iterative construction of an explicit family of expanders. We then present the proof of this fact.

We already know that a given graph is an expander iff the random walk on that graph mixes rapidly. As we discussed in 3.1.2, it is possible to estimate the expansion rate of an  $(n, d)$ -graph  $G$  by considering how rapidly the entropy of the random walk on  $G$  converges to  $\log n$ . This suggests an interesting perspective of zig-zag products which we discuss as well.

This combinatorial method for constructing expanders was suggested by Reingold, Vadhan and Wigderson in [RVW02]. It has led to other constructions, such as expanders which beat the eigenvalue bound (i.e. have better expansion than implied by their spectral gap), and even new Cayley expanders. Both of these will be discussed in later chapters. We conclude this chapter with a remarkable recent application of the zig-zag product to complexity theory. Namely, Reingold's result  $SL = L$ . This means that there is a deterministic algorithm with only  $S$  units of working memory that can explore every graph of size  $e^{O(S)}$ .

### 9.1 Introduction

As in earlier chapters we consider an  $(n, d)$  graph  $G = (V, E)$  with adjacency matrix  $A = A(G)$ . We also recall the notation  $\hat{A} = \frac{1}{d}A(G)$ , the transition matrix of the random walk on  $G$ . Also, if  $G$  is an  $(n, d, \alpha)$ -graph, then Theorem 4.11 gives a lower bound  $(1 - \alpha)d/2 \leq h(G)$  on  $h(G)$ , the expansion ratio of  $G$ .

The  $k$ -th power  $G^k = (V, E')$  is a graph on the same vertex set where we put an edge  $(u, w) \in E'$  for every path of length  $k$  in  $G$  from  $u$  to  $w$ . The adjacency matrix of  $G^k$  is just the  $k$ -th power of the adjacency matrix of  $G$ , whence  $G^k$  is an  $(n, d^k, \alpha^k)$ -graph.

The zig-zag product, which is denoted by  $\mathbb{Z}$ , is an asymmetric binary operation. The product of an  $(n, m)$ -graph and an  $(m, d)$  graph is an  $(nm, d^2)$ -graph. The main result is that the product of two expanders is an expander. We can use this result as a “black box” even before we actually define the zig-zag product.

**Theorem 9.1 (The Zig-Zag Theorem, Reingold-Vadhan-Wigderson [RVW02]).** *Let  $G$  be an  $(n, m, \alpha)$ -graph and  $H$  be an  $(m, d, \beta)$ -graph, then  $G \mathbb{Z} H$  is an  $(nm, d^2, \varphi(\alpha, \beta))$ -graph where the function  $\varphi$  satisfies the following*

1. If  $\alpha < 1$  and  $\beta < 1$  then  $\varphi(\alpha, \beta) < 1$
2.  $\varphi(\alpha, \beta) \leq \alpha + \beta$ .
3.  $\varphi(\alpha, \beta) \leq 1 - (1 - \beta^2) \cdot (1 - \alpha)/2$ .

Since these bounds depend only on the spectral gaps of  $G$  and  $H$ , the first bound indeed has the meaning alluded to above: zig-zag takes two expanders into another expander.<sup>1</sup>

---

<sup>1</sup>The converse is also true: it is easy to see that  $\varphi(\alpha, \beta) \geq \max\{\alpha, \beta\}$  and so zig-zag cannot improve the expansion of the input graphs

The two quantitative bounds are crucial for applications. The second is useful when  $\alpha, \beta$  are small, and the third when they are large. We show below how to use the second bound on  $\varphi$  for the explicit construction of a family of expander graphs, and in the last section we show how Reingold used the third bound on  $\varphi$  for his proof that  $SL = L$ . The definition and analysis of the zigzag product appear in Section 9.3.

## 9.2 Construction of an expander family using zig-zag

Before we proceed to the definition, let us show that zig-zag products can be used to explicitly construct a family of constant degree expanders. To generate that infinite family we need one building block - a fixed size expander of certain parameters.

Let  $H$  be a  $(d^4, d, 1/4)$ -graph, for some constant  $d$ . Note that there is a probabilistic proof (in the spirit of the proof of Lemma 1.9) that such an expander exists, and since  $d$  is constant one can find it by an exhaustive search in constant time. This is a brute-force argument. For a more efficient construction that uses the fact that the degree is quite large in terms of the number of vertices see [RVW02].

Using the building block  $H$  we inductively define the infinite sequence  $G_n$  by:

$$G_1 = H^2, \quad G_{n+1} = (G_n)^2 \mathbb{Z} H \quad \text{for } n \geq 1. \quad (9.1)$$

We claim that this sequence is an expander family:

**Proposition 9.2.**  $G_n$  is a  $(d^{4n}, d^2, 1/2)$ -graph for all  $n$

*Proof.* For  $n = 1$  this follows from the definition. We proceed by induction and assume that  $(G_n)^2$  is a  $(d^{4n}, d^4, 1/4)$ -graph. When we zig-zag it with  $H$  (note that the degree of  $(G_n)^2$  equals the size of  $H$ , as required) it follows from the second bound on  $\varphi$  in Theorem 9.1 that  $G_{n+1}$  is a  $(d^{4(n+1)}, d^2, 1/2)$ -graph.  $\square$

The observant reader must have noticed that this construction is only *mildly* explicit in the sense of Definition 2.3. To make it *strongly* explicit one can interleave the construction in every iteration with another operation – tensoring of  $G_n$  with itself. This provides a much faster growth of these graphs. For more details see [RVW02].

## 9.3 Definition and analysis of the zig-zag product

Let  $G$  be an  $(n, m, \alpha)$ -graph and  $H$  be an  $(m, d, \beta)$ -graph. For every vertex  $v \in V(G)$  we fix some numbering  $e_v^1, \dots, e_v^m$  of the edges incident with  $v$ . Also, we regard the vertex set of  $H$  as the set  $[m] = \{1, \dots, m\}$ . The vertex set of  $G \mathbb{Z} H$  is the Cartesian product  $V(G) \times V(H)$ . It is convenient to think of this vertex set as being created by replacing every vertex  $v$  in  $G$  with a "cloud" of  $m$  vertices  $(v, 1), \dots, (v, m)$ , one for every edge incident with  $v$ . To describe the edges of the product graph, it is easier to first describe the edges of another graph,  $G \mathbb{F} H$  (called the *Replacement product*) on the same vertices. The figure below is useful for the purpose of explaining both constructions.

The edges of  $G \mathbb{F} H$  (shaded lines in the figure), are simply the union of the original edges of  $G$  (these are the wiggly edges – now going between clouds) and  $n$  copies of the edges of  $H$  - one copy per cloud (these are the dashed edges). The edges of  $G \mathbb{Z} H$ , arise from walks of length three of a "zig-zag" nature in  $G \mathbb{F} H$ : dashed-wiggly-dashed. these edges are drawn as solid lines in the figure below. More formally:

**Definition 9.3.**  $G \mathbb{Z} H = (V(G) \times [m], E')$ , where  $((v, i), (u, j)) \in E'$  iff there are some  $k, l \in [m]$  such that  $(i, k), (l, j) \in E(H)$  and  $e_v^k = e_u^l$ . A few remarks are in order about these constructions. The replacement product described above (when  $H$  is a cycle) is well known in graph theory. It is being used often for the purpose of reducing vertex degrees without losing connectivity. Such an argument is used to show for quite a few open problems in graph theory that it suffices to solve the problem for 3-regular connected graphs. (See an example of this below in Section 9.5.) Gromov [Gro83] has studied the expansion of this product of a  $d$ -dimensional cube with a smaller copy of itself. In a different context

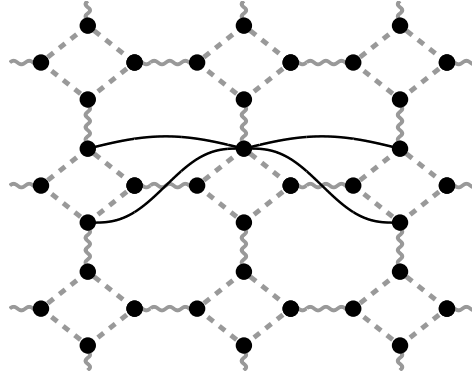


Figure 9.1: The zig-zag product of the grid  $\mathbb{Z}^2$  with the 4-cycle.

(analysis of expansion in Markov chains via decomposition, which may be viewed as reversing the replacement product), Martin and Randal [MR00] proved the third bound on  $\varphi$  in Theorem 9.1, for the replacement product (the second bound need not hold for the replacement product).

Here is an interesting special case of the zig-zag product. Vizing's theorem says that the edges of an  $m$ -regular graph can be colored with  $m + 1$  colors so that incident edges are colored with different colors. Many families of graphs are known where already  $m$  colors suffice. Notice that the graph  $G \mathbb{Z} H$  depends on the arbitrary labels we assign to the edges at each vertex. When  $G$ 's edges can be  $m$ -colored, then we can label each edge by its color. It is easy to verify that in this case  $G \mathbb{Z} H$  is a *lift* of  $H^2$  in the sense of Chapter 6. So far we were not able to take advantage of the similarity between these two methods of constructing graphs.

*Proof of the Zig-Zag theorem.* We prove here only a weaker version of the second bound on  $\varphi$ . In particular this proof gives  $\varphi \leq \alpha + \beta + \beta^2$ .

Clearly  $G \mathbb{Z} H$  is an  $(mn, d^2)$ -graph. It is convenient to estimate the spectral gap by considering the random walk on  $G \mathbb{Z} H$ . Each step in this walk can be conveniently split into three parts: (i) A random step on an edge within a cloud, (ii) A *deterministic* step on an edge connecting two clouds and (iii) Another random step within a cloud. We can now write down the transition matrix  $Z$  of the random walk on  $G \mathbb{Z} H$ . Let  $B$  and  $\hat{B}$  be the adjacency matrix of  $H$  and the transition matrix of the corresponding random walk respectively. The random steps (i) and (iii) are done on  $n$  disjoint copies of  $H$ , so the corresponding transition matrix is  $\tilde{B} = \hat{B} \otimes I_n$ . In the deterministic step (ii) we move from a vertex  $(v, k)$  to the unique vertex  $(u, l)$  for which  $e_v^k = e_u^l$ . Consequently, this step is carried out by multiplying with the matrix  $P$  which is a permutation matrix of an involution.

$$P_{(v,k),(u,l)} = \begin{cases} 1 & \text{if } e_v^k = e_u^l \\ 0 & \text{otherwise} \end{cases}$$

It follows that  $Z = \tilde{B} P \tilde{B}$ . The graph  $G \mathbb{Z} H$  is regular, so the constant vector  $1_{mn}$  is an eigenvector. Therefore what we are claiming is that  $|f Z f| / \|f\|^2 \leq \alpha + \beta + \beta^2$  for all  $f \perp 1_{mn}$ .

We next seek a way to decompose  $f$  so as to reflect the fact that  $V(G \mathbb{Z} H) = V(G) \times [m]$ . Define  $f^\parallel$  as the average of  $f$  on clouds, viz.,  $f^\parallel(x, i) = \frac{1}{m} \sum_{j \in [m]} f(x, j)$ . We also define  $f^\perp$  via:  $f^\perp = f - f^\parallel$ . Clearly  $f^\perp$  sums up to zero on each cloud. Let us expand:

$$\begin{aligned} |f Z f| &= |f \tilde{B} P \tilde{B} f| \\ &\leq |f^\parallel \tilde{B} P \tilde{B} f^\parallel| + 2|f^\parallel \tilde{B} P \tilde{B} f^\perp| + |f^\perp \tilde{B} P \tilde{B} f^\perp| \end{aligned}$$

Two simplifications follow from the fact that  $\tilde{B}$  is a direct sum of  $n$  copies of  $\hat{B}$ .

- Since  $\hat{B} 1_m = 1_m$ , it follows that  $\tilde{B} f^\parallel = f^\parallel$ .

- By assumption,  $\|\hat{B}u\| \leq \beta\|u\|$  whenever  $u \perp 1_m$  and  $f^\perp$  sums to zero on every cloud. Therefore,  $\|\tilde{B}f^\perp\| \leq \beta\|f^\perp\|$ .

In order to deal with  $f^\parallel$  we define the real function  $g$  on  $V(G)$ , via  $g(v) = \sqrt{m}f^\parallel(v, i)$  (this does not depend on  $i$ , of course). Note that with this definition we have  $\|g\|^2 = \|f^\parallel\|^2$ . The definition of  $P$  implies that  $f^\parallel Pf^\parallel = g\hat{A}g$  where  $\hat{A} = \hat{A}_G$  is the transition matrix of the random walk on  $G$ . But  $f^\parallel \perp 1_{mn}$  implies that  $g \perp 1_n$  and hence that  $g\hat{A}g \leq \alpha\|g\|^2$ . Consequently,  $|f^\parallel Pf^\parallel| \leq \alpha\|f^\parallel\|^2$ .

Also, both  $\tilde{B}$  and  $P$  are stochastic matrices, and are therefore contractions in  $l_2$ . Putting all of this together we conclude:

$$|fZf| \leq \alpha\|f^\parallel\|^2 + 2\beta\|f^\parallel\| \cdot \|f^\perp\| + \beta^2\|f^\perp\|^2$$

But  $\|f\|^2 = \|f^\parallel\|^2 + \|f^\perp\|^2$ , so the maximum of this quadratic form is the large eigenvalue of the matrix

$$\begin{pmatrix} \alpha & \beta \\ \beta & \beta^2 \end{pmatrix}$$

The conclusion follows. □

## 9.4 Entropy Analysis

We return to the perspective suggested in Section 3.1.2 and consider the graph  $G \otimes H$  when both  $G$  and  $H$  are good expanders. Why is it that entropy grows with each random step on  $G \otimes H$ ? At issue is the fact that entropy can no longer grow when the distribution is uniform. As above, we view each step as made up of three substeps: A random step ("zig") in one copy of  $H$ , then a deterministic step to a neighboring cloud, and another random step in the new copy of  $H$ . Note that steps 1 and 3 are *independent* random steps on  $H$ . If the conditional distribution, restricted to one of these two clouds is far from uniform, then the entropy grows, by virtue of  $H$ 's expansion. The other two steps cannot harm this, since the entropy never decreases when we multiply by a stochastic matrix.

What is more curious is the situation where on a typical cloud the distribution is nearly uniform. In this case Step 1 cannot increase the entropy since the restriction of the distribution to the cloud remains nearly uniform. To see what happens, recall that  $V(G \otimes H) = V(G) \times V(H)$  and consider  $p$ 's two marginal distributions  $p_G$  and  $p_H$ . Since the distribution on most clouds is near uniform, the (deterministic) middle step on  $G \otimes H$  is then like a real random step on  $G$ . Consequently, and since  $G$  is an expander, the entropy of  $p_G$  increases. But this middle step is a permutation on  $V(G \otimes H)$ , so the entropy of the whole distribution remains unchanged. It follows that the entropy of  $p_H$  must have decreased. That means that in Step 3 we are in the good case, where the conditional distribution on clouds is far from uniform, and the entropy increases due to the expansion of  $H$ . Thus the key is that Step 2 is simultaneously a permutation (so that any progress made in Step 1 is preserved) and an operation whose  $G$ -marginal is simply a random step on  $G$ .

The linear algebra proof we gave for the zig-zag theorem can be seen as a formal statement of this intuition. Other papers in the area such as [RTV05] follow this approach even more closely. This is in particular helpful in establishing the third bound in Theorem 9.1.

## 9.5 An application to complexity theory: $SL = L$

Assume you arrive to an unfamiliar city, and you don't have a map. You would like to get to your hotel (which street name you know). It is natural to start exploring, hoping to cover every street including the one your hotel is on (an equivalent problem is finding the exit from a maze). Naturally, you can create your own map of the city, which will guarantee that you don't go around in loops. But that takes a lot of memory - linear in the size of the city. Suppose you don't have that - all you can remember is where you are, the street name of your hotel, and perhaps a few more items of that magnitude - in short, only *logarithmic* space in the size of the city.

Formally, this is cast as the problem of exploring a given undirected graph  $G$  (or even determining if there is a path between two given vertices  $s$  and  $t$ ).<sup>2</sup>

An important step towards the solution was made by Aleliunas, Karp, Lipton, Lovász, and Rackoff [AKL<sup>+</sup>79]. They showed that this problem can be solved by a *probabilistic* logspace algorithm.<sup>3</sup> To determine if  $s$  and  $t$  are connected, one simply performs a polynomial length random walk starting at  $s$ , and checks if the walk ever reaches  $t$ .

With all the background we have covered so far, the analysis of this result is simple, so let us sketch it. First observe that the algorithm only uses logarithmic memory - all it needs to remember is its current position, and the goal  $t$ . As for the time bound, let us first note that we can assume without loss of generality that the graph is regular (e.g. replace each vertex  $v$  by a cycle of length  $d(v)$ , the degree of  $v$ , as in the replacement product described above. This makes the graph 3-regular, maintaining connectivity in the natural way). Now using the connection between edge expansion  $h(G)$  (which is at least  $\Omega(1/n)$ ) in every connected 3-regular graph with  $n$  vertices) and the eigenvalue, we conclude via Theorem 4.11 that it must be an  $(n, 3, \alpha)$ -graph, with  $\alpha < 1 - \Omega(1/n^2)$ . Thus a random walk of length  $O(n^3)$  will get exponentially close to the uniform distribution, and if we repeat it  $n^2$  times, resulting in a walk of length  $n^5$ , we will not miss a single vertex in this connected component, except with exponentially small probability.

It is a natural approach to this problem to try and *derandomize* this probabilistic algorithm. Namely, to generate deterministically a walk which explores all vertices in every connected graph. This has, indeed, been tried often, (see [Rei05] for background). A key to the success of Reingold's approach was an ingenious application of the zig-zag product.

As mentioned above, we can assume the input graph  $G$  has constant degree  $D$ , of some fixed  $D$  of our choice (we can add self loops to the 3-regular graph above). The proof idea is the following. **If** each connected component of the graph  $G$  is an expander graph, then each component has a logarithmic diameter. Then one can enumerate over all the logarithmically long paths starting at  $s$  and check if one of them arrives at  $t$ . In short, for graphs whose components are expanders, the logspace algorithm is trivial.

But this is indeed a big "if". The question is how to convert, efficiently in logspace, an arbitrary graph into an expander. The answer is by using the zig-zag product. Consider the  $D$ -regular input graph  $G$ , and assume it is connected (otherwise we apply the same argument for each component). We already argued that  $G$  is somewhat expanding, i.e. is an  $(n, D, \alpha)$ -graph for some  $\alpha < 1 - \Omega(1/n^2)$ .

To turn  $G$  into an expander, assume that  $D = d^{16}$  and that we have a  $(d^{16}, d, 1/2)$ -graph  $H$ . We inductively construct the graphs  $G_i$  in a similar manner to the expander construction (9.1), but rather than starting from a fixed size expander as the first graph, we start with  $G$  itself.

$$G_1 = G, \quad G_{i+1} = (G_i \otimes H)^8 \quad \text{for } i \geq 1.$$

We terminate the sequence after only  $k = O(\log n)$  iterations. In each iteration, the size of  $G$  increases by a constant factor which is the size of the graph  $H$ , until the final iteration yields the graph  $G_k$ . The required proposition follows from the following two claims.

**Claim 9.4.**  $G_k$  is an  $(nd^{16k}, d^{16}, 1/2)$ -graph.

**Claim 9.5.** Neighborhood queries for  $G_k$  can be answered in logspace.

Note that the second claim is not obvious. In logspace one cannot keep copies of the graphs  $G_i$  in memory, and therefore one has to evaluate the recursion anew for each query. This means that large expander graphs constructed by the zig-zag product are **very explicit**, even in a stricter sense than the one required by definition 2.3. But indeed the intuition that every one of these steps can be performed with only additional *constant* space is correct. This requires a clever construction of an appropriate data structure. We skip the proof of this claim, and refer the reader to Reingold's paper [Rei05].

To prove the first claim, we use the third bound in Theorem 9.1. Using it we see that (roughly speaking) the spectral gap doubles in each iteration, and thus reaches a constant in logarithmically many iterations.

$$1 - \varphi(\alpha, \beta) \geq (1 - \beta^2) \cdot (1 - \alpha)/2. \tag{9.2}$$

<sup>2</sup>A word to the maven: this problem is complete for the class  $SL$ , which simply means that solving it in logarithmic space (in the class  $L$ ) would imply  $SL = L$ .

<sup>3</sup>Thus proving  $SL \subseteq RL$ .

Let us denote by  $\lambda_i, \mu_i$  the normalized second eigenvalue of  $G_i$  and  $G_i \otimes H$  respectively. Then, using the parameters of the graph  $H$  in Equation (9.2) yields:

$$1 - \mu_i \geq \frac{3}{8} \cdot (1 - \lambda_i).$$

Therefore,

$$\lambda_{i+1} \leq \max\left(\lambda_i^2, \frac{1}{2}\right).$$

Therefore, for  $k = O(\log n)$ , we obtain  $\lambda_k \leq 1/2$  as needed.

We end by noting that a recent paper of Rozenman and Vadhan [RV05] gives a different proof of Reingold's result, using a new notion of *derandomized* graph squaring. This product is closely related to zig-zag (so the proof has the same spirit), but the analysis of the space required in their implementation is more straightforward.

## Chapter 10

# Lossless Conductors and Expanders

Early on in our discussion we had introduced the notion of expansion ratio  $h(G) = \min_{\{S \mid |S| \leq \frac{n}{2}\}} \frac{|E(S, \bar{S})|}{|S|}$ . Indeed, quite a few of the important properties of expander graphs can already be derived from the fact that  $h(G)$  is bounded away from zero. For more refined analysis it becomes necessary to consider the edge isoperimetric parameter  $\Phi_E(G, k) = \min_{S \subset V} \{|E(S, \bar{S})| : |S| = k\}$ . We saw that a comprehensive analysis of the extremal properties of this parameter would be both interesting and very challenging, though some very good explicit constructions are known (see Chapter 5). The analogous questions about the *vertex isoperimetric parameter*:  $\Phi_V(G, k) = \min_{S \subset V} \{|\Gamma(S) \setminus S| : |S| = k\}$  seem even more challenging. Simple considerations from Section 5.1.1 yield that every large set of vertices in a  $d$ -regular graph has vertex expansion at most  $d - 2 + o(1)$ . The Expander Mixing Lemma in Chapter 2 shows that for Ramanujan graphs the vertex expansion is at least  $d/4$  for sets of size at most  $\epsilon n$ , with sufficiently small  $\epsilon > 0$  (we leave this easy calculation to the reader). Kahale [Kah95] showed how to improve this lower bound to approximately  $d/2$ . On the other hand, Kahale also proved that this as far as you can get by knowing the graph's expansion ratio. Namely, he showed that there are Ramanujan graphs which contain sets of bounded size whose vertex expansion is only  $d/2$ .

To understand how good these bounds are, we already know that it's a good idea to look at random graphs first. Indeed, a simple calculation shows that in a random  $(n, d)$  graph every set  $S$  of cardinality  $\leq \epsilon n$  satisfies  $|\Gamma(S) \setminus S| \geq \frac{d}{2}|S|$ . Since we know how to construct  $(n, d)$  graphs in which small set have a vertex expansion of  $d/2$ , it is natural to enquire why we bother about increasing this factor to  $d - 2$  or so. Indeed, this question is motivated by more than sheer curiosity. There is a variety of applications that require  $(n, d)$  graphs in which vertex sets of size  $\epsilon n$  have vertex expansion  $\gamma d$  for  $\gamma > 1/2$ . Such application include networks that can implement fast distributed, routing algorithms e.g. [PU89, ALM96, BFU99] as well the construction of expander-based linear codes e.g. [SS96, Spi96, LMSS01]. Analogous, irregular, even highly unbalanced graphs are used in various storage schemes [UW87, BMRV02] and in proving lower bounds on computational complexity (the generation of hard tautologies for various proof systems [BSW01, ABSRW04, AR01, BOGH<sup>+</sup>03]).

It is still a major challenge to construct families of  $(n, d)$  graphs in which vertex sets of size  $\epsilon n$  have vertex expansion  $\gamma d$  for  $\gamma > 1/2$ . However, in this chapter, we present a construction due to Capalbo, Reingold, Vadhan, and Wigderson [CRVW02], that takes a substantial step in this direction. This is an explicit construction of families of bipartite expander graphs whose left expansion is  $(1 - \delta)d$  for small (but linear size) sets. The construction is based on a generalization of the zig-zag product to *conductors*. This will offer us a glimpse into the realm of randomness-enhancing objects, which, aside of expanders include creatures such as conductors, extractors, and dispersers.

## 10.1 Conductors and lossless expanders

### 10.1.1 Conductors

The main difficulty we face here is that the spectral gap which has served us very well so far seems, by the aforementioned result of Kahale no longer adequate for our purposes. We therefore revert to the language of entropy. A natural

choice would be to use the min-entropy  $H_\infty$  defined in Chapter 3. A lower bound on the min-entropy is a rather strong condition. The inequality  $H_\infty(p) \geq k$  means that no point has probability exceeding  $2^{-k}$  (In contrast, a lower bound on the Shannon entropy means that this holds only on average). In this light we consider a weaker condition and only ask how close, in *total variation distance* (or equivalently in  $l_1$  norm) our distribution is to one with high min-entropy. We introduce now some terminology:

**Definition 10.1.** A  $k$ -**source** is a distribution with min-entropy is at least  $k$ . A distribution is called a  $(k, \epsilon)$ -**source** if there is a  $k$ -source at  $l_1$  distance  $\leq \epsilon$  from it. Also we denote the uniform distribution over  $\{0, 1\}^d$  by  $U_d$ .

A main aspect of the present chapter is to emphasize an idea that has already appeared in Chapters 3,9. Namely, that it useful to view the entropy of a distribution as a physical entity which we transfer around the graph as necessary. We develop an appropriate terminology to advance this idea. In particular, since our main concern here is with bipartite graphs, we will view a bipartite graph as a function. Our bipartite graphs have vertex sets Left and Right, with all Left vertices of the same degree. This function associates with a given Left vertex  $x$ , and an edge label  $i$ , the Right vertex that is the  $i$ -th neighbor of  $x$ . To facilitate our analysis of entropy, we use bit strings to name the vertices and edge labels. As in Sections 3.1.2, 9.4, we are concerned with the following question. Consider a distribution on the Left vertices, of known entropy. Now take a random step along an edge to the Right. This induces a distribution on the Right vertices, the entropy of which we seek to bound from below. This problems is being attacked by means of several classes of graphs which serve as building blocks for the final construction. The common theme is one and the same: Given a bound on the “incoming” entropy, how much is coming out (up to a small  $l_1$  distance). In keeping with the common terminology in the area of pseudorandom number generators [RVW02, Sha04] we refer to the choice of an edge to be taken in the next step as the randomness “injected” into the process or as the “seed” being used. Here is our toolbox.

**Definition 10.2.** A function  $E: \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m$  is a  $(k_{\max}, a, \epsilon)$ -**conductor** if for any  $k \leq k_{\max}$ , and any  $k$ -source  $X$  over  $\{0, 1\}^n$ , the distribution  $E(X, U_d)$  is a  $(k + a, \epsilon)$ -source.

The analysis of the type conducted in Section 9.4 must always be adapted to situations where the entropy reaches its maximum either locally or globally. Recall (Section 3.1.2) that a distribution on an  $M$ -element set has entropy  $\leq \log M$ . Most of the tools we describe below are geared to handle this problem.

1. A function  $E: \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m$  is an  $(a, \epsilon)$ -**extracting conductor** if it is an  $(m - a, a, \epsilon)$ -conductor.

In particular, if the input entropy is  $m - a$ , the output will be uniform. We mention that an explicit construction of an  $(a, \epsilon)$ -extracting conductor can be derived from that of an  $(m - a, \epsilon)$ -extractor, see [Sha04]. This is a well-known and easy to construct object.

2. A function  $E: \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m$  is a  $(k_{\max}, \epsilon)$ -**lossless conductor** if it is a  $(k_{\max}, d, \epsilon)$ -conductor.

We view the specification of the edge label as an “injection of randomness” and wish *none* of this additional entropy to be lost. The constructing of these graphs is our ultimate goal.

3. A pair of functions  $\langle E, C \rangle: \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m \times \{0, 1\}^b$  is a  $(k_{\max}, a, \epsilon)$ -**buffer conductor** if  $E$  is an  $(a, \epsilon)$ -extracting conductor, and  $\langle E, C \rangle$  is an  $(k_{\max}, \epsilon)$ -lossless conductor.

This notion is intended to assure that no entropy is lost when the input randomness is too high. Whatever entropy is lost by the first function, is to be saved completely in the second, which may be viewed as an overflow buffer or bucket.

4. A pair of functions  $\langle E, C \rangle: \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m \times \{0, 1\}^b$ , where  $n + d = m + b$  is an  $(a, \epsilon)$ -**permutation conductor** if  $E$  is an  $(a, \epsilon)$ -extracting conductor, and  $\langle E, C \rangle$  is a permutation over  $\{0, 1\}^{n+d}$ .

This is a special case of a buffer conductor, where the requirement that  $\langle E, C \rangle$  is an  $(k_{\max}, \epsilon)$ -lossless conductor follows since  $\langle E, C \rangle$  is a permutation.

As we shall see, lossless conductance implies lossless expansion, which was our motivating problem to begin with.

### 10.1.2 Lossless expanders

Let  $G = (V_L, V_R; E)$  be a bipartite graph such that  $|V_L| = N$ ,  $|V_R| = M$ , and all left vertices have degree  $D$ .

**Definition 10.3.** The graph  $G$  is a  $(K_{\max}, \epsilon)$ -lossless expander if every set of  $K \leq K_{\max}$  left vertices has at least  $(1 - \epsilon)DK$  neighbors.

That is, in a lossless expander sufficiently small left vertex sets have almost the maximal expansion possible. An alternative view is that most of the neighbors of a small left set are unique neighbors, i.e. neighboring a single vertex of the set. Naturally,  $K_{\max}$  should be somewhat smaller than  $M/D$  for this to be possible.

As mentioned above, we view a conductor  $E: \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m$  as a bipartite graph with  $N = 2^n$  left vertices,  $M = 2^m$  right vertices, where each left vertex  $x$  is connected to  $D = 2^d$  right vertices  $E(x, \cdot)$ . It is easy to check that from this perspective, a  $(k_{\max}, \epsilon)$ -lossless conductor is a  $(K_{\max}, \epsilon)$ -lossless expander, where  $K_{\max} = 2^{k_{\max}}$ .

**Theorem 10.4.** For any  $\epsilon > 0$  and  $M \leq N$ , there is an explicit family of left  $D$ -regular bipartite graphs that are  $(\Omega(\epsilon M/D), \epsilon)$ -lossless expanders, where  $D \leq (N/\epsilon M)^c$  for some constant  $c$ .

We note that the (useful) case where  $M/N$  and  $\epsilon$  are bounded below by a constant, the degree  $D$  will be constant as well. As can be expected, our approach to this theorem is to construct an explicit family of  $(m-d-\log(1/\epsilon)-O(1), \epsilon)$ -lossless conductors, where  $d = O(n - m - \log(1/\epsilon))$ .

## 10.2 The Construction

The required lossless conductors are constructed using a zig-zag product for conductors. However, before we define this product, we need to recall the definition of the zig-zag product given in Chapter 9 and adapt it slightly to case of bipartite graphs. The figure below should help the formal definition.

**Definition 10.5 (Zig-zag product for bipartite graphs).** Let  $H$  be a  $d$ -regular bipartite graph with  $s$  vertices on each side, and let  $G$  be an  $s$ -regular bipartite graph with  $n$  vertices on each side. The zig-zag product  $G \otimes H$  is a  $d^2$ -regular bipartite graph with  $sn$  vertices on each side, where the left and right sides are arranged as  $n$  copies of  $H$ , one per each vertex of  $G$ . The edges emanating from a left vertex  $(x, y) \in [n] \times [s]$  are labeled by  $[d] \times [d]$ . The edge labeled  $(a, b)$  is determined as follows:

1. Take a left to right step in the local copy of  $H$ , using  $a$  to choose an edge.
2. Take a left to right step along an edge of  $G$ , between copies of  $H$ . More precisely, suppose we are at  $(x, y')$ . Let  $x' \in G$  be the  $y'$ -th neighbor of  $x$ . Suppose that on  $x'$  neighbor list,  $x$  is the  $z$ -th neighbor of  $x'$ . Then the second step takes us from  $(x, y')$  to  $(x', z)$ .
3. Take a left to right step in the new local copy of  $H$ , using  $b$  to choose an edge.

Recall that by the zig-zag theorem from Chapter 9,  $G \otimes H$  is an expander if both  $G$  and  $H$  are such. Moreover, the degree of  $G \otimes H$  is controlled by the degrees in  $H$ , while its size and expansion are related to both  $G$  and  $H$ . Unfortunately, while  $\deg(G \otimes H) = \deg^2(H)$ , the expansion of  $G \otimes H$  cannot be better than neither the expansion of  $H$  nor the expansion of  $G$ . In particular, it will never exceed  $d = \sqrt{\deg(G \otimes H)}$ . This can be seen by considering the expansion of a set consisting of a single copy of  $H$  on the left. This is a far cry from the lossless expanders we seek. Clearly, in this example the “injected” entropy is wasted since we have begun with a uniform distribution on a copy of  $H$ , which entropy cannot be increased by a walk inside  $H$ . This problem will be solved by means of a buffer in which we save the injected entropy in such cases. We now formalize it and define the zig-zag product for conductors.

### 10.2.1 The zig-zag product for conductors

Our strategy to avoid entropy loss is to save (or buffer) leftover entropy from the random choices made at each step. Then, we use a lossless conductor with some truly random bits to “condense” the leftovers entropy. As suggested by

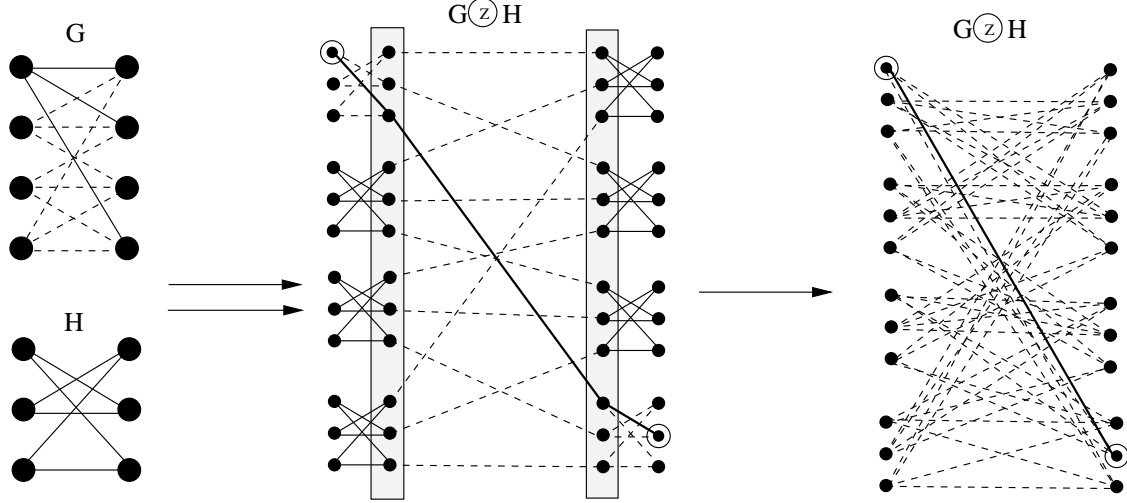


Figure 10.1: Zig-zag product of bipartite graphs

the name "conductor", we make an analogy to electricity or water. Indeed, we think of pouring randomness (water) into a conductor, and of collecting the leftovers (unused randomness beyond the  $k_{\max}$  bound) into a bucket for later use.

The zig-zag product for conductors must be carried out with carefully selected parameters to yield the constant degree lossless expanders/conductors required by Theorem 10.4. To define the product, we need three objects  $E_1$ ,  $E_2$ , and  $E_3$  (which respectively replace the roles of the three steps  $H, G$ , and  $H$  in the original zigzag product. We avoid specifying their most general parameters here, and pick a certain set of parameters which suffices for our purposes. The general achievable sets of parameters for an explicit construction can be found in [CRVW02] ( $E_1$  from Lemma 4.4, and  $E_2, E_3$  from Lemma 4.13). So, let us assume we have in our hands:

1.  $\langle E_1, C_1 \rangle: \{0, 1\}^{n_1} \times \{0, 1\}^{d_1} \rightarrow \{0, 1\}^{m_1} \times \{0, 1\}^{b_1}$ , a permutation conductor.
2.  $\langle E_2, C_2 \rangle: \{0, 1\}^{n_2} \times \{0, 1\}^{d_2} \rightarrow \{0, 1\}^{d_1} \times \{0, 1\}^{b_2}$ , a buffer conductor.
3.  $E_3: \{0, 1\}^{b_1+b_2} \times \{0, 1\}^{d_3} \rightarrow \{0, 1\}^{m_3}$ , a lossless conductor.

The zig-zag product for conductors produces the conductor  $E: \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m$ , where  $n = n_1 + n_2$ ,  $d = d_2 + d_3$ , and  $m = m_1 + m_3$ . Let  $x_1, x_2, r_2, r_3, y_1, y_2, y_3$  be binary strings of respective lengths  $n_1, n_2, d_2, d_3, m_1, d_1, m_3$ . Then, as depicted in Figure 10.2, we evaluate  $^1 y_1 y_3 = E(x_1 x_2, r_2 r_3)$  by the following three steps:

- $(y_2, z_2) = \langle E_2, C_2 \rangle(x_2, r_2)$
- $(y_1, z_1) = \langle E_1, C_1 \rangle(x_1, y_2)$
- $y_3 = E_3(z_1 z_2, r_3)$

Recall that the zig-zag product for bipartite graphs  $G \otimes H$ , uses  $H$  twice. Here, the first use is replaced with  $E_2$  to ensure that when  $x_2$  has high min-entropy,  $y_2$  is close to uniform, and is a good seed for  $E_1$ . The second use of  $H$  is replaced with lossless conductor  $E_3$ , that transfers entropy lost in  $E_1$  and  $E_2$  to the output. The deterministic step of the zig-zag product using the graph  $G$  is replaced with  $E_1$ , which as before doesn't use any new random bits, and whose output is just a permutation of its input (which, however, moves entropy about to allow more to come in later).

To simplify subsequent discussion, we focus on a specific example of achievable parameters for the three building blocks  $E_1, E_2$  and  $E_3$ , whose zig-zag product, depicted in figure 10.2 results in a lossless conductor.

<sup>1</sup>If  $u$  and  $v$  are bit strings, then  $uv$  denotes their concatenation.

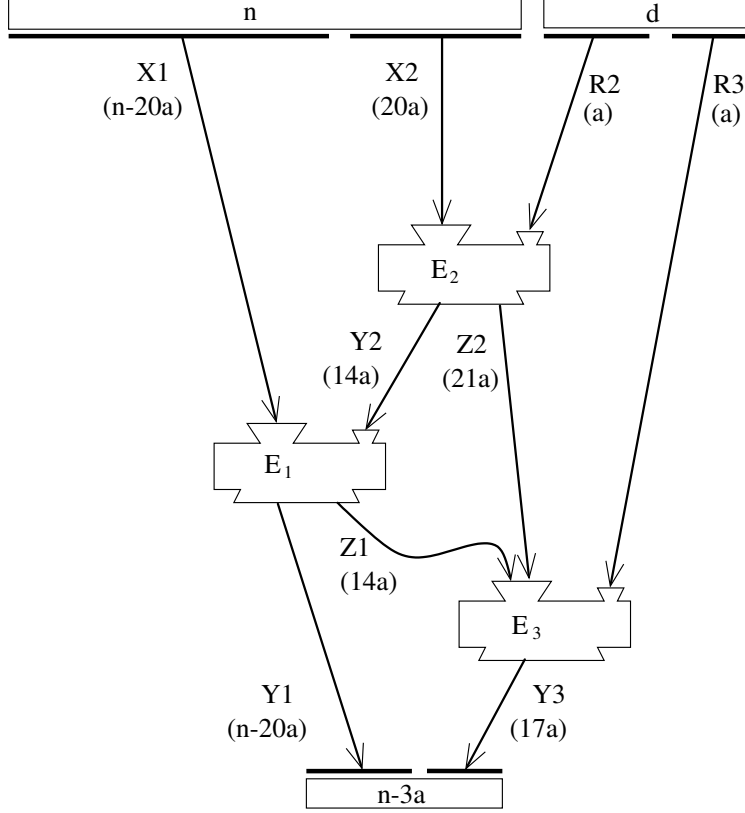


Figure 10.2: Entropy flow in a lossless conductor

### 10.2.2 Proof of the main theorem

We now need a technical lemma, showing how to partition a joint distribution according to conditional min-entropy. Recall that two distributions are called  $\epsilon$ -close if their  $l_1$  distance is at most  $\epsilon$ .

**Lemma 10.6.** *Let  $(X_1, X_2)$  be a probability distribution on a finite product space. Given  $\epsilon > 0$  and  $a$ , there exists a distribution  $(Y_1, Y_2)$  on the same space such that*

- *The distributions  $(X_1, X_2)$  and  $(Y_1, Y_2)$  are  $\epsilon$ -close.*
- *The distribution  $(Y_1, Y_2)$  is a convex combination of two other distributions  $(\hat{Y}_1, \hat{Y}_2)$  and  $(\check{Y}_1, \check{Y}_2)$ , each having min-entropy at least  $H_\infty(X_1, X_2) - \log(1/\epsilon)$ .*
- *For all  $x \in \text{Supp}(\hat{Y}_1)$  we have  $H_\infty(\hat{Y}_2 | \hat{Y}_1 = x) \geq a$ .*
- *For all  $x \in \text{Supp}(\check{Y}_1)$  we have  $H_\infty(\check{Y}_2 | \check{Y}_1 = x) < a$ .*

*Proof.* First, we split  $\text{Supp}(X_1)$  according to  $H_\infty(X_2 | X_1 = x)$ :

$$\hat{S} = \{z : H_\infty(X_2 | X_1 = z) \geq a\}, \quad \check{S} = \{z : H_\infty(X_2 | X_1 = z) < a\}.$$

Then, we define  $(\hat{Y}_1, \hat{Y}_2)$  and  $(\check{Y}_1, \check{Y}_2)$ , so that  $\hat{Y}_1$  and  $\check{Y}_1$  have disjoint supports,  $\hat{S}$  and  $\check{S}$  respectively.

$$\begin{aligned} \Pr [(\hat{Y}_1, \hat{Y}_2) = (z_1, z_2)] &= \Pr [(X_1, X_2) = (z_1, z_2) | X_1 \in \hat{S}], \\ \Pr [(\check{Y}_1, \check{Y}_2) = (z_1, z_2)] &= \Pr [(X_1, X_2) = (z_1, z_2) | X_1 \in \check{S}]. \end{aligned}$$

Let  $p = \Pr[X_1 \in \hat{S}]$ . Then the probability of each value in  $(\hat{Y}_1, \hat{Y}_2)$  is multiplied by  $1/p$ , and the probability of each value in  $(\check{Y}_1, \check{Y}_2)$  is multiplied by  $1/(1-p)$ . Therefore, if  $\epsilon \leq p \leq 1 - \epsilon$  then the min-entropy of  $(\hat{Y}_1, \hat{Y}_2)$  and  $(\check{Y}_1, \check{Y}_2)$  is reduced by at most  $\log(1/\epsilon)$ . In this case, we define  $(Y_1, Y_2) = (X_1, X_2)$ , and we are done, since:

$$(Y_1, Y_2) = (X_1, X_2) = p(\hat{Y}_1, \hat{Y}_2) + (1-p)(\check{Y}_1, \check{Y}_2).$$

Otherwise, assume  $p < \epsilon$  (the case  $p > 1 - \epsilon$  is similar). In this case since  $(\check{Y}_1, \check{Y}_2)$  still has sufficiently high min-entropy, we take  $(Y_1, Y_2) = (\check{Y}_1, \check{Y}_2)$ . This distribution is  $\epsilon$ -close to  $(X_1, X_2)$ , since:

$$\begin{aligned} \sum_{z_1 \in \hat{S}, z_2} |\Pr[(X_1, X_2) = (z_1, z_2)] - \Pr[(\check{Y}_1, \check{Y}_2) = (z_1, z_2)]| &\leq p < \epsilon, \\ \sum_{z_1 \in \check{S}, z_2} |\Pr[(X_1, X_2) = (z_1, z_2)] - \Pr[(\check{Y}_1, \check{Y}_2) = (z_1, z_2)]| &\leq \left(\frac{1}{1-p} - 1\right)(1-p) = p < \epsilon. \end{aligned}$$

□

We now pick specific parameters for our building blocks, which can be explicitly constructed, and prove that combining them using the zig-zag product for conductor yields a lossless expander as stated in Theorem 10.4.

Fix the parameters  $a = 1000 \log(1/\epsilon)$  and  $d = 2a$ . Then:

- $\langle E_1, C_1 \rangle: \{0, 1\}^{n-20a} \times \{0, 1\}^{14a} \rightarrow \{0, 1\}^{n-20a} \times \{0, 1\}^{14a}$ , is an  $(n - 30a, 6a, \epsilon)$ -permutation conductor.
- $\langle E_2, C_2 \rangle: \{0, 1\}^{20a} \times \{0, 1\}^a \rightarrow \{0, 1\}^{14a} \times \{0, 1\}^{21a}$ , is a  $(14a, 0, \epsilon)$ -buffer conductor;
- $E_3: \{0, 1\}^{35a} \times \{0, 1\}^a \rightarrow \{0, 1\}^{17a}$ , is a  $(15a, a, \epsilon)$ -lossless conductor.

We state again the consequence for these choices, which suffices to prove the main theorem 10.4:

**Claim 10.7.** *The resulting conductor  $E: \{0, 1\}^n \times \{0, 1\}^{2a} \rightarrow \{0, 1\}^{n-3a}$ , is an  $(n - 30a, 2a, 4\epsilon)$ -lossless conductor.*

We first deal with the explicitness of our building blocks (more details can be found in [CRVW02]). Note the sizes of  $E_2$  and  $E_3$  are fixed constants, and so there is no issue concerning their explicit construction. They can be shown to exist by a simple probabilistic argument, and can then be found by exhaustive search. The conductor  $E_1$  needs to be arbitrarily large, but its explicit construction follows known explicit expanders. Indeed, any graph with sufficiently small second eigenvalue (close to Ramanujan for these parameters) is such a conductor  $E_1$ , and the proof follows, due to the equivalence of min-entropy and the Rényi entropy:  $H_\infty(X) \leq H_2(X) \leq 2H_\infty(X)$  as shown in ????

Let us follow the entropy flow from the input  $(x_1 x_2, r_2 r_3)$  to the output  $y_1 y_3$ . We would like to prove that if  $H_\infty(X_1, X_2) = k$ , then  $y_1 y_3$  is a  $(k + 2a, 4\epsilon)$ -source. For ease of discussion, we first ignore the small  $l_1$ -errors in the outputs of all conductors (in other words, assume for simplicity that in our building blocks  $\epsilon = 0$ ). These errors will simply be added at the end to give the final error of the lossless conductor  $E$ .

We prove first that  $E_1$  and  $E_2$  together transfer enough entropy into  $Y_1$ , namely:

$$H_\infty(Y_1) \geq k - 14a. \tag{10.1}$$

By lemma 10.6 it suffices to prove this bound *only* in the two extreme cases when  $H_\infty(X_2|X_1 = x_1)$  are *all* large or *all* small, as we vary all attainable values for  $x_1$ .

**Case 1** For all  $x_1 \in \text{Supp}(X_1)$ , we have  $H_\infty(X_2|X_1 = x_1) \geq 14a$ .

In this case, since  $E_2$  is an  $(0, \epsilon)$ -extracting conductor,  $H_\infty(Y_2|X_1 = x_1) = 14a$ , for any  $x_1 \in \text{Supp}(X_1)$ . Therefore  $Y_2$  is uniform, and can be used as a seed for  $\langle E_1, C_1 \rangle$  for any  $x_1 \in \text{Supp}(X_1)$ . Since  $H_\infty(X_1) \geq k - 20a$ , and since  $E_1$  is  $(6a, \epsilon)$ -extracting conductor,  $E_1$  conducts  $6a$  bits of entropy from the seed into  $Y_1$ , and we obtain  $H_\infty(Y_1) \geq k - 14a$ .

**Case 2** For all  $x_1 \in \text{Supp}(X_1)$ , we have  $H_\infty(X_2|X_1 = x_1) \leq 14a$ .

Since  $H_\infty(X_1, X_2) = k$ , it follows that  $H_\infty(X_1) \geq k - 14a$ . Therefore, since  $E_2$  is a lossless extractor,  $H_\infty(Y_2|X_1 = x_1) \geq H_\infty(X_2|X_1 = x_1)$  for any  $x_1 \in \text{Supp}(X_1)$ . It follows that  $H_\infty(X_1, Y_2) \geq H_\infty(X_1, X_2) = k$ . Since  $\langle E_1, C_1 \rangle$  is a permutation, also  $H_\infty(Y_1, Z_1) \geq k$ , and again we get that  $H_\infty(Y_1) \geq k - 14a$ .

Observe that both  $\langle E_1, C_1 \rangle$  and  $\langle E_2, C_2 \rangle$  conserve entropy, since  $\langle E_1, C_1 \rangle$  is a permutation conductor, and  $\langle E_2, C_2 \rangle$  is a buffer conductor. Therefore:

$$k + a = H_\infty(X_1, X_2, R_2) = H_\infty(X_1, Y_2, Z_2) = H_\infty(Y_1, Z_1, Z_2). \quad (10.2)$$

To conclude the argument, consider any  $y_1 \in \text{Supp}(Y_1)$ , and note that by the bound of  $k + a$  on the total entropy, and the lower bound proved above for  $H_\infty(Y_1)$ , we must have  $H_\infty(Z_1, Z_2|Y_1 = y_1) \leq 15a$ .

Thus,  $E_3$ , which is a  $(15a, a, \epsilon)$ -conductor conducts  $a$  bits of entropy from  $R_3$  to  $Y_3$ . That is, all the entropy of  $Z_1, Z_2$  is transferred to the output  $Y_3$  without any entropy loss,  $H_\infty(Y_3|Y_1 = y_1) = H_\infty(Z_1, Z_2|Y_1 = y_1) + a$ . Together with (10.1) and (10.2) this implies that

$$\begin{aligned} H_\infty(Y_3|Y_1 = y_1) - \log \Pr[Y_1 = y_1] &= H_\infty(Z_1, Z_2|Y_1 = y_1) - \log \Pr[Y_1 = y_1] + a \\ &\geq H_\infty(Z_1, Z_2, Y_1) + a = k + 2a. \end{aligned}$$

Therefore,  $H_\infty(Y_1, Y_3) = k + 2a$ , as claimed.

Finally to see the dependence on  $\epsilon$ , note that these  $l_1$ -errors on the extractors outputs add up (since if two random variables are  $\epsilon$ -close, one can be written as a convex combination in which the other has weight at least  $1 - \epsilon$ , and another arbitrary distribution (capturing the error)). The probability of falling in this error event is at most  $\epsilon$  each time we have such  $l_1$ -error. In the above analysis of the function  $E$ , we make four moves from a variable to its  $\epsilon$ -close counterpart, one for each of the building blocks  $E_1, E_2, E_3$ , and one in the use of Lemma 10.6. Thus, the actual value of  $Y_1 Y_3$  is  $4\epsilon$ -close to the value obtained in the error free analysis. This completes the proof that  $E$  is an  $(n - 30a, 2a, 4\epsilon)$ -lossless conductor.

### 10.2.3 Final comments

We note again that there is no known algebraic way for constructing lossless expanders. Such strong vertex expansion does not seem to be implied by simple algebraic parameters of the graph. Indeed, the construction in this chapter only provides bipartite graphs which are losslessly expanding in *one* direction. While this seems to suffice for all known applications, it is a very interesting challenge to explicitly construct simple graphs which are lossless expanders for all (small enough) vertex sets. In particular, we do not know if this can be achieved for any Cayley graphs.

We note that a somewhat easier problem, unique vertex expansion (which is implied by lossless expansion, but likewise eludes algebraic attacks), was studied by Alon and Capalbo [AC02] who gave explicit constructions of such graphs.



# Chapter 11

## Error Correcting Codes

Connections between error correcting codes - a central area in communication and engineering – and expander graphs, have seen an enormous growth in the past two decades, even though their roots go back to the 1960's. Here we describe in detail only one connection - the application of explicit lossless expanders of the previous section to the construction of the simplest known efficient asymptotically good codes. We also briefly review other variants and connections, and give some references. But first we give some background to this important field.

Consider the problem of sending information through a noisy channel that may flip some bits we send. One way to overcome channel errors is to use error correcting codes. Let us restrict our attention to schemes that transmit the information in  $n$ -bit blocks. To make sure that the receiver can recover from bit-flip errors, we agree in advance that only a subset of the  $2^n$  possible  $n$ -bit strings, may be transmitted. We call such possible strings codewords, and would like our set of codewords to have two conflicting properties. The first is that many bit flips are needed to transform one codeword into another, and the second is that there are many codewords. These properties guaranty (respectively) the robustness of our transmission scheme to channel errors, and the efficiency of the channel utilization. One further property that we crucially need, is algorithmic in nature. We would like an efficient implementation of the encoder and the decoder in such a scheme.

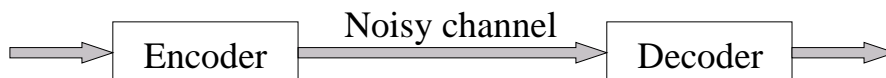


Figure 11.1: Transmitting a message over a noisy channel.

To proceed, we need better formulation of the concepts involved. For reference, there are many excellent texts, e.g. the book of van Lint [vL99]

### 11.1 Definition of Error Correcting Codes

Before starting, we note that we restrict ourselves here only to binary codes, and only to adversarial errors. We note that codes over other alphabets, and great variety of error models exist which are not even mentioned here.

A code is a set of  $n$ -bit binary strings  $C \subseteq \{0, 1\}^n$ . The **distance** of  $C$ , denoted  $\text{dist}(C)$ , is the minimum Hamming distance between a pair of distinct codewords  $x, y \in C$ . That is  $\min_{x \neq y \in C} d_H(x, y)$ , where the Hamming distance  $d_H(x, y)$  is the number of coordinates  $x$  and  $y$  differ on. The **rate** of  $C$  is  $\text{rate}(C) = \log |C|/n$ .

Suppose that  $k$  bit-flips occurred to a message. Then, to guarantee that it won't be confused with another codeword, we need to have  $2k < \text{dist}(C)$ . I.e., we can guarantee correct decoding as long as the number of bit-flips is bounded by  $\lfloor (\text{dist}(C) - 1)/2 \rfloor$  bit-flips. Also, since we transmit  $\log |C|$  bits of information for every  $n$  channel bits, the efficiency of the code is  $\log |C|/n$ , which is the code rate.

When defining a code, we desire both the rate and the distance to be as large as possible. On top of that, for practical applications, we would like the code to be efficiently encodable and decodable, hopefully linear time algorithms for both. But a minimal requirement in this setting is the following.

**Efficient, asymptotically good codes** A family  $C_n \subset \{0, 1\}^n$  of codes is *asymptotically good* for some fixed constants  $r > 0$  and  $\delta > 0$  we have, for all  $n$  both  $\text{dist}(C) > \delta n$  and  $\text{rate}(C) > r$ . The family is called *efficient* if encoding and decoding (from  $\delta n$  errors) can be performed in polynomial time in  $n$ .

The existence of asymptotically good codes can be easily proved by a probabilistic argument. The strive to find efficient ones took over two decades. Here we will see a simple such construction, from expander graphs. But first we need to introduce linear codes.

## 11.2 Linear Codes

It is often convenient to use **linear codes**. That is, subspaces of the  $n$ -dimensional vector space  $\{0, 1\}^n$ , or in other words, subsets of  $\{0, 1\}^n$  that are closed under coordinate-wise mod 2 addition. Such codes can always be described concisely by specifying a basis, and therefore can be efficiently encoded in  $O(n^2)$  time. On the other hand the decoding problem is already NP-hard - we sketch here the simple reduction from the MAXCUT problem, whose NP-hardness and importance to expansion was already mentioned.

Given a linear code  $C$  (say by a basis), and a vector  $y$ , the problem of finding the closest codeword to  $y$ . We reduce MAXCUT to it as follows. Given a graph  $G(V, E)$ , we consider the code  $C_G$  which is the cut-space of  $G$ . Namely the code  $C_G \subset \{0, 1\}^E$  consists of all the characteristic vectors of cuts in  $G$ ,  $E(S, \bar{S})$  for some vertex set  $S$ . It can be easily verified that  $C_G$  is a subspace, namely a linear code. Then upon receiving the all one vector  $(1, 1, \dots, 1)$ , the decoder must return a maximal weight codeword, which is equivalent finding the largest cut in  $G$ .

Finally, we end with an observation. The distance of any linear code is equal to the smallest weight (number of ones) of a non-zero codeword. This follows from the fact that the Hamming distance between two vectors is the equal to the weight of their bit-wise sum.

$$\text{dist}(C) > (2p + \epsilon)n.$$

## 11.3 Asymptotic Bounds

Here we state the basic upper and lower bounds on the relation between the distance of a code and its size (and hence, also its rate). For this, let us denote by  $v(n, r)$  the volume of the radius  $r$  ball in the Hamming cube (which is simply  $v(n, r) = \sum_{i=0}^r \binom{n}{i}$ ).

### 11.3.1 Lower bounds on size: The Gilbert-Varshamov Bound

**Theorem 11.1.** *There exists a length  $n$  code with distance  $d$  and size  $2^n/v(n, d)$ . Moreover, this bound holds even for linear codes.*

*Proof.* Consider the following (exponential time) greedy algorithm to construct a distance  $d$  code.

Init:

$$S = \{0, 1\}^n, C = \emptyset.$$

Repeat until  $S$  is empty:

Pick any point  $x \in S$ .

Add  $x$  to  $C$ , and remove all the points in  $S$  that are within distance  $d$  from  $x$ .

The bound follows by observing that the initial size of  $S$  is  $2^n$ , and we delete at most the volume of a radius  $d$  Hamming ball in each iteration.

For linear codes,  $C$  is maintained as a linear subspace, and at every step, when we pick an  $x \in S$ , we add to  $C$  not only  $x$ , but the whole coset  $x + C$ , thus doubling its size, and then remove its  $r$ -neighborhood. The counting is the same as above.  $\square$

Let  $\delta = d/n$  be the normalized distance of the code, for some  $\delta \leq 1/2$ . Then the size of the radius  $d$  Hamming ball is  $\sum_{i=0}^{\delta n} \binom{n}{i}$ , which is dominated by the last term  $\binom{n}{\delta n}$ . Therefore, the rate of the resulting code is at least

$\log(2^n / \binom{n}{\delta n})/n$ . Since  $\log \binom{n}{\delta n}/n$  is approximately the binary entropy function  $H(\delta) = -\delta \log \delta - (1 - \delta) \log(1 - \delta)$ , we obtain the following asymptotic version of the Gilbert-Varshamov bound.

**Corollary 11.2.** *For large  $n$ , there exists codes with normalized distance  $\delta$  and rate  $r \geq 1 - H(\delta)$ .*

It is not known to explicitly construct such codes, or if one can significantly improve this bound even existentially - as we'll see in a second, the best upper bound is exponentially far from this lower bound on code size.

### 11.3.2 Upper bounds on size: The Sphere Packing and Linear Programming bounds

**Theorem 11.3.** *Any code  $C$  of length  $n$  and distance  $d$  satisfies  $|C| \leq 2^n/v(n, d/2)$ .*

**Corollary 11.4.** *Every code of relative distance  $\delta$  has rate  $r \leq 1 - H(\delta/2)$ .*

*Proof.* (of theorem) For a code  $C$  of distance  $d$ , all radius  $d/2$  balls around the points of  $C$  must be disjoint. The bound follows by observing that the size of the union cannot surpass  $2^n$ , which is the size of the whole space.  $\square$

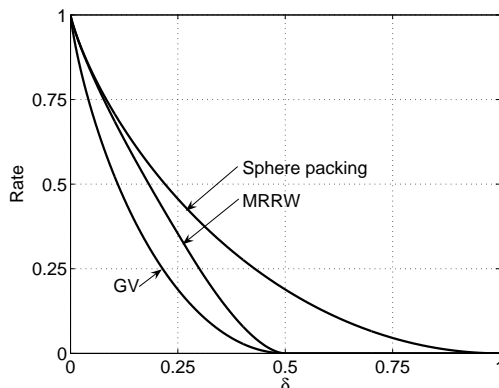


Figure 11.2: Asymptotic upper and lower bounds on rate vs. the relative distance

A much stronger upper bound was shown by McEliece, Rodemich, Rumsey, and Welch [MRRW77].

**Theorem 11.5.** *Every code of relative distance  $\delta$  has rate  $r \leq H(1/2 - \sqrt{\delta(1 - \delta)})$ .*

The relations between the lower bound and two upper bounds is illustrated in Figure 11.2.

## 11.4 Codes from Graphs

We now explain the basic connection between graphs and linear codes, which was one of the initial motivations for the very definition of expanders and the quest for their explicit construction.

Any length  $n$  linear code can be represented by a set of linear equations over  $\text{GF}(2)$  in  $n$  variables (which are often called *parity check* equations). They are satisfied by all codewords, and span the dual space to the linear subspace comprised by the code.

Such a set of equations can be represented as a bipartite graph with the variables on the left and the equations on the right, where each equation is connected to the variables occurring in it. Note that a code  $C$  may have many such representations, since one can find many different bases for the dual space of  $C$ . Remarkably, properties of the underlying bipartite graph, most notably expansion properties, are crucial for establishing lower bounds on its distance, and designing efficient decoding of the code.

Given a bipartite graph  $G$  with  $n$  vertices on the left and  $m$  on the right, we define the code  $C(G) \subseteq \{0, 1\}^n$  as the set of all solutions for the  $m$  equations defined by the graph. The size of  $C(G)$  is at least  $2^{n-m}$ , with equality holding if the equations are independent.

As an example, the graph in Figure 11.3 represents a code of length 10, defined by 5 equations, where the first equation is  $v_1 + v_2 + v_3 + v_6 + v_7 + v_8 = 0 \pmod 2$ . It can be verified that all equations are independent in this case, and therefore this is a code of rate  $1/2$ .

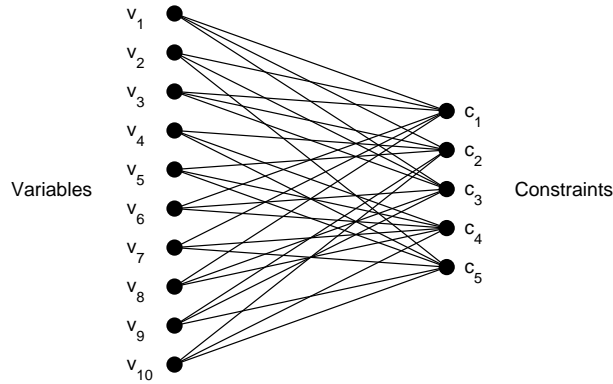


Figure 11.3: The Constraints Graph of a Code

Below we briefly (and incompletely!) sketch some of the history of this branch of error correction.

This idea to construct codes this way was first suggested by Gallager [Gal63], who used sparse bipartite graphs, giving rise to the term *LDPC* codes, for Low Density Parity Check codes (as each parity check equation has a few nonzero coefficients). Gallager used random graphs, as explicit expanders did not exist at the time. This idea was picked up by Pinsker and Bassalygo [BP73] in Russia, who sought out explicit constructions, realized that expansion was at the heart of Gallager’s arguments, formally defined expanders and saw their utility in other basic applications, most notably to robust communication networks. It sparked the first two basic results in expander theory: Pinsker’s probabilistic proof that most sparse graphs are expanders [Pin73] referred to in the first chapter, and Margulis’ first explicit construction [Mar73] which is mentioned all over this book (see also his early paper on LDPC codes and girth, in [Mar82]).

The area went dormant for quite a few years, and coding theorists concentrated on algebraic techniques, till graph codes stormed again into consciousness in the 80’s and 90’s, with important works of Tanner [Tan81], Alon et al [ABN<sup>+</sup>92], Sipser and Spielman [SS96], and more. This particular direction culminated in the paper of Spielman [Spi96], which achieved optimal efficiency for asymptotically good codes – linear time encoding and decoding!

All these constructions use regular graphs. An important breakthrough was the idea of using bipartite graphs with *irregular* degree distributions by Luby, Mitzenmacher, Shokrollahi and Spielman [LMSS01]. These ideas were pushed further, by Richardson and Urbanke [RSU01, RU01], resulting in nearly linear time decoding of codes which approach the capacity - the best rate at which information can flow in some natural error channels. We note that the underlying graphs here are typically chosen at random - the required properties that guarantee the superior behavior of the resulting codes cannot at present be obtained explicitly, and present an excellent challenge.

We now turn to describe a prototype of this family, achieving the goal of efficient asymptotically good codes, via the lossless expanders of the previous chapter.

## 11.5 Efficient asymptotically good codes from lossless expanders

Let  $G = (V_L; V_R, E)$  be a bipartite graph that is  $k$ -regular<sup>1</sup> on the left, such that  $|V_L| = n$ , and  $|V_R| = m$ . We will need the following variant of vertex expansion for bipartite graphs.

**Definition 11.6.** The *left vertex expansion ratio*  $L(G, d)$ , to be the minimum of  $|\Gamma(S)|/|S|$ , over all sets  $S$  in  $V_L$  of size at most  $d$ . In other words, every such set satisfies  $|\Gamma(S)| \geq L(d)|S|$ .

<sup>1</sup>We use  $k$  for the degree here as  $d$  is reserved for distance of the codes

Note that  $L(G, d)$  cannot exceed  $k$  for any graph  $k$ -left-regular graph  $G$  and any  $d$ . On the other hand, the paper of [CRVW02] explained in the previous chapter provides us with explicit bipartite graphs  $G$  for any  $m = \Omega(n)$  (and hence constant rate codes  $C(G)$ ) and expansion  $L(G, d) > .99k$  for  $d = \Omega(n)$ . We first see how these imply asymptotically good codes, and later show that they actually suggest natural efficient (indeed, linear time!) decoding (efficient encoding comes free with the codes being linear). The two theorems below are due to Sipser and Spielman, were proved before explicit construction of lossless expanders were available.

**$C(G)$  has large distance**

**Theorem 11.7 (Sipser-Spielman [SS96]).** *If  $L(G, d) > k/2$ , then  $\text{dist}(C(G)) \geq d$ .*

*Proof.* We first prove a simple consequence of  $> k/2$  left expansion, namely that every set  $S \subseteq V_L$  of size at most  $d$  has a **Unique Neighbor**, i.e a left vertex  $c \in V_R$  such that  $|\Gamma(c) \cap S| = 1$ . To see this, consider the bipartite graph between  $S$  and  $\Gamma(S)$ . As the left degree is  $k$ , and expansion exceeds  $k/2$  the average right degree is less than 2, implying there must be at least one vertex with only one neighbor in  $S$ .

We now use the unique neighbor property of sets of size at most  $d$  to prove that  $C(G)$  has no codewords of weight smaller than  $d$ . Let  $x$  be a non-zero word of weight at most  $d$ , and let  $S \subseteq V_L$  be the set of coordinates  $v$  where  $x_v = 1$ . We saw above that  $\Gamma(S)$  must contain a vertex  $c$  which has only one neighbor in  $S$ . But then the parity check equation  $c$  represents evaluates to 1 on  $x$ , and so  $x$  is not in the code  $C(G)$ .  $\square$

**$C(G)$  can be efficiently decoded**

We now turn to decoding of graph codes. Consider the following naive iterative decoding algorithm: Upon receiving the input  $n$ -bit string  $y$ , as long as there exists a variable such that *most* of its neighboring constraints are not satisfied, flip it.

This algorithm (and variants thereof) are often called *belief propagation*, and are a powerful theme not only in error correction, but in artificial intelligence and computational learning theory. However, often they are only heuristics, while here this strategy always works!

**Theorem 11.8 (Efficient Decoding, Sipser-Spielman [SS96]).** *If  $L(G, d) > \frac{3}{4}k$ , and if the naive decoding algorithm is given as input an  $n$ -bit string  $y$  whose distance from a codeword  $x$  is at most  $d/2$ , then the algorithm will return  $x$  after a linear number of iterations.*

*Proof.* Let  $y^{(i)}$  be the state of the algorithm after  $i$  iterations, where  $y = y^{(0)}$ . Let  $A_i$  be the set of errors at iteration  $i$ , i.e.  $A_i = \{v : y_v^{(i)} \neq x_v\}$ . Then we have to prove that  $A_T$  is empty for  $T = O(n)$ . Consider the set  $A = A_i$  at iteration  $i$  (we discard the index  $i$  to avoid cumbersome notation), and assume that  $A$  is not empty and that  $|A| \leq d$ . Partition  $\Gamma(A)$  to satisfied neighbors  $S$  and unsatisfied neighbors  $U$ . Then:

$$|S| + |U| = |\Gamma(A)| > \frac{3}{4}k|A|. \tag{11.1}$$

Now, count the edges between  $A$  and  $\Gamma(A) = U \cup S$ . There are at least  $|U|$  edges leaving  $U$  and at least  $2|S|$  edges leaving  $S$  (every vertex in  $S$  has at least two neighbors in  $A$ ). Therefore,  $|U| + 2|S| \leq k|A|$ . Combining with (11.1) it yields

$$k|A| - |U| \geq 2|S| > 2 \cdot \left(\frac{3}{4}k|A| - |U|\right),$$

and therefore,

$$|U| > \frac{1}{2}k|A|. \tag{11.2}$$

Therefore, more than half the neighbors of  $A$  are unsatisfied, and by the pigeon principal, there is a variable in  $A$  with more than  $k/2$  unsatisfied neighbors. This implies that as long as there are errors and  $|A| \leq d$ , some variable will be flipped by the algorithm. Since we flip a vertex with more unsatisfied neighbors than satisfied ones,  $|U|$  decreases with every step. We deduce that if the distance from  $x$  does not exceed  $d$  throughout the run of the algorithm, then the algorithm will halt with the codeword  $x$  after a linear number of iterations.

To show that  $A_i$  can never exceed  $d$ , note that at any step  $|A_i|$  changes by  $\pm 1$ . Therefore, if at any iteration  $|A_i|$  exceeds  $d$ , there must be an index (say  $i$ ) such that  $|A_i| = d$ . Then by (11.2),  $|U_i| > kd/2$ . On the other hand, by assumption, in the beginning  $|A_0| \leq d/2$ , and therefore  $|U_0| \leq |\Gamma(A_0)| \leq kd/2$ . Since  $|U|$  is decreasing,  $|U_i| \leq |U_0| \leq k\frac{\alpha}{2}n$ , which is a contradiction.  $\square$

We conclude with mentioning that a similar analysis can be given to a parallel "belief propagation" algorithm, in which at every phase *all* variables with a majority of violated constraints flip their value. This algorithm, under the similar assumptions, will converge to the correct codeword in  $O(\log n)$  phases, and with total work (=number of flips)  $O(n)$ . The analysis (which we leave for the reader) utilizes lossless expansion can show that the error sets  $A_i$  shrink by a constant factor with each iteration.

## Chapter 12

# Cayley expander graphs

Cayley graphs offer a combinatorial depiction of groups and their generators. In this chapter we describe some methods to compute the second eigenvalue of Cayley graphs, or a relative of the spectral gap, the *Kazhdan constant*. Our methods include the Fourier Transform (as featured already in Chapter 8) and Group Representations. Serre's classical book [Ser77] is a good introduction to this beautiful theory. We explain how to use these methods to establish expansion for some examples seen in earlier chapters, and mention the remarkable recent progress in understanding expansion in Cayley graphs of *simple* groups.

We then describe a beautiful connection between a well-known group operation, the *semidirect product*, and the zigzag product of graphs (described in Chapter 9). This connection is a basis of two elementary constructions of Cayley expander graphs for non-simple groups with bounded, or very slowly growing degrees. This follows the work of Alon, Lubotzky, and Wigderson [ALW01], Meshulam, and Wigderson [MW02], and Rozenman, Shalev and Wigderson [RSW04].

**Definition 12.1.** Let  $H$  be a group and  $S \subseteq H$  a subset thereof. The corresponding **Cayley graph**  $C(H, S)$  has  $H$  as a vertex set and  $(g, h)$  is an edge if  $g \cdot s = h$  for some  $s \in S$ . On occasion we consider  $S$  as a multiset, with several copies of each element. In this case the Cayley graph has (the appropriate number of) parallel edges between vertices.

This definition yields a directed graph. However, we will usually assume that the set  $S$  is **symmetric**, i.e.  $s \in S$  implies  $s^{-1} \in S$ . In this case  $C(H, S)$  is undirected and  $|S|$ -regular. We say that  $S$  *generates*  $H$  if every  $h \in H$  can be written as a word in  $S$ , i.e.  $h = s_1 \cdot s_2 \cdot \dots \cdot s_k$  with  $s_i \in S$ . Note that  $S$  generates  $H$  iff the Cayley graph  $C(H, S)$  is connected.

**Definition 12.2.** We will use the shorthand notation  $\lambda(H, S)$  for  $\lambda(C(H, S))$ , the second eigenvalue of the normalized adjacency matrix of this Cayley graph. We also define its spectral gap  $g(H, S) = 1 - \lambda(H, S)$ .

Here are some very simple examples of Cayley graphs:

- The additive cyclic group  $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$  of integers modulo  $n$  is generated by  $S = \{+1, -1\}$ . The graph  $C(\mathbb{Z}_n, S)$  is the cycle on  $n$  vertices.
- The additive group  $\mathbb{Z}_2^d$  of the  $d$ -dimensional vector space over the field  $\mathbb{Z}_2$  with its standard set of generators  $S = \{e_1 = (1, 0, \dots, 0), e_2, \dots, e_d\}$ . Here  $C(\mathbb{Z}_2^d, S)$  is the graph of the  $d$ -cube (see Section 4.2.1).
- For any group  $H$ , when  $S = H$  the Cayley graph  $C(H, S)$  is the complete graph on vertex set  $H$  (plus a loop at each vertex).

Cayley graphs form a rather special class of graphs. For example, as we now show they are all vertex-transitive.

**Definition 12.3.** An *automorphism* of a graph is a permutation  $\pi$  of the vertices such that  $(v, w)$  is an edge if and only if  $(\pi(v), \pi(w))$  is an edge. A graph is called *vertex transitive* if for any ordered pair of vertices  $v, w$  there is some automorphism of the graph that sends  $v$  to  $w$ .

**Claim 12.4.** *Every Cayley graph is vertex transitive.*

*Proof.* Let  $G = C(H, S)$  be a Cayley graph, and let  $g, h$  be two vertices. The mapping  $x \rightarrow hg^{-1}x$  is an automorphism of  $G$  that sends  $g$  to  $h$ .  $\square$

**Constructions of Cayley graph expanders - an overview** The special structure of Cayley graphs makes it easier to bound their second eigenvalues in many cases, or at least derive such bounds from known theorems in other areas.

The first construction of an explicit expander graph family was given by Margulis [Mar73] in 1973. Margulis defined a sequence of Cayley graphs on the groups  $SL_3(p)$  of  $3 \times 3$ -matrices with determinant 1 over the field with  $p$  elements (where  $p$  varies over all primes). The generating sets are all the *elementary matrices*, namely those with 1's on the main diagonal and an extra  $\pm 1$  at one other entry. This yields 12-regular Cayley graphs. They are shown to be expanders with second eigenvalue at most  $1 - \varepsilon$  for some fixed  $\varepsilon > 0$ , independent of  $p$ . The proof relies on a theorem of Kazhdan that the “mother group” of these finite groups, namely  $SL_3(\mathbb{Z})$ , has the Kazhdan property (T). From this the spectral gap follows simultaneously for all these quotient groups.

Lubotzky [Lub94] proved a similar result for  $SL_2(p)$ . While the “mother group” here,  $SL_2(\mathbb{Z})$ , fails to have Kazhdan property (T), Lubotzky observes that Selberg’s celebrated 3/16 theorem suffices to establish the spectral gap on all these quotients in an identical way. He calls this weaker property ( $\tau$ ), and his book [Lub94] (as well as his recent book with Zuk [LZ]) elaborates on the usefulness of this property in a variety of contexts.

The same family of groups feature in the celebrated Ramanujan graphs [LPS88, Mar88], mentioned in Section 5.3. Here different generators are used. The optimal eigenvalue bounds are based on deep results of Deligne solving the Ramanujan conjecture for varieties over finite fields, as well as subsequent estimates.

Quite surprisingly, about 20 years have elapsed till the next slew of constructions for Cayley expander has appeared. These have a very diverse nature and use a variety of new techniques. The zigzag product and its connection to semi-direct product were used [MW02, RSW04] to construct expander Cayley graphs. These constructions work iteratively and apply for some non-simple groups. This is described at the end of this chapter.

We say that an infinite family of groups  $\{H_i\}$  can be made into a family of expanders if there is a constant  $d$  and a generating set  $S_i$  of size  $d$  in each  $H_i$  so that the family  $C(H_i, S_i)$  is a family of expanders as in Definition 2.2.

We recall that a finite group is *simple* if it has no non-trivial normal subgroups. Such groups can be viewed as the building blocks from which all finite groups can be constructed. The complete classification of finite simple groups is one of the major achievements of modern mathematics. An understanding of expansion in almost all finite simple groups has recently evolved. Kassabov [Kas05a], following previous work of Shalom [Sha99] and others, showed that the family of groups  $SL_n(p^m)$  for all  $n \geq 3, m \geq 1$  and  $p$  prime can be made into a family of expanders. Lubotzky [Lub] derived similar results for the family  $SL_2(p^m)$ .

In another breakthrough [Kas05b], Kassabov showed that the family of alternating and symmetric groups  $A_n$  and  $S_n$ , can also be made into a family of bounded-degree expanders. His construction combines ingenious combinatorial arguments with estimates on characters of the symmetric group due to Roichman [Roi96].

Using results by Nikolov [Nik05], Lubotzky [Lub] showed that a family consisting of most simple groups can be made into a family of expanders. (We only know this for “most” simple groups, since the claim is not known to hold for the so-called simple groups of Suzuki type). All these exciting developments are explained in [KLN05].

It is natural to seek other families of groups which can be made into a family of expanders. The next proposition shows that this is not always possible.

**Proposition 12.5.** *Let  $H$  be an Abelian group, and let  $S$  be a generating set such that  $\lambda(H, S) \leq 1/2$ . Then  $|S| \geq \log |H|/3$ .*

A deeper result of Lubotzky and Weiss [LW93] shows that solvable groups of bounded derived length cannot be made expanders with generating sets of bounded size.

*Proof.* As mentioned in Section 2.4, an  $(n, d, \alpha)$ -graph has a logarithmic diameter. Specifically, if the cardinality of  $H$  is  $n$ , the Cayley graph  $C(H, S)$  has diameter smaller than  $3 \log n$ . In particular, every element of  $H$  can be written as a product of at most  $3 \log n$  elements of  $S$ . In an Abelian group, the number of different products of length at most  $\ell$  of elements of  $S$  is equal to the number of partitions of  $\ell$  to  $|S|$  parts. Define  $m = |S|$ . The number of such partitions is then  $\binom{\ell+m-1}{\ell}$ . Consequently,  $\binom{\ell+m-1}{\ell} \geq n$ . Substitute  $\ell = 3 \log n$ , to conclude that

$|S| = m \geq \frac{1}{10} \log n$ . This calculation uses a standard estimate of binomial coefficients, viz.  $\binom{k}{\alpha k} = 2^{(1+o(1))kH(\alpha)}$ , where  $H(x) = -x \log_2 x - (1-x) \log_2(1-x)$  is the entropy function.  $\square$

The following theorem of Alon and Roichman [AR94] proves that Abelian groups are extreme in the sense considered in Proposition 12.5. Indeed *any* group, Abelian or not, has a choice of logarithmically many generators which yield an expanding Cayley graph. In fact, a *random* subset of this size suffices.

**Theorem 12.6 ( [AR94]).** *Let  $H$  be a group and let  $S$  be a subset chosen randomly uniformly in  $H$  with size  $100 \log |H|$ . Then  $\lambda(C(H, S)) < 1/2$  with probability at least 0.5,*

A recent paper of Wigderson and Xiao [WX05] shows that such generating sets can be found efficiently deterministically, in time that is polynomial in the size of the group. The proof uses and extends the ideas of derandomization based on expander walks described in Chapter 3.

## 12.1 Representations of finite groups

Here we give a short introduction to this theory and explain its usefulness for understanding the eigenvalues of Cayley graphs (and Schreier graphs which we define below). For an excellent text on this theory we refer the reader to [Ser77].

The simplest representations of a group are *characters*, whose definition and basic properties were given in Subsection 8.1.1. The important connection of characters to eigenvectors and eigenvalues of Cayley graphs is given below.

**Proposition 12.7.** *Let  $M$  be the normalized adjacency matrix of a Cayley graph  $C(H, S)$ . Let  $\chi$  be a character of  $H$ . Then the vector  $(\chi(h))_{h \in H}$  is an eigenvector of  $M$ , with eigenvalue  $1/|S| \cdot \sum_{s \in S} \chi(s)$ . In particular, the trivial character corresponds to the trivial eigenvalue 1.*

*Proof.* The proof follows from the following simple calculation:

$$(M \cdot \chi)(x) = \frac{1}{|S|} \cdot \sum_{s \in S} \chi(xs) = \frac{1}{|S|} \sum_{s \in S} (\chi(x) \cdot \chi(s)) = \frac{1}{|S|} \left( \sum_{s \in S} \chi(s) \right) \cdot \chi(x).$$

$\square$

Together with proposition 8.5 we obtain a simple method to determine the eigenvalues of  $C(H, S)$  when  $H$  is Abelian. We simply have to compute  $1/|S| \sum_{s \in S} \chi(s)$  for all the characters  $\chi$  of  $H$ . Here are two examples, which had already appeared in Section 4.6.5.

**Eigenvalues of the discrete cube** We mentioned in Section 8.1.1 the characters of  $\mathbb{Z}_2^d$ . For the character  $\chi_a$  it holds that  $\chi_a(e_i) = -1$  if  $a_i = 1$  and  $\chi_a(e_i) = 1$  otherwise. The corresponding eigenvalue is  $(1/d) \cdot \sum_i (\chi_a(e_i)) = 1 - 2w(a)/d$ , where  $w(a)$  is the (Hamming) weight of  $a$ , namely the number of 1's in it. The eigenvalue set is therefore  $\{1 - \frac{2k}{d} \mid k = 0, \dots, d\}$ , where the eigenvalue  $1 - \frac{2k}{d}$  appears with multiplicity  $\binom{d}{k}$ . The largest nontrivial eigenvalue is thus  $1 - 2/d$  which corresponds to every character  $\chi_a$  where  $a = e_j$  for some  $d \geq j \geq 1$ .

**Eigenvalues of the cycle** From the remarks in Section 8.1.1 it follows that the eigenvalues corresponding to the character  $\chi_k$  is  $(e^{2\pi k} + e^{-2\pi k})/2 = \cos(2\pi k)$ . Thus the eigenvalues of the  $n$ -cycle are the numbers  $\cos(2\pi k)$  for  $k = 0, \dots, n-1$ . The largest nontrivial eigenvalue is  $\cos(2\pi/n) = 1 - \Theta(1/n^2)$ .

### 12.1.1 Representations and Irreducible Representations

The method just discussed that works so well for Abelian groups does not, in general, apply to non-Abelian groups, for those may fail to have any nontrivial characters altogether. To deal with this more complicated case we need to introduce a generalization of characters that exist for every group, namely, *group representations*. Rather than characters which are homomorphisms into the complex plane, we consider homomorphisms into matrix groups.

Recall that a linear operator  $A$  acting on a complex inner-product space  $V$  is called *unitary* if  $\langle Av, Aw \rangle = \langle v, w \rangle$  for every  $u, v \in V$  (namely  $A$  is a rotation). The set  $U(V)$  of all unitary operators forms a group under operator composition.

**Definition 12.8.** A *unitary representation* of a group  $H$  is a pair  $(V, \rho)$  where  $V$  is a complex linear space with an inner product and  $\rho$  is a group homomorphism  $\rho : H \rightarrow U(V)$ . We often omit  $V$ , when it is obvious or unimportant, and denote the representation by  $\rho$  alone. The *dimension* of  $\rho$  is  $\dim(V)$ .

Thus, representations are the natural extension of characters, which are simply 1-dimensional representations.

To be more concrete, a  $d$ -dimensional representation of a group  $H$  is an assignment of a  $d \times d$  unitary matrix  $A_h$  to every  $h \in H$ , so that  $A_g A_h = A_{gh}$  for every pair of elements  $g, h \in H$ .

We note that representation theory deals as well with non-unitary representations, but for finite groups, which are our main concern here, there is no loss of generality in restricting ourselves to unitary representations, as we do.

Now we define the main representation of a group: *regular representation*. We will later see that it "captures" all representations of the group, in a well defined way.

**Definition 12.9.** Let  $V$  be the  $|H|$ -dimensional vector space of all complex functions  $f : H \rightarrow \mathbb{C}$ , with the standard inner product. The *regular representation*  $r = r_H$  of  $H$  is defined by  $[r(h)f](x) = f(xh)$  for every  $h \in H$ .

We now introduce "irreducible" representations which can be considered as building blocks from which all representations can be constructed. To this end we first introduce the notion of invariant subspaces of a given representation, which are preserved under the action of all matrices in it. Our first step in this direction is to define invariant vectors. These will feature later on as well in our analysis of expansion in Cayley and Schreier graphs.

**Definition 12.10.** An *invariant vector* of a representation  $(\rho, V)$  is a (nonzero) vector  $v \in V$  such that  $\rho(h)v = v$  for all  $h \in H$ . For example, in the regular representation the invariant vectors are the constant functions on the group  $H$ .

An *invariant subspace* is a linear subspace  $W \subset V$  such that  $\rho(h)W = W$  for all  $h \in H$ . (This means that  $\rho(h)(w) \in W$  for every  $w \in W$ , but not necessarily that  $\rho(h)(w) = w$ ).

For example, in the regular representation, constant functions on  $H$  are invariant, and there are no other invariant vectors. The orthogonal complement of the space of constant functions - the functions whose coordinate sum is zero - form an invariant subspace.

**Definition 12.11.** A representation is called *irreducible* if it has no nontrivial invariant subspaces.

Representations which are identical up to a basis change in the linear space are considered identical for the purpose of representation theory.

**Definition 12.12.** Two representations  $(V, \rho), (W, \eta)$  are called *isomorphic* if there is an isomorphism of vector spaces  $f : V \rightarrow W$  such that  $f(\rho(g)v) = \eta(f(v))$ . Equivalently, there is a common change of basis which moves the matrices in  $\rho$  to the corresponding matrices in  $\eta$ .

Therefore, if  $(V, \rho)$  is not irreducible and has a nontrivial invariant subspace  $V'$ , we can carry out a change of basis to make the action of  $\rho$  on  $W$ , and on its orthogonal complement  $V''$  take place in disjoint sets of coordinates. This yields a representation isomorphic to  $(V, \rho)$  which now decomposes into two representations of smaller dimension  $(V', \rho')$  and  $(V'', \rho'')$ . This process can be iterated on the resulting representations till no more invariant subspaces are found. Thus every representation can be decomposed to irreducible components. We will soon see that the outcome of this process is unique.

The following lemma states that every representation decomposes (uniquely) into irreducible ones.

**Proposition 12.13.** Let  $(V, \rho)$  be a unitary representation of a group  $H$ . Then there is an orthogonal decomposition  $V = \oplus V_i$  with  $\rho(h)V_i = V_i$  for all  $i$  and  $h \in H$ , and  $(V_i, \rho)$  are irreducible representations. Moreover, this decomposition is unique up to isomorphism.

We next see that the *regular representation* contains all possible irreducible representations in its decomposition.

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<sup>1</sup>For our purposes  $V$  can simply be taken to be  $\mathbb{C}^d$  with usual inner product of complex vectors.

**Proposition 12.14.** *Let  $(V, r)$  be the regular representation of a group  $H$ , and let  $(U, \rho)$  be any irreducible representation of  $H$ . Then  $(U, \rho)$  appears in the decomposition of  $(V, r)$  to irreducible representations.*

We finally get to use this apparatus to analyze Cayley graphs. We proceed in a way analogous to the method used in Proposition 12.7. Fix a generating set  $S$  for  $H$  and a representation  $(V, \rho)$  of  $H$ , and consider the matrix

$$A_\rho = (1/|S|) \cdot \sum_{s \in S} \rho(s).$$

The following lemma is simple but useful. It shows how the matrices above, when we range over all *irreducible* representations, capture *all* eigenvalues of the Cayley graph with generators  $S$ .

**Lemma 12.15.** • *The normalized adjacency matrix of  $C(H, S)$  is  $A_r$  for  $r$  the regular representation of  $H$ .*

- *Every eigenvalue of  $A_r$  is an eigenvalue of  $A_\rho$  for some irreducible representation  $\rho$ .*
- *The converse also holds: If  $\rho$  is a representation of  $H$ , then every eigenvalue of  $A_\rho$  is also an eigenvalue of  $A_r$ .*

In principle, at least, this is a recipe by which we can calculate all the eigenvalues of  $C(H, S)$ . In practice, however, it is usually quite hard to analyze all the matrices  $A_\rho$ . It is more manageable, though, for generating sets  $S$  of special structure. Also, as mentioned above, it is often the case that results from various areas of mathematics can be used to this end.

### 12.1.2 Schreier graphs

Much of what he have just done applies in a context more general than that of Cayley graphs of groups. Namely for graphs corresponding to the action of a group on a set. Let  $H$  be a group, and  $X$  be some set. An *action* of  $H$  on  $X$  is a group homomorphism  $\pi : H \rightarrow \text{Sym}(X)$  that sends each element  $h$  to a permutation of the elements of  $X$ . For any subset  $S$  of  $H$  we define the *Schreier graph*  $\text{Sch}(H, X, S)$  whose vertex set is  $X$  and whose edges are  $(x, \pi(s)x)$  for every  $s \in S$  and every  $x \in X$ . Here are a few examples.

- Schreier graphs are indeed more general than Cayley graphs, since any group  $H$  acts on itself by sending an element  $h$  to the permutation  $g \rightarrow gh$ . In this case  $\text{Sch}(H, X, S)$  is simply the Cayley graph  $C(H, S)$ .
- The symmetric group  $S_n$  acts on the set  $[n]$  by  $\pi(\sigma)(i) = \sigma(i)$ .
- The group  $\text{GL}_n(p)$  of invertible matrices over the field  $\mathbb{F}_p$  acts on the set  $\mathbb{F}_p^n$  of  $n$ -dimensional vectors, by  $\pi(M)v = M \cdot v$ .

While these examples still look special and "algebraic", the following folklore result shows that essentially every regular graph is a Schreier graph of some group.

**Theorem 12.16.** [Gro77] *Every finite regular graph of even degree is a Schreier graph corresponding to some finite group acting on some finite set.*

We now explain how the eigenvalues of the adjacency matrix of a Schreier graph are obtained from those of the naturally associated Cayley graph of the acting group.

An action  $\pi$  of  $H$  on a set  $X$  naturally defines a *permutation representation*  $(V, \rho)$  of  $H$ . Here  $V$  is the vector space of all complex-valued functions on  $X$ , and  $(\rho(h)f)(x) = f(\pi(h)x)$ . The adjacency matrix of  $\text{Sch}(H, X, S)$  is equal  $A_\rho = 1/|S| \cdot \sum_{s \in S} \rho(h)$ . By proposition 12.15 every eigenvalue of  $A_\rho$  is an eigenvalue of  $A_r$ , where  $r$  is the regular representation. Consequently, every eigenvalue of the Schreier graph is an eigenvalue of  $C(H, S)$ . In particular

**Proposition 12.17.** *Let  $Z$  be a connected component of  $\text{Sch}(H, S, X)$ . Then  $\lambda(Z) \leq \lambda(H, S)$ .*

The proposition implies that if a Cayley graph of  $H$  is an expander, then so are all the corresponding connected Schreier graphs. We apply this to show that two families of graphs we previously met in Section 2.2 are expanding. Let  $\mathrm{SL}_2(m)$  be the group of  $2 \times 2$  matrices with determinant 1 over the ring of integers modulo  $m$ . It can be proved (see [Lub94]) that the set

$$S = \left( \begin{array}{cc} 1 & \pm 1 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ \pm 1 & 1 \end{array} \right)$$

generates a Cayley graph with second eigenvalue at most  $1 - 1/2000$ . The group  $\mathrm{SL}_2(\mathbb{Z}_m)$  acts on the set  $(\mathbb{Z}_m)^2$  (multiply the matrix by the vector). Consider the corresponding Schreier graph  $\mathrm{Sch}(\mathrm{SL}_2(m), (\mathbb{Z}_m)^2, S)$ . Here a vertex  $(x, y)$  is adjacent to  $(x, y \pm x)$  and  $(x \pm y, y)$ . This is a 4-regular subgraph of the 8-regular graph defined in Chapter 2.2 and a close relative of the graphs discussed in Chapter 8. By proposition 12.17, every connected component of this Schreier graph has second eigenvalue at most  $1 - 1/2000$ , and it is not hard to see that the connected components are the vector  $(0, 0) \in (\mathbb{Z}_m)^2$  and its complement.

Let  $p$  be a prime. The *projective line*  $\mathbf{P}^1(\mathbb{Z}_p)$  is the punctured plane  $(\mathbb{Z}_p)^2 \setminus \{(0, 0)\}$  modulo the equivalence relation  $(x, y) \equiv (ax, ay)$  for all  $0 \neq a \in \mathbb{Z}_p$ . The set  $\mathbf{P}^1(\mathbb{Z}_p)$  consists of  $p + 1$  points, which are the equivalence classes of  $(x, 1)$  for all  $x \in \mathbb{Z}_p$ , plus the equivalence class of  $(1, 0)$ .

The group  $\mathrm{SL}_2(\mathbb{Z}_p)$  acts on  $\mathbf{P}^1(\mathbb{Z}_p)$ , since it acts on  $(\mathbb{Z}_p)^2 \setminus \{(0, 0)\}$  and respects the equivalence relation. Identify the equivalence class  $(x, 1)$  of the projective plane with the element  $x \in \mathbb{Z}_p$ , and define the equivalence class  $(1, 0)$  to be “the point at infinity”. Then the action of  $\mathrm{SL}_2(\mathbb{Z}_p)$  on the projective plane can be defined as follows (we extend the arithmetic operations to include infinity in the obvious way).

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) x = \frac{ax + b}{cx + d}.$$

We say that  $\mathrm{SL}_2(p)$  acts on the projective plane by *fractional linear transformations*. (also known as *Möbius transformations*). Now consider the following generating set  $S$  of  $\mathrm{SL}_2(p)$ .

$$S = \left( \begin{array}{cc} 1 & \pm 1 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$$

In the corresponding Cayley graph on the projective line  $\mathbf{P}^1(\mathbb{Z}_p)$ , the neighbors of  $x$  are the points  $(-1/x)$  and  $(x \pm 1)$ . It can be shown that  $\lambda(\mathrm{SL}_2(p), S) < 1 - 1/104$ , so the same holds for the connected graph  $\mathrm{Sch}(\mathrm{SL}_2(p), \mathbf{P}^1(\mathbb{Z}_p), S)$ . If you remove the point at infinity, and add a self-loop to the point 0 (to compensate for the missing edge) you obtain the 3-regular expander graph described in Section 2.2.

### 12.1.3 Kazhdan Constant and expansion of Cayley graphs

Let  $(V, \rho)$  be an irreducible unitary representation of the group  $H$ . Consider some  $v \in V$  of unit norm and let us investigate the distance from  $v$  to its images  $\{\rho(h)v \mid h \in H\}$ . Since  $\rho(h)$  is always a unitary matrix, all the vectors  $\rho(h)v$  have unit norm, and in particular  $\|\rho(h)v - v\| \leq 2$ . On the other hand, it is easy to verify that the vector  $\sum_{h \in H} \rho(h)v$  is invariant, so by irreducibility it must be zero. But the term corresponding to  $h = \mathrm{id}$ , the identity in  $H$  gives  $\rho(h)v = v$ , so there must be some  $s \in H$  for which  $\|\rho(s)v - v\| > \sqrt{2}$ . The Kazhdan constant of a subset  $S \subseteq H$  quantifies the extent to which a similar conclusion still holds when  $s$  must be selected only from  $S$ .

This discussion illustrates the significance of smallest displacement of any vector by any nontrivial representation acting on it. In this section we point out that the spectral gap of a Cayley graph as well as the *Kazhdan constant* are closely related to this minimum displacement and are therefore closely related to each other. Kazhdan [Kaz67] has originally introduced this constant as well as the closely related Kazhdan’s property (T) for *infinite* groups. These concepts have played a key role in the solution of several important open problems. It turns out that these concepts are very useful for finite groups as well. Although the Kazhdan constant and the spectral gap in Cayley graphs are so closely related, there are different situations where either one is easier to work with. This dual perspective was a key feature in Kassabov’s recent breakthrough result [Kas05b] that the symmetric groups can be made into a family of expanding graphs.

We have already often used the variational definition of eigenvalues to estimate the spectral gap of graphs. In the case of a Cayley graph  $C(H, S)$ , this can be stated as follows:

$$\lambda(H, S) = \min_{v \perp 1} \frac{v^T A_r v}{\|v\|^2} = \min_{v \perp 1} \frac{1}{|S|} \sum_{h \in S} \frac{v^T r(h)v}{\|v\|^2}$$

where  $r$  is the regular representation. We use the expression for the normalized adjacency matrix of  $C(H, S)$  from Lemma 12.15. Since  $r$  is unitary, it is easily verified that  $\frac{\|r(h)v - v\|^2}{\|v\|^2} = 2(1 - \frac{v^T r(h)v}{\|v\|^2})$ . Consequently,

$$g(H, S) = 1 - \lambda(H, S) = \min_{v \perp 1} \frac{1}{2|S|} \sum_{h \in S} \frac{\|r(h)v - v\|^2}{\|v\|^2}.$$

As long as we do not seek optimal expansion rate and consider mostly qualitative concepts such as infinite families of expander graphs, it suffices to consider the largest term in this sum. This leads us to the following definition.

**Definition 12.18.** Let  $S$  be a subset of a group  $H$ . The *Kazhdan Constant* of  $S$  is defined by

$$K(H, S) = \min_{v \perp 1} \max_{h \in S} \frac{\|r(h)v - v\|^2}{\|v\|^2}$$

We already know by Lemma 12.13 that the regular representation decomposes to an orthogonal sum of irreducible representations, and that all the irreducible representations of  $H$  appear in this decomposition. The vector with minimal displacement will therefore appear in some irreducible representation, so the Kazhdan constant may also be defined as

$$K(H, S) = \min_{\rho} \min_{v \perp 1} \max_{h \in S} \frac{\|\rho(h)v - v\|^2}{\|v\|^2}$$

Where  $(\rho, V)$  ranges over all nontrivial irreducible representations of  $H$ .

The spectral gap of  $C(H, S)$  and the Kazhdan constant are therefore related as follows:

$$K(H, S)/(2|S|) < g(H, S) < K(H, S)/2.$$

Therefore, when we consider generator sets of bounded cardinality and if we do not strive for optimal parameters, the spectral gap and the Kazhdan constant can be used interchangeably. The Kazhdan constant is sometimes more convenient to handle than the spectral gap. For example, enlarging the generating set  $S$  may decrease  $g(H, S)$ , while  $K(H, S)$  can only increase. Here is a useful observation in the same spirit.

**Claim 12.19.** *Let  $H$  be a group, and let  $S, \tilde{S}$  be subsets of  $H$  such that each element of  $\tilde{S}$  can be written as a product of at most  $m$  elements of  $S$ . Then  $K(H, S) \geq K(H, \tilde{S})/m$*

Note that this property is not shared by the spectral gap. If the set  $\tilde{S}$  is obtained from  $S$  by adding many copies of the identity element, then the assumption of the claim holds with  $m = 1$ , but  $g(H, S)$  may be much smaller than  $g(H, \tilde{S})$ .

## 12.2 The Replacement Product and Semidirect Product

In chapter 9 we defined the zig-zag product of graphs, and used it to construct expander families. In this section we construct families of expander Cayley graphs using the zig-zag product. In general, a zig-zag product of two Cayley graphs is not a Cayley graph. Nevertheless, under certain conditions on the underlying groups and on the generating sets, the zig-zag product of two Cayley graphs  $C(A, S_A) \otimes C(B, S_B)$  is indeed a Cayley graph  $C(C, S_C)$  when  $C$  is the *semidirect* product of  $A$  and  $B$ . This connection and its utility in establishing expansion were first described in [ALW01]. We first present the connection in this section, and then the applications of this and other papers in the subsequent sections.

We start by defining semidirect product, and then turn to describe the relation to the zigzag product.

**Definition 12.20.** An *action* of a group  $B$  on a group  $A$  is a group homomorphism  $\phi : B \rightarrow \text{Aut}(A)$ . In other words, each element  $b \in B$  corresponds to an automorphism  $\phi_b$  of  $A$ , where  $\phi_{b_1 \cdot b_2} = \phi_{b_1} \phi_{b_2}$ .

For example, the cyclic group  $\mathbb{Z}_d$  acts on the group  $\mathbb{Z}_2^d$ , the additive group of the  $d$ -dimensional vector space over  $\mathbb{Z}_2$ , by cyclically shifting the coordinates of the  $d$ -dimensional vectors. When a group  $B$  acts on  $A$  we can define a group structure on the set  $A \times B$ .

**Definition 12.21.** Suppose a group  $B$  acts on a group  $A$ . The *semidirect product*  $A \rtimes B$  is a group whose elements are pairs  $(a, b)$  where  $a \in A$  and  $b \in B$ . We define

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot \phi_{b_1}(a_2), b_1 \cdot b_2).$$

The simplest example of the semidirect product is the special case of the direct product  $A \times B$ . In this case  $\phi_b$  is the identity automorphism of  $A$  for all  $b \in B$ , and so  $(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot a_2, b_1 \cdot b_2)$ . Let us move to a more interesting example. As we saw the group  $\mathbb{Z}_d$  acts on  $\mathbb{Z}_2^d$ . The element set of the resulting semidirect product is  $(\bar{v}, x) \in \mathbb{Z}_2^d \times \mathbb{Z}_d$ . The multiplication rule is

$$((v_i)_{i=0}^{d-1}, x) \cdot ((w_i)_{i=0}^{d-1}, y) = ((v_i + w_{i+x})_{i=0}^{d-1}, x + y).$$

Notice that this group is not commutative although the constituent groups are.

We now return to Cayley graphs. Continuing with our example, pick  $\{+1, -1\}$  as a generating set of  $\mathbb{Z}_d$ , and  $e_0, \dots, e_{d-1}$  for  $\mathbb{Z}_2^d$ . The corresponding Cayley graphs are the  $d$ -cycle and the  $d$ -dimensional binary cube respectively. Since the degree of the  $d$ -dimensional cube is equal to the number of vertices in the  $d$ -cycle, we can form a zig-zag product of the two. Let us consider the simpler **replacement product**, defined in Section 9.3. In this product we replace every vertex of  $\mathbb{Z}_2^d$  by a cloud of  $d$  vertices representing the set  $\mathbb{Z}_d$ . On each cloud we preserve the edges of the original  $d$ -cycle. We also connect each vertex in the cloud to one of the  $d$  neighbors of the cloud in the binary cube. In our discussion in Chapter 9 it did not matter much how the clouds are connected to each other. Now we choose to connect the vertex  $(v, h)$  to  $(v + e_h, h)$ , the usefulness of this specific choice will become clear later. Like the zig-zag construction, the replacement product is an expander if the original two graphs are expanders. The resulting graph for  $d = 3$  is depicted in Figure 12.1.

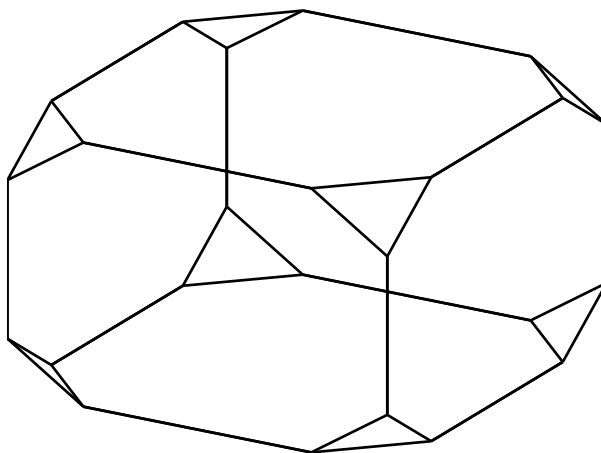


Figure 12.1: A Cayley graph of  $\mathbb{Z}_2^3 \rtimes \mathbb{Z}_3$

We have described two operations above: The semidirect product takes groups  $A, B$  and puts a group structure on the set  $A \times B$ . The replacement product (or zig-zag product) takes graphs on vertex sets  $A, B$ , and constructs a graph on vertex set  $A \times B$ . The two constructions are closely related. Under certain assumptions, the replacement product of the Cayley graphs of the groups  $A, B$  is also a Cayley graph of the semidirect product of  $A \times B$ . In our running example, consider the replacement product of the  $d$ -cycle and the  $d$ -dimensional binary cube we defined above. It is a Cayley graph of the semidirect product  $\mathbb{Z}_2^d \rtimes \mathbb{Z}_d$ , with the three element generating set  $\{(\bar{0}, \pm 1) \cup (e_0, 0)\}$ .

We now turn to the general statement relating replacement products and semidirect products. Suppose a group  $B$  acts on a group  $A$ . The *orbit* of an element  $a \in A$  under the action of  $B$  is the set  $\{\phi_b(a) | b \in B\}$ . For example, the orbit of  $v \in \mathbb{Z}_2^d$  under the action of  $\mathbb{Z}_d$  is the set of all cyclic shifts of  $v$ . Suppose that some (possibly very large) generating set  $S_A$  of  $A$  is such an orbit of some element  $x \in A$ . We prove in the following lemma that we can find a generating set for  $A \rtimes B$  consisting of some generating set  $S_B$  for  $B$  (embedded in  $A \rtimes B$ ), plus the single element  $x$ , instead of the whole set  $S_A$ . Furthermore, the resulting Cayley graph is the replacement product of  $C(A, S_A)$  and  $C(B, S_B)$ . The replacement product is an expander if the constituent graphs are expanders, and the degree of the replacement product is dominated by the size of  $S_B$ , while  $S_A$  may be as large as  $B$ . We do not care, as long as  $S_A$  is a single  $B$  orbit. We state the conclusion formally in the following theorem.

**Theorem 12.22.** [ALW01] *Let  $A, B$  be two groups with generating sets  $S_A, S_B$ , such that  $|B| = |S_A|$ . Furthermore, suppose that  $B$  acts on  $A$  in such a way that  $S_A$  is the orbit of one of its elements  $x \in S_A$  under this action. Then  $S := \{(1, s) | s \in S_B\} \cup \{(x, 1)\}$  generates  $A \rtimes B$ , and  $C(A \rtimes B, S)$  is a replacement product of  $C(A, S_A)$  and  $C(B, S_B)$ . More generally, if  $S_A$  is a union (taken with multiplicities)  $S_A^1 \cup \dots \cup S_A^k$  of  $k$  orbits under the action of  $B$ , and if  $x_1, \dots, x_k$  are representatives of these orbits, then  $S := \{(1, s) | s \in S_B\} \cup \{(x_1, 1), \dots, (x_k, 1)\}$  is a generating set for  $A \rtimes B$ , and  $C(A \rtimes B, S)$  is the union of the  $k$  replacement products of  $C(A, S_A^i)$  and  $C(B, S_B)$  when  $1 \leq i \leq k$ .*

*Proof.* We will prove the single-orbit case, and leave the general case to the reader. Suppose then that  $S_A$  is a single orbit under the action of  $B$ , with representative  $x$ . To see that  $S$  generates  $A \rtimes B$ , first note that each element  $(a, b) \in A \rtimes B$  can be written as the product  $(a, 1) \cdot (1, b)$ , and can therefore be written as a product of elements from  $(1, S_B) \cup (S_A, 1)$ . By assumption, any  $s_a \in S_A$  is equal to  $\phi_{s_b}(x)$  for some  $s_b \in S_B$ , and therefore  $(1, s_b) \cdot (x, 1) \cdot (1, s_b^{-1}) = (\phi_b(x), 1) = (s_a, 1)$ . This implies that indeed  $S$  generates the group  $A \rtimes B$ .

Consider now the graph  $C(A \rtimes B, S)$ . This graph consists of clouds of the elements of  $B$ , where each cloud is interconnected by the edges of  $C(B, S_B)$ , since  $(a, b) \cdot (1, s_b) = (a, b \cdot s_b)$ . Between the clouds we have edges of the form  $(a, b) \cdot (x, 1) = (a \cdot \phi_b(x), b)$ . So the cloud of the element  $a$  is connected by one edge to each of the clouds corresponding to  $a$ 's neighbors in the graph  $C(A, S_A)$ , as required by the replacement product definition.  $\square$

**Remark 12.23.** *It is not difficult to check that under the assumptions of the lemma, that (instead of the replacement product) we can also describe the zig-zag product of two Cayley graphs as a Cayley graph on  $A \rtimes B$  with the generating set  $\{(1, s_1) \cdot (x, 1) \cdot (1, s_2) : s_1, s_2 \in S_B\}$ .*

## 12.3 Constructing expander families by iterated semidirect products

We have already seen how to construct a family of constant degree expanders using the zig-zag product. We have just seen that the zigzag product of two Cayley graphs is sometimes a Cayley graph. Can we combine these two ideas to construct a family of Cayley expander graphs? This is indeed possible, and we will present two constructions. One, by Meshulam and Wigderson [MW02] constructs a family of expanders with non-constant, but very slowly growing degree. The other construction, by Rozenman, Shalev, and Wigderson [RSW04], obtains a family of constant-degree expanders.

### 12.3.1 Cayley expanders from group rings [MW02]

For a group  $H$  let  $F_p[H]$  be the *group ring* over  $F_p$ , namely the set of all formal sums  $\{\sum_{h \in H} \alpha_h h : \alpha_h \in F_p\}$ , with the obvious addition and multiplication. We think of  $F_p[H]$  as an additive group, so the group  $H$  naturally acts on  $F_p[H]$ . Therefore, we can iteratively construct a sequence groups by  $H_{i+1} = F_{p_i}[H_i] \rtimes H_i$ , provided that we can find a constant number of orbits under the action of  $H_i$  that make  $F_{p_i}[H_i]$  an expander. This is indeed possible with a proper choice of  $p_i$  and  $H_1$ :

**Theorem 12.24 (Meshulam and Wigderson [MW02]).** *One can find a group  $H_1$  and a sequence of generating sets  $U_i$  such that  $\lambda(H_n, U_n) \leq 1/2$  and  $|U_n| \leq \log^{(n/2)} |H_n|$ , where  $\log^{(n/2)}$  is the iterated logarithm.*

It turns out that for this sequence of groups, this construction is almost the best that one can hope for, since the groups  $H_n$  are solvable with huge Abelian subgroups. More precisely, group  $H_n$  is a solvable group with solvability index at most  $n$  (as  $H_{n-1}$  is a normal subgroup with Abelian quotient  $F_{p_{n-1}}[H_{n-1}]$ ). In this case it is known that any generating set achieving  $\lambda_2 \leq 1/2$  must have cardinality at least  $\log^{(n)} |H_n|$  [LW93]

To achieve the above construction, Meshulam and Wigderson prove a sufficient condition to make  $F_p[H]$  an expander with a generating set consisting of a constant number of  $H$  orbits. Interestingly, this condition relies on the distribution of dimensions of the irreducible representations of  $H$ .

**Theorem 12.25.** *Let  $d_r$  be the number of irreducible representations of  $H$  of dimension smaller than  $r$ . Then if  $d_r < c^r$  for all  $r$  and some constant  $c$ , then there is a constant number of orbits of  $F_p[H]$  that make it an expander.*

One family of groups that have this property are the **monomial groups**.<sup>2</sup> Such groups have the nice property that if  $H_i$  is monomial then  $H_{i+1}$  is monomial as well, and one can explicitly find generating orbits for  $F_p[H_i]$ . This gives a sequence of explicit expanding Cayley graphs  $X_i$  on the groups  $H_i$ , where the degree of  $X_n$  is exponential in  $n$  and the size of  $X_{n+1}$  is a tower function in  $n$ . The degree of  $X_n$  is therefore a indeed an iterated logarithm in the size of  $X_n$ , yielding an "almost constant" degree family of expanders.

### 12.3.2 Cayley expanders from iterated wreath products

Given a group  $H$ , the symmetric group  $S_d$  (or the alternating group  $A_d$ ) acts naturally on the group  $G^d$ , the direct product of  $d$  copies of  $H$ , by permuting the coordinates. The resulting semidirect product  $H^d \rtimes S_d$  is called the *wreath product* of  $H$  and  $S_d$ . The wreath product may also be defined for any subgroup of  $S_d$ , for example the alternating group  $A_d$ . Take  $H_1 = A_d$ , and define  $H_{i+1} = H_i \rtimes A_d$ . Rozenman, Shalev and Wigderson [RSW04] prove that this family of groups can be made into an expander family. The main lemma in the proof is a construction of a single  $A_d$  orbit that generates  $H^d$  as an expander (under a certain condition on  $H$ ).

Here is the idea of this main lemma. Let  $U$  be some expanding generating set for  $H$ . It is not hard to check that  $U^d$ , the set of all  $U$ -valued vectors of length  $d$  is an expanding generating set for  $H^d$  (with the same second eigenvalue). However, the set  $U^d$  is far from being a single  $A_d$  orbit. Consider the subset  $U^{(d)} \subset U^d$ , consisting of  $U$ -valued vectors with length  $d$  in which every element of  $U$  appears the same number of times (assume that  $|U|$  divides  $d$ ). This subset is indeed a single  $S_d$  orbit, but is it still expanding? The next theorem answers positively, if  $|U|$  is small relative to  $d$ , and if every element of  $H$  can be written as a commutator.

**Theorem 12.26 ([RSW04]).** *Let  $H$  be a group in which every element  $h$  can be written as a commutator  $h = aba^{-1}b^{-1}$ . Let  $U$  be an expanding generating set for  $H$  with second eigenvalue at most  $1/4$ , and suppose  $|U| < d/106$  then  $\lambda(H^d, U^{(d)}) < 1/2$ .*

Theorem 12.26 performs part of the induction step of the expander family construction. We want to show that when  $H_i$  has a small enough expanding generating set, the same holds for  $H_{i+1} = H_i^d \rtimes A_d$ . We found a generating set for  $H^d$  which is a single  $A_d$  orbit. We now need to find an expanding generating set for  $A_d$ . Since the generating set of  $H_{i+1}$  will have size dominated by the size of the generating set of  $A_d$ , we need  $A_d$  to have a small expanding generating set. The same requirement is needed to find an expanding generating set for the first group  $H_1$  of the construction (which is also  $A_d$ ).

**Theorem 12.27 ([Kas05b]).** *The alternating group  $A_d$  has an explicit generating set  $U$  with size independent of  $d$  such that  $\lambda(A_d, U) < 1/100$ .*

These two theorems imply that the construction above works for large enough  $d$ .

**Corollary 12.28 ([RSW04]).** *Let  $H_i$  be the group sequence defined by  $H_1 = A_d$  and  $H_{i+1} = H_i \rtimes A_d$ . Then when  $d$  is large enough, There exists a sequence of generating sets  $U_i$  such that  $\lambda(H_i, U_i) \leq 1/2$  and  $|U_i|$  is independent of  $i$ .*

[S].

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<sup>2</sup>i.e., all their irreducible representations are induced from one-dimensional representations of subgroups of  $H$ .

## 12.4 Expansion is not a group property

Suppose that for some family of groups  $H_i$  we can find expanding generating sets  $U_i$  of bounded size, independent of  $i$ . Now suppose that  $V_i$  is another family of generating sets for  $H_i$ , also with bounded size. Can we perhaps deduce that  $V_i$  also generates  $H_i$  as an expander? If true, this would imply that expansion is a property of the groups  $G_i$ , independent of a specific generating set. This question was raised by Lubotzky and Weiss [LW93]. Unfortunately, this is untrue, as was first demonstrated via the connection of semi-direct product to the zigzag product of [ALW01] discussed above. We explain this example, and then the far simpler example which became available once Kassabov's work

**First counterexample** : The idea is the following. Consider the vector space  $A_p = F_2^{p+1}$  for some prime  $p$ . This is an Abelian group, and of course does not have a bounded-size expanding generating set. We have seen in section 12.1.2 that  $SL_2(p)$  acts on the projective plane  $\mathbf{P}^1(\mathbb{Z}_p)(F_2)$  which has  $p + 1$  points. Therefore,  $SL_2(p)$  acts on  $F_2^{p+1}$  by permuting the coordinates of the vectors. It can be shown that (i) The standard basis of  $F_2^{p+1}$  is a single  $SL_2(p)$  orbit. (ii) The union of two *random* orbits is an expanding generating set for  $F_2^{p+1}$ . As we have mentioned in section 12.1.2, there exists a set  $S$  of four matrices in  $SL_2(p)$  which are expanding.

By Lemma 12.22, combining our two generating sets on  $F_2^{p+1}$  with the generating set of  $SL_2(p)$  create two Cayley graphs on  $F_2^{p+1} \rtimes SL_2(p)$ . The first Cayley graph comes from a set of six generators - the four generators of  $SL_2(p)$  and representatives of two random orbits. By Lemma 12.22 this graph is a replacement product of two expander graphs, which is therefore an expander. The second Cayley graph comes from a set of five generators for  $F_2^{p+1} \rtimes SL_2(p)$  - the four generators of  $SL_2(p)$  and a representative of the orbit of standard basis vectors). This Cayley graph is a replacement product, where one of the constituent graphs is the Cayley graph of  $F_2^{p+1}$  with the standard basis vectors. This graph is simply the discrete  $p + 1$ -dimensional cube, which has second eigenvalue at  $1 - 1/(p + 1)$ . A replacement product can never have a smaller second eigenvalue than any of the constituent graphs, which proves the non-expansion result.

**Second counterexample** Consider the Symmetric group  $S_d$ . By theorem 12.27,  $A_d$  has bounded generating set  $U_d$  for every  $d$ . On the other hand, consider the permutation  $(12)$  which switches the first two elements, and the permutation  $(12 \dots d)$  which cyclically shifts the  $d$  elements, and the inverse of this cycle. It is not hard to check that the resulting 3-regular Cayley graph has second eigenvalue which tends to zero when  $d$  grows.



# Chapter 13

## Metric Embedding

Any metric space can be embedded into Euclidean space with some distortion of the distances. We prove that the graph metric of expander graphs is the hardest metric to embed, in the sense that of all finite metric spaces with the same number of points, expanders require the largest distortion.

The study of finite metric spaces and their embeddings has been undergone tremendous growth in the last decade (although much important work was done earlier, mainly within the local theory of Banach spaces, e.g. [MS86]). These objects are studied from at least three different perspectives: Geometric, combinatorial and computational. At the end of this chapter we will elaborate on these, and in particular on low distortion embeddings into other spaces, notably  $l_1$ , which has a major algorithmic significance.

### 13.1 Basic Definitions

A metric space is the pair  $(X, d)$ , where  $X$  is a set of points and  $d$  is the distance function  $d : X \times X \rightarrow \mathbb{R}^+$ . The distance function is a symmetric non-negative function satisfying the triangle inequality. That is,  $d(x, y) \geq 0$  with equality if  $x = y$ ;  $d(x, y) = d(y, x)$ ; and  $d(x, y) \leq d(x, z) + d(z, y)$  for every  $x, y, z \in X$ .

In this chapter we consider how well a finite metric space  $(X, d)$  can be approximated by a Euclidean metric. That is, our “model space” is  $\mathbb{R}^n$  with  $l_2$  norm. For an embedding  $f : X \rightarrow \mathbb{R}^n$  of the metric space  $(X, d)$  into  $(\mathbb{R}^n, l_2)$ , we define:

$$\begin{aligned} \text{expansion}(f) &= \max_{x_1, x_2 \in X} \|f(x_1) - f(x_2)\| / d(x_1, x_2), \\ \text{contraction}(f) &= \max_{x_1, x_2 \in X} d(x_1, x_2) / \|f(x_1) - f(x_2)\|, \\ \text{distortion}(f) &= \text{expansion}(f) \cdot \text{contraction}(f). \end{aligned}$$

It is clear that there are metric spaces that need to be embedded with some distortion. For example, consider the metric  $(\{1, 2, 3, 4\}, d)$  of the star graph, depicted in Figure 13.1, where  $d(1, 4) = d(2, 4) = d(3, 4) = 1$ , and  $d(i, j) = 2$  otherwise.

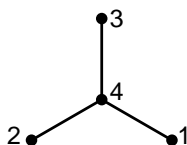


Figure 13.1: A graph that cannot be isometrically embedded into a Euclidean metric

Embedding this space into  $\mathbb{R}^n$  with no distortion, implies that each of the triplets  $\{1, 2, 4\}$ ,  $\{1, 3, 4\}$ , and  $\{2, 3, 4\}$  are on a single line. Therefore, all four point must be one the same line which is a contradiction.

## 13.2 Finding the Minimal Distortion

Let  $C_2(X, d)$  denote the least distortion of the metric space  $(X, d)$  into  $(\mathbb{R}^n, l_2)$ . It is not hard to see that for finite  $X$ , the smallest distortion achievable can always be achieved with dimension  $n \leq |X|$ . In this section we present some bounds on the dimension and distortion achievable for arbitrary metric spaces, and derive an efficient algorithm to calculate  $C_2(X, d)$ . The algorithm relies on a quadratic semi-definite characterization of the problem. Using semi-definite duality, it leads to a simple way to prove lower bounds on  $C_2(X, d)$ .

We start with the existential upper bound - the grand ancestor of this area.

**Theorem 13.1 (Bourgain [Bou85]).** *Any  $n$ -point metric space  $(X, d)$  can be embedded into Euclidean space with distortion  $O(\log n)$ .*

The original theorem was actually proved about embedding into  $l_1$ , which is a bigger space than  $l_2$ , but the same proof works to give the theorem above. The relation between  $l_2$  and  $l_1$  will return later in this chapter. We now see that in  $l_2$  the dimension can be significantly reduced without significant loss in distortion (a fact false in  $l_1$  - see [BC03, LN04]).

**Theorem 13.2 (Johnson-Lindenstrauss [JL84]).** *Any  $n$ -point Euclidean metric can be embedded into a  $O(\log n)$ -dimensional Euclidean space with constant distortion.*

The proof of this extremely useful theorem is simple. It uses random linear projections to a low dimensional subspace, and follows directly from the concentration of measure under such mappings. Combining the two theorems, we see that a logarithmic distortion can be achieved even in logarithmic dimension. We will not be too concerned with dimension here, however. We now turn to the algorithmic problem of computing distortion.

**Theorem 13.3 (Linial-London-Rabinovich [LLR95]).** *There is a polynomial time algorithm, that given a metric  $(X, d)$  (say by a matrix of distances) computes  $C_2(X, d)$ .*

*Proof.* The proof is based on semi-definite programming. Let  $(X, d)$  be a metric space with  $|X| = n$ . Let  $f : X \rightarrow l_2$ . Since one can always scale  $f$  so that  $\text{contraction}(f) = 1$ , we have  $\text{distortion}(f) \leq \gamma$  if and only if

$$d(x_i, x_j)^2 \leq \|f(x_i) - f(x_j)\|^2 \leq \gamma^2 d(x_i, x_j)^2 \quad \text{for all } 1 \leq i < j \leq n. \quad (13.1)$$

Let us recall some standard facts from linear algebra next. A symmetric matrix  $Z$  is said to be positive semi definite if  $v^T Z v \geq 0$  for all  $v \in \mathbb{R}^n$  (or equivalently, all eigenvalues of  $Z$  are nonnegative). We denote by  $\text{PSD} = \text{PSD}_n$  the collection of all  $n \times n$  positive semidefinite matrices. It is not difficult to show that  $Z \in \text{PSD}$  iff  $Z = W W^T$  for some matrix  $W$ . We consider any embedding  $f$  of  $X$  in  $\mathbb{R}^n$  where  $f(x_i) = u_i$  and let  $U$  be the matrix whose  $i$ -th row is  $u_i$ , and let  $Z = U U^T$ .

It follows that finding  $u_i = f(x_i)$  that satisfy (13.1) is equivalent to finding  $Z \in \text{PSD}$  such that

$$d(x_i, x_j)^2 \leq z_{ii} + z_{jj} - 2z_{ij} \leq \gamma^2 d(x_i, x_j)^2 \quad \text{for all } 1 \leq i < j \leq n, \quad (13.2)$$

since  $\|u_i - u_j\|^2 = z_{ii} + z_{jj} - 2z_{ij}$  for  $Z = U U^T$ . Thus we conclude that  $C_2(X, d) \leq \gamma$  if and only if there is a positive semi-definite matrix  $Z$  satisfying (13.2). This is a convex programming problem that can be solved in polynomial time by the ellipsoid algorithm. (See the book [GLS93] for general background in discrete optimization and the ellipsoid algorithm.)  $\square$

The algorithm above constructs a primal problem and solves it by the ellipsoid algorithm. The dual problem gives us an interesting characterization of  $C_2(X, d)$ , that is handy in proving lower bounds on the distortion. When we transform a primal problem to its dual we take a non negative combination of its constraints. But how do we look at the constraint  $Z \in \text{PSD}$ ? Again we need a simple but useful fact from linear algebra.

**Claim 13.4.**  $Z \in \text{PSD}$  if and only if  $\sum_{i,j} q_{ij} z_{ij} \geq 0$  for all  $Q \in \text{PSD}$ .

*Proof.*

$\Leftarrow$  For  $v \in \mathbb{R}^n$ , let  $Q$  be the PSD matrix defined by  $(Q)_{ij} = v_i \cdot v_j$ . Then,  $v^T Z v = \sum_{i,j} q_{ij} z_{ij} \geq 0$ , implying that  $Z \in \text{PSD}$ .

⇒ If  $Q$  is a PSD matrix of rank 1, then it has the form  $(Q)_{ij} = v_i \cdot v_j$  for some  $v \in \mathbb{R}^n$ . Therefore,  $\sum_{i,j} q_{ij} z_{ij} \geq 0$  for any PSD matrix  $Z$ . As mentioned above, any PSD matrix  $Q$  can be written as  $WW^T$  for some matrix  $W$ , and is therefore the sum of rank 1 PSD matrices. This implies the required claim. □

**Theorem 13.5 (Linial-London-Rabinovich [LLR95]).**

$$C_2(X, d) = \max_{P \in \text{PSD}, P \cdot \bar{1} = 0} \sqrt{\frac{\sum_{p_{ij} > 0} p_{ij} d(x_i, x_j)^2}{-\sum_{p_{ij} < 0} p_{ij} d(x_i, x_j)^2}}$$

*Proof.* As we have seen, the primal problem is:

$$\begin{aligned} \sum_{ij} q_{ij} z_{ij} &\geq 0 && \text{for all } Q \in \text{PSD}, \\ z_{ii} + z_{jj} - 2z_{ij} &\geq d(x_i, x_j)^2 && \text{for all } i, j, \\ \gamma 2d(x_i, x_j)^2 &\geq z_{ii} + z_{jj} - 2z_{ij} && \text{for all } i, j. \end{aligned}$$

The dual program is the statement that for  $\gamma < C_2(X, d)$ , there must exist a non-negative combination of the constraints of the primal problem that yields a contradiction.

We are looking for a linear combination of the constraints that yields the contradiction  $0 \geq 1$ . By Claim 13.4 this combination comes down to selecting a specific  $Q \in \text{PSD}$  and writing  $\sum_{ij} q_{ij} z_{ij} \geq 0$ . The rest of the inequalities should be combined with coefficients so as to eliminate all  $z_{ij}$  from our inequalities. To eliminate the off-diagonal entries,  $z_{ij}$  for  $i \neq j$ , these coefficients are necessarily as follows:

- If  $q_{ij} > 0$  then we add the constraint  $z_{ii} + z_{jj} - 2z_{ij} \geq d(x_i, x_j)^2$  multiplied by  $q_{ij}/2$ .
- If  $q_{ij} < 0$  then we add the constraint  $\gamma 2d(x_i, x_j)^2 \geq z_{ii} + z_{jj} - 2z_{ij}$  multiplied by  $-q_{ij}/2$ .

In order that at the end of this process, the coefficients of the diagonal entries  $z_{ii}$  will be zero, we need that all the row sums of  $Q$  will be zero. Therefore, assuming that  $\sum_j q_{ij} = 0$  for all  $i$ , we obtained the following inequality

$$0 \geq \sum_{q_{ij} > 0} q_{ij} d(x_i, x_j)^2 + \gamma 2 \sum_{q_{ij} < 0} q_{ij} d(x_i, x_j)^2.$$

The Theorem follows by observing that this is a contradiction if  $\gamma 2$  is smaller than

$$\sum_{q_{ij} > 0} q_{ij} d(x_i, x_j)^2 / \left( - \sum_{q_{ij} < 0} q_{ij} d(x_i, x_j)^2 \right).$$

□

### 13.3 Distortion bounds via semidefinite duality

We now demonstrate the power of the characterization above to obtain optimal bounds on the distortion of two natural metrics. Both will be graph metrics: the shortest distance metric on the vertices of a graph. For a graph  $G$  we denote by  $C_2(G)$  the best distortion of this metric. We will give optimal bounds on  $C_2(G)$  for the Boolean cube, as well as for expanders (showing them to be the hardest to embed).

### 13.3.1 Embedding the cube into $l_2$

Let  $Q_r$  denote the  $r$ -dimensional binary cube on  $\{0, 1\}^r$ . Note that Hamming distance is the shortest path metric in this graph. The natural, identity embedding of this graph into  $l_2$  (every vertex is viewed as an  $r$ -dimensional vector) can be easily seen to have distortion  $\sqrt{r}$ . Indeed, the embedding  $id : \{0, 1\}^r \rightarrow \mathbb{R}^r$  defined by  $id(x) = x$ , has  $contraction(id) = \sqrt{r}$  and  $expansion(id) = 1$ . We use Theorem 13.5 to show that this embedding is the best possible. Let define the  $2^r \times 2^r$  matrix  $P$ :

$$P(x, y) = \begin{cases} -1 & \text{if } d(i, j) = 1 \\ r - 1 & \text{if } i = j \\ 1 & \text{if } d(i, j) = r \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that  $P\vec{1} = 0$ , and that  $P \in \text{PSD}$ . The latter holds, since  $P$  has the same eigenvectors as the cube. Since  $\sum_{p_{ij} > 0} p_{ij} d(x_i, x_j)^2 = 2^r \cdot r/2$ , and  $-\sum_{p_{ij} < 0} p_{ij} d(x_i, x_j)^2 = 2^r \cdot r$ , we get that  $C_2(Q_r) \geq \sqrt{r}$ .

### 13.3.2 Embedding expander graphs into $l_2$

Consider some  $k$ -regular<sup>1</sup> graph  $G = (V, E)$  of size  $|V| = n$  and  $\lambda_2 \leq k - \epsilon$ , where  $k > 2$  is fixed. As for the cube, it is simple to see that an expander can be embedded with distortion  $O(\log n)$  in  $l_2$ . In fact, any graph can be embedded with distortion equal to its diameter, which is  $O(\log n)$  for an expander. To see this, consider embedding a graph to the vertices of a simplex in  $\mathbb{R}^n$ . Since every two vertices of the simplex have distance 1, the expansion is one and the contraction is equal to the diameter of the graph. We prove that for expander graphs, such embedding is optimal up to a multiplicative constant independent of  $n$ .

**Lemma 13.6.** *The graph  $H = (V, E')$  on the same vertex set as  $G$ , where two vertices are adjacent if their distance in  $G$  is at least  $\lfloor \log_k n \rfloor$ , has a perfect matching when  $n$  is even.*

*Proof.* Since  $G$  is a  $k$ -regular graph then every vertex has at most  $k^r$  vertices within distance  $r$ . If  $r = \lfloor \log_k n \rfloor - 1$  then there at most  $n/2$  vertices within distance  $r$ , and therefore  $H$  has minimal degree at least  $n/2$ . Therefore, it has a Hamiltonian cycle by Dirac's Theorem, (E.g., Theorem 10.1.1 in [Die97]). It follows that  $H$  has a perfect matching, as claimed.  $\square$

**Theorem 13.7 (Linial-London-Rabinovich [LLR95]).** *Let  $G = (V, E)$  be a  $k$ -regular graph of even size  $n$ , with  $\lambda_2 \leq k - \epsilon$ . Then  $C_2(G) = \Omega(\log n)$ .*

*Proof.* Let  $B$  be the adjacency matrix of a perfect matching in  $H$ , whose existence is guaranteed by Lemma 13.6. Let  $P = kI - A_G + \frac{\epsilon}{2}(B - I)$ . It is easy to verify that  $P\vec{1} = 0$ . To check that  $P$  is PSD, it suffices to consider the form  $x^T P x$  for some vector  $x \perp \vec{1}$ . Then

$$\begin{aligned} x^T(kI - A_G)x &\geq (k - \lambda_2)\|x\|^2 \geq \epsilon\|x\|^2, \\ x^T(B - I)x &= \sum_{(i,j) \in B} (2x_i x_j - x_i^2 - x_j^2) \geq -2 \sum_{(i,j) \in B} (x_i^2 + x_j^2) = 2\|x\|^2. \end{aligned}$$

Therefore,  $x^T P x = x^T(kI - A_G)x + x^T(\epsilon/2)(B - I)x \geq 0$ . To obtain the lower bound on  $C_2(G)$  we evaluate:

$$\begin{aligned} -\sum_{p_{ij} < 0} d(i, j)^2 p_{ij} &= kn \\ \sum_{p_{ij} > 0} d(i, j)^2 p_{ij} &\geq \frac{\epsilon}{2} \cdot n \lfloor \log_k n \rfloor^2, \end{aligned}$$

where the inequality follows since distances of edges in  $B$  are at least  $\lfloor \log_k n \rfloor$ . This implies that  $C_2(G) = \Omega(\log n)$ , as claimed.  $\square$

<sup>1</sup>We use  $k$  for the degree in this chapter as  $d$  is reserved for the distance

## 13.4 Distortion bounds via the spectral gap

In some cases, a simpler inequality already gives tight bounds on the distortion of graph metrics. We mention it since it directly follows the Rayleigh quotient definition of the spectral gap of a graph, which is a key player here. It suffices to generate the bounds above for the cube and expanders, but instances in which it fails to give tight bounds are far more interesting for the algorithmic questions below.

**Theorem 13.8.** *Let  $G$  be any  $k$ -regular graph and  $\lambda_2$  its 2nd eigenvalue. Then for every embedding  $f : V \rightarrow \mathbb{R}^n$  we have*

$$\mathbb{E}_{(u,v) \in V \times V} \|f(u) - f(v)\|^2 \leq k/(k - \lambda_2) \mathbb{E}_{(u,v) \in V \times V} \|f(u) - f(v)\|^2$$

The proof follows directly three observations - that it is sufficient to prove it for real functions  $f$  (as it extends to any dimension coordinate-wise), that it is sufficient to consider functions  $f$  orthogonal to uniform (as shifting  $f$  by constant does not affect the inequality), and that now it is the variational definition of the second eigenvalue.

The usefulness of this theorem comes in when we have a good bound on the average distance (say  $D$ ) between a random pair of points in our graph. Now take an optimal embedding  $f$  that is non-expanding, and has contraction at most  $C_2(G) = \gamma$ . The LHS is lower bounded by  $(D/\gamma)2$ , while the RHS is upper bounded by  $k/(k - \lambda_2)$ . This gives  $C_2(G) \geq D\sqrt{(k - \lambda_2)/k}$ .

In the Boolean cube  $Q_r$ , using  $\lambda_2 = r - 2$  (see Chapter 12 and average distance  $D = \Omega(r)$ ) one derives the first bound above (at least up to a constant). For expanders, one easily sees that the average distance is  $\Omega(\log n)$ , obtaining the second bound.

## 13.5 Algorithms for cut problems via embeddings

Let us consider the following natural computational problem: Given an  $n$ -vertex graph  $G$ , compute, or estimate its expansion ratio  $h(G)$ . We mention that there are many variants, like balanced cut (estimating the number of edges to separate the graph into roughly equal pieces), and others. These arise naturally as subroutines in many graph algorithms, as part of a "divide-and-conquer" approach, where small cuts guarantee smaller interference between sub-solutions in the separate pieces.

It has been known in a long time that the exact determination of  $h(G)$  is difficult (co-NP hard) [BKV<sup>+</sup>81] (and such is the case for most related cut problems). How well it can be approximated in polynomial time is still an open and fascinating problem. A first breakthrough was achieved by Leighton and Rao [LR99] who presented a polynomial time algorithm that approximates  $h(G)$  to within an  $O(\log n)$  factor for an  $n$ -vertex graph.

The connection of this problem to metric embeddings was made in [LLR95], where an alternative proof for this result was given. We turn to explain this, by which way we'll see how low-distortion embeddings of metric spaces come together with this set of algorithmic problems. Before we do that, we should mention a recent breakthrough [ARV04] which gives a polynomial-time approximation which computes  $h(G)$  up to a factor of only  $O(\sqrt{\log n})$ . The question of whether an efficient algorithm exists for a constant factor approximation is fascinating. Some indications that this may be too optimistic were given in [CKK<sup>+</sup>05]. We won't be able to review here any of these exciting recent news.

As mentioned above, there is a host of important and mostly difficult optimization problems where the input is a graph  $G = (V, E)$  and where the objective is to find a subset  $S \subseteq V$  so as to optimize some quantity related to the cut  $E_G(S, \bar{S})$ . Up to a small factor, the question of finding  $h(G)$  falls into this category, since it is easily verified that:

$$2 \frac{h(G)}{n} \geq \min \frac{|E_G(S, \bar{S})|}{|S|(n - |S|)} \geq \frac{h(G)}{n}$$

where the minimum is over all subsets  $S \subseteq V$ . So let us concentrate on the problem of finding  $\min \frac{|E_G(S, \bar{S})|}{|S|(n - |S|)}$ . It is closely related to another classical computational problem called the *all-pairs multicommodity flow* problem, which we turn to define now.

The input to this problem is an  $n$ -vertex graph  $G = (V, E)$ . Between every pair of vertices we should "ship"  $\delta > 0$  units of a commodity, and we should maximize  $\delta$  subject to the following constraints:

- There are  $\binom{n}{2}$  distinct and unexchangeable commodities, one per each pair of vertices in  $G$ .

- Edge capacities are 1. Namely, it is possible to ship different commodities through each edge as long as their total amount does not exceed 1.
- The flow of each commodity satisfies preservation of matter at each vertex.

Again, the problem is to find the largest  $\delta > 0$  for which this can be done.

It is fairly easy to show that for every  $S \subseteq V$ ,

$$\delta \leq \frac{|E_G(S, \bar{S})|}{|S|(n - |S|)}.$$

The main claim, a proof of which we sketch below is that, in fact,

$$\frac{|E_G(S, \bar{S})|}{|S|(n - |S|)} \leq O(\delta \cdot \log n). \quad (13.3)$$

The all-pairs multicommodity flow problem is a linear programming problem and can, therefore, be solved in polynomial time. Consequently, we achieve a polynomial-time algorithm that approximate  $h(G)$  to within  $O(\log n)$ .

Now using linear programming duality, it is not hard to show that the optimum of the all-pairs multicommodity flow problem equals

$$\max \delta = \min \frac{\sum_{(i,j) \in E} d_{i,j}}{\sum_{i,j \in V} d_{i,j}}. \quad (13.4)$$

Here  $d$  is any metric on the set  $V$ .

To prove Inequality 13.3 we note first that every metric that's realizable in  $l_2$  is realizable in  $l_1$  as well.<sup>2</sup> In fact, Bourgain's original proof maps  $(X, d)$  into  $l_1$  and it is an easy observation that the same mapping works for  $l_2$  just as well. Now a simple property of  $l_1$  metrics is that they form a polyhedral convex cone. Namely, if  $d_1$  and  $d_2$  are two  $l_1$  metrics on the same finite set  $X$ , then so is  $a_1 d_1 + a_2 d_2$  for every two nonnegative constants  $a_1, a_2$ . This cone is called the *cut cone* for reasons which we will soon see. There is a whole book dedicated to the basic properties of this cone and its relatives [DL97].

The extreme rays of the cut cone are easy to determine: Associated with any proper  $S \subseteq [n]$  is a metric  $d_S$  on  $[n]$  as follows:  $d_S(x, y) = 1$  if exactly one of  $x, y$  is in  $S$  and 0 otherwise. This is the so-called *the cut metric associated with  $S$*  which is easily seen to be an  $l_1$  metric. We can now approach Inequality 13.3 as follows: By Bourgain's Theorem only a factor of  $O(\log n)$  is lost if in minimizing the r.h.s. in Equation 13.4 we insist that the metric be in  $l_1$ . Since the cut cone is convex, the optimum among  $l_1$  metrics is obtained at an extreme ray, i.e.,<sup>3</sup> by a cut metric  $d_S$ . In this case the r.h.s. in Equation 13.4 becomes  $\frac{|E_G(S, \bar{S})|}{|S|(n - |S|)}$ , as claimed.

This type of argument seems very alluring from a computational perspective. Consider any optimization problem where we are given a graph  $G = (V, E)$  and seek a cut  $E_G(S, \bar{S})$  which is optimal in some sense. By the same reasoning, such a problem can be turned into a problem of convex optimization, in which we are trying to optimize some objective function on the cut cone. There is a general theory of discrete optimization as laid out in [GLS93], in which we try to optimize some linear function over a convex domain  $\Omega$ . In order to use this machinery we must be able to efficiently solve two basic questions for  $\Omega$ : (i) *Membership* - To determine, given a point  $x$  whether it belongs to  $\Omega$ , and (ii) *Separation* - Same as above, but if  $x \notin \Omega$ , find a hyperplane that separates  $x$  from  $\Omega$ . Unfortunately, the cut cone is computationally quite bad. Even the membership problem (which is the simpler of the two) for the cut cone is NP-hard. In words: Given a metric, it is difficult to decide whether it is an  $l_1$  metric.

This suggests the following natural problem first raised independently by Linial and by Goemans:

**Open problem 13.9.** *Is there a cone that is a good (preferably only constant distortion) approximation to the cut cone and for which membership and separation can be solved in time polynomial in the dimension?*

<sup>2</sup>This is another beautiful issue we cannot develop here and only mention its connections with Dvoretzky's Theorem (e.g. [Mat02]). The (simple) proof of this fact can be found e.g. in the book [DL97].

<sup>3</sup>There is a small technicality which we neglect here: We actually intersect the cone with some hyperplane, so as to do the optimization on some bounded domain.

They have also suggested a candidate for the this job. We say that a metric  $(X, d)$  is of *negative type* if  $\sqrt{d}$  is an  $l_2$  metric. It is easy to show that metrics of negative type form a convex cone for which both the membership and separation problems can be solved efficiently. Also, every  $l_1$  metric is of negative type. In this light, Linial and Goemans have posed the following conjecture, a positive answer to which would realize the above-mentioned program.

**Conjecture 13.10.** *Every metric of negative type can be mapped to an  $l_1$  metric with bounded distortion.*

In a fascinating and still quite mysterious paper, Khot and Vishnoy [KV05] have recently refuted this conjecture. However the lower bound they derive is very small  $(\log \log n)^c$  for some  $1 > c > 0$  for  $n$ -point metrics. There is clearly still a good story here waiting to unfold.

## 13.6 A glimpse into the bigger picture

As mentioned above, the previous sections give only a very brief summary of a fascinating and highly active area of research. For general reviews of this area see [Lin02], chapter in the book [Mat02], and [IM04]. The quadratic programming method presented here has been used in several additional cases [LMN02, LM00]. Progress in this area has been extremely fast and already these fairly recent surveys are by now outdated. To keep track of the present state of affairs, the reader is encouraged to look up a list of open problems in this area that is being curated by J. Matousek: <http://kam.mff.cuni.cz/matousek/>.

We also mention briefly another connection of (finite) expanders to metric embeddings of (infinite) graphs. There are conjectures in topology, most notably the Baum-Connes Conjecture [Val03], which address the embeddability of manifolds into Hilbert space with "bounded" distortion (a term which has to be properly defined). Some extensions of these conjectures were recently disproved by Gromov [Gro03] in an ingenious way which spirit is the following. The "hard-to-embed" manifolds will be (like in the finite setting) derived from expanders. The manifold is defined via their "fundamental group", an infinite group whose Cayley graph carries the metric structure of the manifold. The definition of this group via generators and relations guarantees that the above Cayley graph contains in it larger and larger finite expanders, which (as we already know) do not embed well into  $l_2$ . Carrying out this plan is highly nontrivial, and we refer the reader to [Val03].



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