Universal Traversal Sequences for Expander Graphs

Shlomo Hoory
Avi Wigderson

Hebrew University, Jerusalem

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1 Introduction

Graph reachability is a key problem in the study of various logarithmic space complexity classes. Its version for directed graphs is logspace complete for $NSPACE(\log n)$, and hence if proved to be in $DSPACE(\log n)$, the open question $DSPACE(\log n) = NSPACE(\log n)$ will be settled. Seemingly the problem is easier for undirected graphs. In [1] it was shown to be in RLP (1-sided error, logspace, polynomial expected time). Recently it was shown by [3] to be in ZPLP (no-error, logspace, polynomial expected time).

In [1] it was also proved that polynomial length universal traversal sequence exist. However the explicit construction of such a sequence is somewhat more difficult. Recently [7] gave such an explicit construction of length $O(n^{\log n})$. This implies that the undirected reachability problem can be solved in $DSPACE(\log^2 n)$ which is the same as the upper bound given by Savitch [8] when showing that $NSPACE(\log n) \subseteq DSPACE(\log^2 n)$. It is not known if Savitch's upper bound is tight, and therefore it would be very interesting if shorter universal sequences could be explicitly constructed. Recently an explicit polynomial universal sequence for the cycle was constructed by [5].

In this paper we give an explicit construction of a polynomial length universal sequence for a subclass of constant degree expander graphs. The extra requirement is that the labeling of the edges be consistent, i.e. that for each vertex not only the outgoing labels be distinct, but the incoming labels as well. Consistent labeling is an intermediate requirment between the weeker unrestricted labeling and the stronger requirment of symmetric labeling (where both labels on each edge are identical). For any of them, obtaining universal traversal sequences in uniform logspace will put undirected reachability in $DSPACE(\log n)$. For expanders we don't know how to handle unrestricted labling, while the symmetric case is trivial (since the diameter is small). Here we handle the intermediate case of consistent labling.

2 Definitions and Notations

- A (d, n) graph is a connected undirected d-regular graph on n vertices - we may sometimes think of each undirected edge as a pair of directed edges.
- A (d, n) graph is *labeled* if every directed edge (u, v) in the graph is assigned a label $l(u, v) \in \Sigma$ where $\Sigma = \{1, 2, ..., d\}$, so that for every vertex v all the outgoing edges are assigned different labels.
- A (d, n) graph is *labeled consistently*, if it is a labeled (d, n) graph and for every vertex v all the incoming edges are assigned different labels.
- For a graph $G = (V, E)$ and $A, B \subseteq V$ define

  $E(A, B) = \{(a, b) \in E : a \in A \text{ and } b \in B\}$.
- A (d, n) graph is a *c-expander* if for every subset $U \subseteq V$ $|E(U, V-U)| \geq c|U|(n - |U|)/n$ (compare with [4]). Note that for $d \geq 3$ there is a $c = c(d)$ such that all but an exponentially small fraction of the (d, n) graphs are c-expanders.

In the following definitions $G = (V, E)$ is a labeled (d, n) graph with a distinguished vertex v_0.

- For $s \in \Sigma^*$ and $v \in V$

  Define $s(v)$ as the vertex, reached by a walk starting from v, following the labeled edges according to the instructions given by s.

- For $s \in \Sigma^*$ and $U \subseteq V$ define $s(U) = \{s(v) : v \in U\}$.

- $s \in \Sigma^*$ is a *traversal sequence* for G, if a walk starting from any $v \in V$, following the instructions of s, will visit all the vertices in V.

  s is a *universal traversal sequence* for a set of (d, n) labeled graphs $\mathcal{G}$, if it is a traversal sequence for any $G \in \mathcal{G}$.

- For $s \in \Sigma^*$ define $R(s) = \{v \in V : s \text{ started from } v \text{ will visit } v_0\}$

- For $S \subseteq \Sigma^*$ define $r(S) = \max_{s \in S} |R(s)|$.

- A set of strings $S \subseteq \Sigma^*$ is *k-good* if $r(S) \geq k$.

3 Construction of the Universal Sequence

Theorem 3.1 *There is an explicitly constructible universal traversal sequence of length $n^{O((d \log d)/c)}$ for all the consistently labeled (d, n) c-expander graphs.*

The construction of the universal sequence borrows its basic structure from the [6] construction for cliques.

Lemma 3.1 *If $S \subseteq \Sigma^*$ is n-good for the pair (G, v_0), for every $G \in \mathcal{G}$ and every v_0 in G, then the string $\bar{s} \in \Sigma^*$ obtained by concatenating all the strings in S in any order, is a universal traversal sequence for $\mathcal{G}$.*

Proof:

Let $S \subseteq \Sigma^*$, $\bar{s} \in \Sigma^*$ be as in the lemma. Then because S is n-good for all the graphs in $\mathcal{G}$, then for every graph $G \in \mathcal{G}$ and any starting vertex, the distinguished vertex v_0 will be reached. Since any vertex could be the distinguished one, every vertex will be reached, i.e. $\bar{s}$ is a universal traversal sequence for $\mathcal{G}$. ▮

We will construct a sequence of sets $S_i \subseteq \Sigma^*$:

$S_0 = \{\epsilon\}$, $S_{i+1} = \{utu : u \in S_i, t \in \Sigma \cup \{\epsilon\}\}$.

Let G be a fixed consistently labeled (d, n) c-expander. Let $r_i = r(S_i)$, which is the largest number of vertices starting from which a single sequence from S_i will visit v_0.

Then the sequence $\{r_i\}$ starts at $r_0 = 1$. The next lemma will show that it grows quickly to n.

Lemma 3.2 *$r_{i+1} \geq r_i + \delta$, where $\delta = c\frac{r_i(n-r_i)}{2dn}$, $i \geq 0$*

Proof:

Fix $i \geq 0$, and choose $s \in S_i$ such that $r_i = |R(s)|$. Recall that $R(s)$ are the vertices from which s visits v_0.

Also since G is consistently labeled $|s(V - R(s))| = n - r_i$.

Intuitively, we want that an application of s or sl for some label l will lead many vertices from $V - R(s)$ to $R(s)$.

Formally, define $\alpha = |s(V - R(s)) \cap R(s)|$. Then one of the following two possibilities is true :

1. $\alpha \geq \delta$. Then since $ss \in S_{i+1}$:

   $r_{i+1} \geq |R(ss)| \geq r_i + \delta$.

2. $\alpha < \delta$. Then:

   $|E(s(V - R(s)), R(s))| \geq |E(V - R(s), R(s))| - d\alpha$

   using the expansion property: $|E(V - R(s), R(s))| \geq 2d\delta$

we get $|E(s(V-R(s)),R(s))| \geq 2d\delta - d\alpha > d\delta$.

Since there has to be a label $l \in \Sigma$ that labels at least $1/d$ of the edges in $E(s(V-R(s)),R(s))$ we have

$r_{i+1} \geq |R(sls)| > r_i + \delta$. ▮

The lemma above gives a recurrence that establishes the growth rate of r_i. The number of steps it takes to grow to n is determined next.

Lemma 3.3 *Let $k = 2\lceil \log(n/2)/\log(1+c/4d) \rceil$. Then $r_k > n-1$ (and by integrality $r_k = n$).*

Proof:

Let $p(i) = r_i/n$, the fraction of "good" vertices, and $q(i) = 1 - p(i)$ the fraction of "bad" ones. Let $c' = c/(2d)$. Then $p(0) = 1/n$, and by lemma 3.2

$$p(i+1) \geq p(i) + c'p(i)(1-p(i)) = p(i)(1 + c' - c'p(i))$$

We separately analyze the number of steps till $p(i)$ exceeds $1/2$, and then the number of steps to reach 1.

1. if $p(i) < 1/2$ then:
   $p(i+1) \geq p(i)(1 + c'/2)$
2. if $p(i) \geq 1/2$ then:
   $q(i+1) \leq 1-(1-q(i))(1+c'q(i)) = c'q(i)^2 + q(i)(1-c') \leq q(i)(1-c'/2)$

therefore $q(\lceil \log_{1+c'/2}(n/2) \rceil + \lceil \log_{1/(1-c'/2)}(n/2) \rceil) \leq 1/n$.

Since $1/(1-c'/2) > 1 + c'/2$ we have

$q(2\lceil \log(n/2)/\log(1+c'/2) \rceil) < 1/n$ ▮

Proof: (theorem 3.1)

In lemma 3.3 we have shown that for every consistently labeled (d, n) graph S_k is n-good when $k = 2\lceil \log(n/2)/\log(1 + c/4d)\rceil$. By lemma 3.1 the sequence $\bar{s}$ obtained by the concatenation of all the strings in S_k is a universal traversal sequence for all the consistently labeled (d, n) graphs.

To show that $\bar{s}$ is short, just multiply the length of each string in S_k by the cardinality of this set:

$$|\bar{s}| \leq (2^k - 1)(d + 1)^k \leq (3d)^{2\lceil \log(n/2)/\log(1+c/4d)\rceil} = n^{O((d \log d)/c)}.$$

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