

Monotone Circuits for Matching Require Linear Depth

Ran Raz

Avi Wigderson *

The Hebrew University

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Abstract

We prove that monotone circuits computing the perfect matching function on n -vertex graphs require $\Omega(n)$ depth. This implies an exponential gap between the depth of monotone and nonmonotone circuits.

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1 Introduction

In 1985 Razborov [R1] proved a superpolynomial lower bound on the **size** of monotone circuits computing the perfect matching function. As matching is in P , this result first showed a superpolynomial gap between the size of monotone and non-monotone circuits. This gap was shown to be exponential by Tardos [T] (with a different monotone function in P).

Another exponential gap for size, when the circuits are restricted to have constant depth, was proved by Ajtai and Gurevich [AG].

Karchmer and Wigderson [KW] studied the **depth** of boolean circuits, and characterized it in communication complexity terms. This approach was used in proving an $\Omega((\log n)^2)$ lower bound on the depth of monotone circuits computing st -connectivity. No nontrivial gap was known between the depth of monotone and non-monotone circuits.

Our main result is that every monotone circuit that decides if an n node graph has a matching of size $n/3$ must have depth $\Omega(n)$. The proof uses

the communication complexity approach of [KW]. It consists of a sequence of simple reductions from the probabilistic communication complexity of the set disjointness function, for which a linear lower bound was known [KS].

The same lower bound is derived for the bipartite perfect matching and clique functions.

From the main result we deduce a hierarchy theorem for monotone depth, and an exponential gap between monotone and nonmonotone depth.

The next section contains the necessary definitions. In section 3 we prove the main result, and in section 4 we describe some of its consequences.

2 Preliminaries

Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a monotone boolean function. We consider here boolean circuits for f over the boolean $\{\wedge, \vee, \neg\}$ and monotone $\{\wedge, \vee\}$ bases. All gates have fanin 2. Negations, if any, occur only at the inputs.

Let $d(f)$ (resp. $d^m(f)$) denote the minimal depth of any circuit (resp. monotone circuit) computing f .

Let X, Y, Z be finite sets, and let $R \subseteq X \times Y \times Z$ be a relation. A *deterministic communication protocol* A over (X, Y, Z) specifies the exchange

of information bits by two players, I and II , that initially receive as inputs $x \in X$ and $y \in Y$, respectively, and finally agree on a value $A(x, y) \in Z$. Denote by $c_A(x, y)$ the number of bits exchanged by I and II on the input pair (x, y) when using protocol A . Let $c_A(R) = \max_{(x,y) \in X \times Y} c_A(x, y)$.

We say that A *computes* R if for all $(x, y) \in X \times Y$ we have $(x, y, A(x, y)) \in R$. Then the *deterministic communication complexity* of the relation R is $c(R) = \min\{c_A(R) \mid A \text{ computes } R\}$.

We accommodate the more common notion of computing functions (rather than relations) in a natural way. If R has the property that for every (x, y) there is a unique $z(x, y) \in Z$ with $(x, y, z) \in R$, then we identify the relation R with the function $R : X \times Y \rightarrow Z$ where $R(x, y) = z(x, y)$.

A *probabilistic protocol* A is simply a probability distribution over deterministic protocols. Equivalently, we can think of A as a deterministic protocol in which the two players I and II *share* an (infinite) random string. Then $c_A(x, y)$ is the *expected* number of bits exchanged by I and II on input pair (x, y) . As before, $c_A(R) = \max_{(x,y) \in X \times Y} c_A(x, y)$.

A probabilistic protocol A ϵ -*computes* the relation R if for all $(x, y) \in X \times Y$, $\text{Prob}[(x, y, A(x, y)) \in R] \geq 1 - \epsilon$. The ϵ -*error probabilistic communication complexity* of R is thus defined by $c_\epsilon(R) = \min\{c_A(R) \mid A \epsilon\text{-computes } R\}$.

A trivial observation that we shall use is:

Fact 2.1 *If a relation R' is a restriction of a relation R to a subset $X' \subset X$, $Y' \subset Y$, then $c(R') \leq c(R)$, and for every ϵ , $c_\epsilon(R') \leq c_\epsilon(R)$.*

Our starting point for proving a monotone depth lower bound is the relationship, discovered by [KW], between circuit depth of a Boolean function and the communication complexity of an associated relation. Specifically, let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a monotone Boolean function. Recall that a minterm (resp. maxterm) of f is a minimal subset of $[n]$, such that setting the associated variables to 1 (resp. 0) forces the function f to take the value 1 (resp. 0). Let $MIN(f)$, $MAX(f)$ denote its sets of minterms and maxterms, respectively. Let the relation $R_f^m \subseteq MIN(f) \times MAX(f) \times [n]$ be defined by $(p, q, i) \in R_f^m \iff i \in p \cap q$. (In words, player I gets a minterm, player II gets a maxterm, and their task is to find a member of the intersection of these two sets, which always exists). Then

Theorem 2.1 (KW)

$$d^m(f) = c(R_f^m)$$

3 The Lower Bound

All functions and relations defined hereafter will be parametrized by the integer m . Let N be a set of vertices, $|N| = n = 3m$. Let $MATCH$ be the monotone function that on input graph $G = (N, E)$, $MATCH(G) = 1$ iff G has a matching of m edges. Our goal is to prove:

Main Theorem:

$$d^m(MATCH) = \Omega(n)$$

Let $DISJ : \{0, 1\}^m \times \{0, 1\}^m \rightarrow \{0, 1\}$ be the disjointness function, i.e. $DISJ(x, y) = 1$ iff $\forall i \in [m] x_i \wedge y_i = 0$. It was first proved by [KS] (and simplified in [R3]) that even probabilistic protocols for $DISJ$ require a linear number of bits.

Theorem 3.1 [KS]

$$c_{\frac{1}{3}}(DISJ) = \Omega(m)$$

We shall use a sequence of deterministic and probabilistic reductions to derive $d^m(MATCH) \geq c_{\frac{1}{3}}(DISJ)$, from which the Main Theorem follows.

Let P_N be the set of all m -matchings over N . (Each member of P_N is a set of m disjoint pairs (edges) from N). Let \hat{Q}_N be the set of all $(m-1)$ -subsets

of N . Let E_N denote of all pairs (edges) from N . Finally define the relation $\hat{M} \subset P_N \times \hat{Q}_N \times E_N$ by $(p, \hat{q}, e) \in \hat{M}$ iff both $e \in p$ and $e \cap \hat{q} = \emptyset$.

Proposition 3.1

$$c(\hat{M}) \leq d^m(MATCH)$$

Proof: Note that P_N is the set of minterms of $MATCH$. Also, $\hat{q} \in \hat{Q}_N$, interpreted as a clique on the nodes $(N \setminus \hat{q})$ is a maxterm of $MATCH$. Finally, an edge e such that $e \in p$ and $e \cap \hat{q} = \emptyset$ is in the intersection of this minterm-maxterm pair. Hence \hat{M} is a restriction of the relation R_{MATCH}^m , and the proposition follows from Theorem 2.1 and Fact 2.1.



We will now define a decision problem (function) M related to the search problem (relation) \hat{M} . Let Q_N be the set of all m -subsets of N . Let $M : P_N \times Q_N \rightarrow \{0, 1\}$ be the function defined by $M(p, q) = 1$ iff there exists no edge $e \in p$ such that $e \cap q = \emptyset$. Before stating the formal relationship between M and \hat{M} we need some more notation.

For $j \in \{0, 1, 2\}$, $p \in P_N, q \in Q_N$, let $\alpha_j(p, q) = |\{e \in p : |e \cap q| = j\}|$.

For $0 \leq i \leq m$ define

$W_i = \{(p, q) \in P_N \times Q_N \mid \alpha_0(p, q) = i, \alpha_1(p, q) = m - i, (\alpha_2(p, q) = 0)\}$. Now

M can be defined equivalently by $M(p, q) = 1$ iff $(p, q) \in W_0$.

We say that a (probabilistic) communication protocol A is *good on average for M* if it has the following two properties: (which informally state that A is always correct on W_0 , and is correct on half the inputs in each $W_i, i \geq 1$)

1. $Prob_{W_0}[A(p, q) = M(p, q) = 1] = 1$
2. For every $i, 1 \leq i \leq m, Prob_{W_i}[A(p, q) = M(p, q) = 0] \geq \frac{1}{2}$

Here $Prob_{W_i}[A(p, q) = \dots] = E[Prob[A(p, q) = \dots]]$ where the expectation E is over the pairs $(p, q) \in W_i$ with the uniform distribution.

Let $\tilde{c}(M) = \min\{c_A(M) \mid A \text{ is good on average for } M\}$.

Proposition 3.2

$$\tilde{c}(M) \leq c(\hat{M}) + 1$$

Proof: We use an optimal deterministic protocol A for \hat{M} to construct a probabilistic protocol B for M that is good on average and has essentially the same complexity.

Let $(p, q) \in P_N \times Q_N$ be an input of M . Recall that the players wish to determine if $(p, q) \in W_0$. The player holding q removes one point $u \in q$ at random, resulting in an $(m - 1)$ -subset \hat{q} . The players use the optimal

protocol A on (p, \hat{q}) to find an edge $e = A(p, \hat{q})$ such that $e \in p$ and $e \cap \hat{q} = \emptyset$.

Then they compute $B(p, q) = 1$ if $u \in e$ and $B(p, q) = 0$ otherwise.

Clearly, if $(p, q) \in W_0$, then $B(p, q) = 1$ (correctly). Consider now an input $(p, q) \in W_1$. (For $W_i, i > 1$ the proof is similar). If B is lucky enough to remove the unique "free" point in q (which misses all edges of p), then it answers '0' correctly.

Otherwise, in the resulting (p, \hat{q}) there are exactly two edges $e_1, e_2 \in p$ that are disjoint from \hat{q} . Exactly one of them is (consistently) returned by algorithm A (which is deterministic).

There are four possible (p, q) that could result in this (p, \hat{q}) , namely when the removed point u is an endpoint of either e_1 or e_2 . Then clearly in exactly two of these four cases B answers '0' (correctly).



We will now define the 3-letter distinctness function $DIST$, that somewhat resembles the disjointness function $DISJ$. Let $V = \{a, b, c\}^m$. $DIST : V \times V \rightarrow \{0, 1\}$ is defined by $DIST(u, v) = 1$ iff

$\forall i \in [m] u_i \neq v_i$.

Proposition 3.3

$$c_{\frac{1}{3}}(DIST) \leq \tilde{c}(M)$$

Proof: For $u, v \in V$ we define $\beta(u, v) = |\{k \in [m] : u_k = v_k\}|$. For each $i, 0 \leq i \leq m$ let $Z_i = \{(u, v) \in V \times V \mid \beta(u, v) = i\}$. (Note that $DIST$ can be equivalently defined by $DIST(u, v) = 1$ iff $(u, v) \in Z_0$.)

We use an optimal protocol B that is good on average for M , to construct a probabilistic protocol C that $\frac{1}{3}$ -computes $DIST$, and has the same complexity.

The players consider their random string as an encoding of a pair (S, ρ) , all pairs equally likely. $S = (S_1, S_2, \dots, S_m)$ is a partition of the vertex set N into a sequence of triples $S_i, (|S_i| = 3)$.

$\rho = (\rho_1, \rho_2, \dots, \rho_m)$ is a sequence of random 1-1 mappings

$\rho_i : \{a, b, c\} \rightarrow S_i$. For $\gamma \in \{a, b, c\}$ define the "complementary" mapping

$$\bar{\rho}_i(\gamma) = S_i \setminus \{\rho_i(\gamma)\}.$$

Each pair (S, ρ) defines two mappings, $\sigma_I : V \rightarrow P_N, \sigma_{II} : V \rightarrow Q_N$ as follows.

$$\sigma_I(u) = \{\bar{\rho}_1(u_1), \bar{\rho}_2(u_2), \dots, \bar{\rho}_m(u_m)\}$$

$$\sigma_{II}(v) = \{\rho_1(v_1), \rho_2(v_2), \dots, \rho_m(v_m)\}$$

On input pair (u,v) , player I computes $p = \sigma_I(u)$, player II computes $q = \sigma_{II}(v)$ and they use protocol B to compute $B(p,q)$. Now they flip a biased coin with $Prob[H] = \frac{1}{3}$, and decide as follows. If H then $C(u,v) = 0$ else $C(u,v) = B(p,q)$.

Claim: For every $(u,v) \in V \times V$, $Prob[C(u,v) = DIST(u,v)] \geq \frac{2}{3}$

Proof of Claim: Note that $V \times V$ is the disjoint union of the sets Z_i , $0 \leq i \leq m$. We prove the claim separately for each i .

For each $(u,v) \in Z_i$ note that $(p,q) = (\sigma_I(u), \sigma_{II}(v))$ is a uniformly distributed random variable in $W_i \subseteq P_N \times Q_N$. Recall that B is good on average for M , which means that it is always correct on W_0 , and is correct with probability at least $\frac{1}{2}$ for random pairs in each $W_i, i \geq 1$. Hence we have:

$$Prob[C(u,v) = DIST(u,v)] = \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 = \frac{2}{3} \text{ when } (u,v) \in Z_0$$

$$Prob[C(u,v) = DIST(u,v)] \geq \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{3} \text{ when } (u,v) \in Z_i, 1 \leq i$$



Finally we relate the probabilistic communication complexity of the functions $DIST$ and $DISJ$.

Proposition 3.4

$$c_{\frac{1}{3}}(DISJ) \leq c_{\frac{1}{3}}(DIST)$$

Proof: We describe a probabilistic protocol D for $DISJ$ that uses an optimal probabilistic protocol C for $DIST$. Players I and II get respectively inputs x and y from $\{0, 1\}^m$. Define two mappings

$\tau_I, \tau_{II} : \{0, 1\} \rightarrow \{a, b, c\}$ by $\tau_I(0) = a, \tau_{II}(0) = b, \tau_I(1) = \tau_{II}(1) = c$. These mappings naturally extend to $\tau_I, \tau_{II} : \{0, 1\}^m \rightarrow V$.

Player I computes $u = \tau_I(x)$, player II computes $\tau_{II}(y)$, and use protocol C to compute $D(x, y) = C(u, v)$. Clearly, $DISJ(x, y) = DIST(\tau_I(x), \tau_{II}(y))$, so if C ϵ -computes $DIST$ then D ϵ -computes $DISJ$.



4 Consequences

4.1 Lower Bounds on Other Functions

Define three more monotone functions on graphs, with $|N| = n$:

- **PM:** Does G on vertex set N have a perfect matching?
- **BPM:** Does a bipartite G on vertices $N \cup \bar{N}$ have a perfect matching?

- **CL(k)**: Does G on vertex set N have a clique of size $> k$?

Theorem 4.1

$$d^m(PM) = \Omega(n)$$

Proof: Immediate from the Main Theorem by a standard reduction from *MATCH*: simply add m new vertices to the input graph, and connect each of them to all the original vertices. The new graph has a perfect matching iff the input graph had a matching of size m .



Theorem 4.2

$$d^m(BPM) = \Omega(n)$$

Proof: The communication problem \hat{M} on (P_N, \hat{Q}_N) is easily reduced to the relation R_{BPM}^m associated with *BPM* on nodes $N \cup \bar{N}$: Duplicate each vertex (\bar{i} for each original $i \in N$) and put edges (i, \bar{j}) and (j, \bar{i}) for each original (i, j) . The two players transform their inputs accordingly, and any solution (i, \bar{j}) or (j, \bar{i}) to the new problem corresponds to a solution (i, j) for the original problem \hat{M} .

Warning: this is not a general reduction of *PM* to *BPM* (which could not

work!). This reduction works only because the lower bound used just \hat{Q}_N as maxterms (they are not all maxterms).



Theorem 4.3

$$d^m(CL((2n/3) + 1)) = \Omega(n)$$

Proof: A simple reduction from the search problem \hat{M} to the relation R_{CL}^m associated with CL . All we have to observe is that \hat{Q}_N are cliques of size $(2n/3) + 1$, i.e. minterms of CL . Also, P_N is a subset of the maxterms of CL . Hence, a protocol for R_{CL}^m is in fact a protocol for \hat{M} , with the names of the players switched.



As a corollary to Theorem 4.3 we get a depth bound on $CL(k)$, for every k . For large k it is better than the one derived from the size bounds of [R1] and [AB], and the direct bound of [GH].

Corollary 4.1 *For every $k = k(n) \leq n/2$,*

$$d^m(CL(k)) = \Omega(k)$$

Finally, all these lower bounds hold even for the probabilistic communication complexity of the associated boolean relations. As in [RW], let $C_0(R)$ denote the Las Vegas (error free) probabilistic communication complexity of a relation R . Then, for example, we have:

Theorem 4.4 $C_0(R_{BPM}^m) = \Omega(n)$.

(In words, the probabilistic communication complexity of the monotone relation of BPM is $\Omega(n)$).

Similar results hold for the other functions discussed above, even in the Monte Carlo probabilistic model that allows ϵ error probability per input (see [RW]).

4.2 Hierarchy and Gap Theorems

Let $L \subseteq \{0, 1\}^*$ be a language, and let $L_n : \{0, 1\}^n \rightarrow \{0, 1\}$ be the boolean function mapping exactly those strings in L of length n to ‘1’. Say that the language L is *monotone* if for every integer n the function L_n is monotone. For a function g on the integers we define two complexity classes of monotone languages, $MDEPTH(g(n))$ and $MDEPTH^m(g(n))$. A monotone language L is in $MDEPTH(g(n))$ (resp. $MDEPTH^m(g(n))$) iff $d(L_n) = O(g(n))$

(resp. $d^m(L_n) = O(g(n))$). The obvious inclusions between these classes is summarized below.

Fact 4.1 *For every two functions $h(n), g(n)$ with $h(n) \leq g(n)$ we have*

$$MDEPTH^m(h(n)) \subseteq MDEPTH^m(g(n)) \subseteq MDEPTH(g(n)) \subseteq MDEPTH^m(n)$$

Theorem 4.5 (Hierarchy theorem for monotone depth) *For every two functions $g(n) \leq \sqrt{n}$, and $h(n) = o(g(n))$,*

$$MDEPTH^m(h(n)) \not\subseteq MDEPTH^m(g(n)) \cap P$$

Proof: It is easy to see that $d^m(MATCH) = O(m)$, i.e. the lower bound of the main theorem is tight. The number of boolean variables in *MATCH* is $O(m^2)$, so together with the lower bound we get the separation for $g(n) = \sqrt{n}$. For smaller functions $g(n)$ the result follows by padding.



The theorem above shows that there are families of functions on n bits in P that are computable in monotone depth $g(n)$ but not $o(g(n))$ (for $g(n) < \sqrt{n}$). By counting we know that such families exist that separate monotone depth $g(n)$ from $g(n) - 1$. Can such a tight hierarchy be formed by explicit functions, say in P or NP ? Noga Alon [Al] recently used our results to give

an answer which is almost as good - his separating functions are projections of NP functions.

Theorem 4.6 (A1) *There is a family of functions on $n(n+3)$ variables in NP , such that for every $1 \leq g(n) \leq n/1000$ there is a restriction of the n th function whose monotone depth is exactly $g(n)$.*

Proof sketch: The function is the following variant of the clique function: The input is a graph on n vertices, and two integers k and l (where both $n-k$ and $n-l$ are given in unary for monotonicity). The question is whether the graph has a k -clique containing the first (in some fixed ordering) l vertices.

The theorem follows easily from the following claim. For every t , the set of monotone depths of circuits for computing the functions with parameters $1 \leq l, k, n \leq t$ forms an interval of integers between 1 and at least $t/1000$. Clearly, taking $n = t, k = (2n/3) + 1, l = 0$ requires by theorem 4.3 depth at least $t/1000$. Also, the theorem clearly follows from the claim, as every depth in the interval is achieved.

The proof that indeed this set is an interval is by induction on t . For $t \leq 3$ it is easy to check. In moving from $t-1$ to t , proceed by backwards

induction on $k + l$. When $l = k = n = t$, the question is whether the input graph is a clique, which needs depth $\log((n - 1)n/2)$, and one can check that with all depths obtained from smaller values of t it is still an interval. This gives the base case, $l + k = 2n$. When $l + k < 2n$, the cases $k = n$ and $l = k$ are easy, and otherwise one can decrease $l + k$ by removing a vertex. The exact details are left to the reader.



Razborov's lower bound [R2] on the size of monotone circuits for matching proved that for size, nonmonotone circuits are superpolynomially stronger than monotone circuits. Tardos [T] improved this to an exponential gap by considering a function in P related to the clique function. We prove a similar gap for circuit depth.

Theorem 4.7 (Exponential gap between monotone and nonmonotone depth) *For every function $g(n) \leq \sqrt{n}$*

$$MDEPTH^m(g(n)) \not\leq MDEPTH(\log^2 g(n))$$

Proof: Borodin, von zur Gathen and Hopcroft [BGH] observed that a randomized algorithm for matching, proposed by Lovasz [L], can be implemented by shallow circuits, i.e. $d(MATCH) = O(\log^2 m)$. This result with

our lower bound give the separation for $g(n) = \sqrt{n}$. Again, for smaller functions $g(n)$ the result follows by padding.



Corollary 4.2 *There is a monotone function in NC^1 that has no monotone NC circuits.*

Proof: Choose $g(n) = 2^{\sqrt{\log n}}$ in the gap theorem.



4.3 Branching Programs

A monotone branching program over $X = \{x_1, \dots, x_n\}$ is a directed graph H with distinguished nodes s, t , and some arcs labeled with elements from X . An assignment σ to the variables in X defines in a natural way a subgraph H_σ of H . Let $H(\sigma) = 1 \iff H_\sigma$ has an $s - t$ path. Then H defines a monotone boolean function, and for every monotone function f , let $mbp(f)$ be the size of the smallest H computing f .

Theorem 4.8

$$mbp(BPM) = 2^{\Omega(\sqrt{n})}$$

Proof: A monotone branching program of size s can be easily converted into a monotone circuit of f of depth $O((\log s)^2)$.



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