Combinatorial Characterization of Read-Once Formulae

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Abstract

We give an alternative proof to a characterization theorem of Gurvich for Boolean functions whose formula size is exactly the number of variables. Those functions are called Read-Once. We use methods of combinatorial optimization and give as a corollary an alternative proof for some results of Seymour.

1 Introduction

Let X be a finite set (interpreted as Boolean variables). A monotone formula is a rooted tree whose leaves are labeled with members of X, and whose internal nodes are labeled with the Boolean operations AND, OR. The root of the tree computes a monotone Boolean function $f : \{0,1\}^X \mapsto \{0,1\}$ in the natural way.

Let f be a monotone Boolean function. A minimal set $S \subseteq X$ is a *minterm (maxterm)* if setting all variables in S to 1 (0), forces the value of f to 1 (0). Let MIN(f) (MAX(f)) denote the set of all minterms (maxterms).

The monotone formula complexity of a monotone Boolean function f, denoted by $L_m(f)$, is the minimum number of leaves in any formula computing f. *Read-Once Formulae* are

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formulae in which every variable of X appears exactly once. These are, of course, the smallest possible for functions that depend on all their variables. A Boolean function f is *read-once* if it has a read-once formula. A monotone Boolean function $f : \{0,1\}^X \mapsto \{0,1\}$ depends on all its variables if $\bigcup_{S \in MIN(f)} S = X$. We give the following simple characterization of monotone read-once functions, originally proved by Gurvich in 1977 [2], [3].

Theorem 1.1 *A monotone Boolean function f that depends on all its variables is read-once if and only if*
$T \in MAX(f),\ S \in MIN(f) \implies |S \cap T| = 1 \qquad (*)$

We get theorem 1.1 as a corollary of theorem 2.6 in which we state some connections between certain clutters, their blockers and antiblockers and their 'graphs'.

We have recently been informed that the characterization theorem was proved by Gurvich [2] back in 1977 and more recently by Beynon and Paterson as well [1]. (Both cases use somewhat different proofs that does not go through the the eqivalence of 2 and 3 in 2.6). We also note here that the results on the blocker antiblocker relation (in theorem 2) can be deduced from the work of P. Seymour [6], [7], studying binary clutters, cuts and paths of serial-parallel graphs, with different methods.

We will need some definitions and notations.

2 definitions and Main Theorem

Let $f : \{0,1\}^X \mapsto \{0,1\}$ be a Boolean function, $|X| = n$. We identify the arguments of f with subsets of X in the natural way. ($f(S) = f(x_1, ..., x_n)$, where $x_i = 1$ if $x_i \in S$ and $x_i = 0$ otherwise, $1 \le i \le n$).

Let $\mathcal{C} \subseteq 2^X$ be an antichain of sets, $\mathcal{C}$ is called *a clutter*. Note, by the definition of $MAX(f),\ MIN(f)$, both these families are clutters.

Let $\mathcal{C}$ be a clutter, we will denote by $V(\mathcal{C}) = \bigcup_{S \in \mathcal{C}} S$, that is, the set of all elements of X that actually appear in $\mathcal{C}$.

Let $G = (V, E)$ be a graph, G^C will denote the graph which is the complement of G, i.e: $V(G^C) = V,\ \ E(G^C) = \{(u,v)|\ \ (u,v) \notin E\}$.

A P_4 is the simple path with 4 vertices and 3 edges. We say that a graph $G = (V, E)$ is P_4- free if it has no induced subgraph isomorphic to P_4.

Definition 2.1 *For a clutter $\mathcal{C}$ on a set of points X. define:*

1. *The blocker of $\mathcal{C}$, $\mathcal{C}^B$, is a clutter defined by*

$$\mathcal{C}^B = \{S \subseteq X|\ \forall T \in \mathcal{C}\ |S \cap T| \ge 1, \quad \textit{and } S \textit{ is minimal}\}$$

2. *The anti-blocker of $\mathcal{C}$, $\mathcal{C}^A$, is a clutter defined by*

$$\mathcal{C}^A = \{S \subseteq V(\mathcal{C}) | \ \forall T \in \mathcal{C} \ |S \cap T| \leq 1, \quad \textit{and } S \textit{ is maximal}\}$$

Definition 2.2 *For a clutter $\mathcal{C}$ on a set of points X, define a Boolean function $f_{\mathcal{C}} : \{0,1\}^X \mapsto \{0,1\}$ by $f(S) = 1$ if for some $T \subseteq S, \ T \in \mathcal{C}$.*

In fact $f_{\mathcal{C}}$ is a monotone Boolean function, (write $f(x_1, ..x_n) = \bigvee_{S \in \mathcal{C}} \bigwedge_{x_i \in S} x_i$), and $\mathcal{C}$ is $MIN(f_{\mathcal{C}})$. (Unless $V(\mathcal{C}) = \phi$).

Lemma 2.3 *[4] Let $\mathcal{C} \subseteq 2^X$ be a clutter, then $(\mathcal{C}^B)^B \ = \ \mathcal{C}$.*

Proof Easy and appears in [4]. □

Claim 2.4 *If f is a monotone Boolean function with $MIN(f) = \mathcal{C}$ then $\mathcal{C}^B = MAX(f)$.*

Proof: Follows directly from the definitions.

Definition 2.5 *For a clutter $\mathcal{C} \subseteq 2^X$, define the graph $G(\mathcal{C}) = (V(\mathcal{C}), E)$ where $(u, v) \in E$ if there is some $T \in \mathcal{C}$, with $u, v \in T$.*

We can state now our main theorem;

Theorem 2.6 *Let $f : \{0,1\}^X \mapsto \{0,1\}$ be a monotone Boolean function with $\mathcal{C} = MIN(f)$ and $X = V(\mathcal{C}) \neq \phi$, then the following conditions are equivalent:*

1. *f is read-once.*
2. *$\mathcal{C}^B \subseteq \mathcal{C}^A$.*
3. *$\mathcal{C}^B = \mathcal{C}^A$*
4. *$\mathcal{C}$ is the set of maximal cliques of $G(\mathcal{C})$ and, for any induced subgraph $G' \subseteq G(\mathcal{C})$, every maximal clique intersects every maximal independent set.*
5. *$\mathcal{C}$ is the set of maximal cliques of $G(\mathcal{C})$ and, $G(\mathcal{C})$ is P_4- free.*

We will need the following two definitions,

Definition 2.7 *For a clutter $\mathcal{C}$ on a set of points X and $W \subset X$;*

1. *The deletion of W from $\mathcal{C}$ is the clutter*

$$\mathcal{C} \setminus W \ = \ \{S \in \mathcal{C} | \ S \cap W = \phi \ \}$$

2. *The contraction of $\mathcal{C}$ by W is the clutter $\mathcal{C}/W$, obtained by taking all minimal sets of $\mathcal{C}(W)$ where $\mathcal{C}(W) \ = \ \{S - W \mid S \in \mathcal{C}\}$.*

3 Proof of the main Theorem

Lemma 3.1 *($1 \to 2$)*

Let f be a monotone read-once Boolean function on a variable set X, with $MIN(f) = \mathcal{C}$ then $\mathcal{C}^B \subseteq \mathcal{C}^A$.

Proof: We prove that f satisfies condition 2 by induction on the size of the variable set $|X|$. By claim 2.4, an equivalent reformulation of condition 2 is property (*) stated in theorem 1.1.

If $|X| = 1$ clearly f has property (*). Assume first that f has a read-once formula in which the output gate is an OR gate, i.e $f = g \vee h$. Clearly g and h are both monotone read-once on some disjoint sets $X_g,\ X_h$ respectively. We have $MIN(f) = MIN(g) \cup MIN(h)$ and

$$MAX(f) = \{S \mid S = T \cup Q\ , T \in MAX(g),\ Q \in MAX(h)\ \}$$

It is easily verified that f has property (*) by the fact that $X_g,\ X_h$ are disjoint. In the other case where the output gate of the read-once formula of f is an AND gate, a similar argument holds. □

Lemma 3.2 *($2 \to 3$) For a clutter $\mathcal{C}$, $If\ \ \mathcal{C}^B \subseteq \mathcal{C}^A$, then $\mathcal{C}^B = \mathcal{C}^A$.*

Proof First we observe that if $\mathcal{C}$ meets the assumptions of the lemma so does $\mathcal{C} \setminus W$ for any $W \subset X$, and that the lemma is trivial for $|X| = 1$.

Assume then, for contradiction, that $\mathcal{C}$ is a counterexample with $|X|$ minimal. Let $T \in \mathcal{C}^A - \mathcal{C}^B$.

We may assume that $|T| \neq 1$. Otherwise, $T = \{x\}$. Take some $B \in \mathcal{C}^B,\ x \in B$, (It is always possible to find such B), thus by condition 2, $B \in \mathcal{C}^A$. But $T \subset B,\ \ T \neq B$ which contradicts the fact that $\mathcal{C}^A$ is a clutter.

Define for every $t \in T,\ \ \mathcal{C}_t = \{S \in \mathcal{C},\ \ S \cap T = \{t\}\}$, and define $\mathcal{D} = \mathcal{C} - (\bigcup_{t \in T} \mathcal{C}_t)$. By the definition of $\mathcal{C}^A$ we get that the families $\mathcal{D}, and\ \mathcal{C}_t,\ t \in T$ are pairwise disjoint, and by the assumption $\mathcal{D} \neq \phi$. Define, for $t \in T$, $W_t = (V(\mathcal{D})\ -\ \bigcup_{s \neq t} V(\mathcal{C}_s))\ \cap V(\mathcal{C}_t)$, that is, W_t is the set of all the points from X that appear in $\mathcal{D}$ and in $\mathcal{C}_t$ but not in any other $\mathcal{C}_s,\ s \neq t$. Note that $W_t \cap W_s = \phi$ for $s \neq t$.

Claim 3.3 *For any $t \in T$ and any $D \in \mathcal{D}$, $W_t \cap D \neq \phi$.*

Proof: If not, say $W_t \cap D = \phi$ for some $D \in \mathcal{D}$, look at $\mathcal{C}' = \mathcal{C} \setminus (W_t \cup \{t\})$. Note the following:

- $D \in \mathcal{C}'$.
- $T - \{t\} \in \mathcal{C}'^A$ since $|(T - \{t\}) \cap S| \leq 1$ for any $S \in \mathcal{C}'$. If there exists $x \in V(\mathcal{C}')$ such that $(T - \{t\}) \cup \{x\}$ has the same property, then clearly $x \notin \bigcup_{s \neq t} V(\mathcal{C}_s)$. Thus,

$x \in V(\mathcal{D})$, but $T \cup \{x\} \notin \mathcal{C}^A$ so $x \in V(\mathcal{C}_t)$. It follows that $x \in W_t$ which contradicts the fact that $x \in V(\mathcal{C}')$.

- $(T - \{t\}) \cap D = \phi$ so $T - \{t\} \notin \mathcal{C}'^B$.

We get that $\mathcal{C}'$ is a counter example on $X - (W_t \cup \{t\})$. (A smaller counter example). □

Thus, by the claim we have that $W_t \cup T$ intersects every set in $\mathcal{C}$, that is, there is some $S_t \in \mathcal{C}^B,\ \ S_t \subseteq W_t \cup T$. Clearly, $T - \{t\} \ \subseteq S_t$ since W_t does not intersect the sets of $\mathcal{C}_s,\ s \neq t$, so we have $S_t = (T - \{t\}) \cup V_t$, where $V_t \subseteq W_t$, and V_t intersects every set of $\mathcal{D} \cup \mathcal{C}_t$. Since $S_t \in \mathcal{C}^B$ and therefore $S_t \in \mathcal{C}^A$ by condition 2, it implies that V_t is a minimal set of points that intersect every set in $\mathcal{C}_t$. Thus $V = \bigcup_{t \in T} V_t$ is a member of $\mathcal{C}^B$ with $|V| \geq |T| \geq 2$. But, since every V_t intersects every set of $\mathcal{D}$, V intersects any set of $\mathcal{D}$ in more then one point, contradicting the fact that $V \in \mathcal{C}^A$ (by condition 2). That completes the proof of the lemma. □

Remark 3.4 *If $\mathcal{C}$ satisfies condition 2, so does $\mathcal{C}' = \mathcal{C} \setminus \{x\}$, $(\mathcal{C}' = \mathcal{C}/\{x\})$ for any $x \in X$ such that $V(\mathcal{C}') \neq \phi$. Thus by the lemma, these clutters satisfy condition 3 too.*

Lemma 3.5 *(3 → 4)*

Let $\mathcal{C}$ be a clutter for which $\mathcal{C}^B = C^A$, then $\mathcal{C}$ is the set of maximal cliques of $G(\mathcal{C})$ and, for any induced subgraph $G' \subseteq G(\mathcal{C})$, every maximal clique intersects every maximal independent set.

Proof

Claim 3.6 *If $\mathcal{C}^B = \mathcal{C}^A$ then $\mathcal{C} = (\mathcal{C}^A)^A$.*

Proof (of the claim): We have $\mathcal{C}^B = \mathcal{C}^A$. But since $\mathcal{C} = (\mathcal{C}^B)^B$, $\mathcal{C}^B$ as a clutter, also satisfies condition 2, thus $(\mathcal{C}^B)^B = (\mathcal{C}^B)^A$ (lemma 3.2). Substituting $(\mathcal{C}^B)^B$ with $\mathcal{C}$, $\mathcal{C}^B$ with $\mathcal{C}^A$ gives the required claim.

Claim 3.7 *[5] Let $\mathcal{C}$ be a clutter, if $(\mathcal{C}^A)^A = \mathcal{C}$ then $\mathcal{C}$ is the set of maximal cliques of $G(\mathcal{C})$, and, in that case, $\mathcal{C}^A$ is the set of maximal independent sets of G.*

proof (of the claim): By the definition of $G(\mathcal{C})$, every $T \in \mathcal{C}$ induces a clique of $G(\mathcal{C})$ and every $S \in \mathcal{C}^A$ is a maximal independent set. Moreover, every maximal independent set is in $\mathcal{C}^A$, thus $\mathcal{C}^A$ is the set of maximal independent set of $G(\mathcal{C})$. It follows that every maximal clique of $G(\mathcal{C})$ is in $(\mathcal{C}^A)^A$, but $(\mathcal{C}^A)^A = \mathcal{C}$ which completes the proof of the claim.

Claim 3.8 *Let $\mathcal{C}$ be a clutter satisfying condition 3, let $G = G(\mathcal{C}) = (X, E)$, then $G - \{x\}$ is the graph of $\mathcal{C}/\{x\}$ or the graph of $\mathcal{C} \setminus \{x\}$.*

Proof Observe that these properties are true for any clutter, [4].

1. $(\mathcal{C}/\{x\})^B \ = \ \mathcal{C}^B \setminus \{x\}$
2. $(\mathcal{C} \setminus \{x\})^B \ = \ \mathcal{C}^B/\{x\}$

Assume $V(\mathcal{C} \setminus \{x\}) \ = \ X - \{x\}$ then $G - \{x\}$ is the graph of $\mathcal{C} \setminus \{x\}$ since for every $(u,v) \in E(G - \{x\})$ if $(u,v) \notin G(\mathcal{C} \setminus \{x\})$ then $(u,v) \in G^C(\mathcal{C} \setminus \{x\}) \ = \ G((\mathcal{C} \setminus \{x\})^B) \ = \ G(\mathcal{C}^B/\{x\})$ that is, u,v are in some $T \in \mathcal{C}^B$. But $(u,v) \in E(G - \{x\})$ implies that there is a set $S \in \mathcal{C}$ such that $u,v \in S$, thus $\{u,v\} \subseteq T \cap S$ contradicting condition 3.

We will show that if $V(\mathcal{C} \setminus \{x\}) \ \neq \ X - \{x\}$ then $V(\mathcal{C}^B \setminus \{x\}) \ = \ X - \{x\}$, thus by the above $G^C - \{x\}$ is the graph of $\mathcal{C}^B \setminus \{x\}$ and thus $G - \{x\}$ is the graph of $(\mathcal{C}^B \setminus \{x\})^B \ = \ \mathcal{C}/\{x\}$.

Note: if for some $z \in X - \{x\}$, $z \notin V(\mathcal{C} \setminus \{x\})$,then $\forall S \in \mathcal{C}$ $(z \in S \Longrightarrow x \in S)$. Similarly, if for some $w \in X - \{x\}$, $w \notin V(\mathcal{C}^B \setminus \{x\})$,then $\forall T \in \mathcal{C}^B$ $(w \in T \Longrightarrow x \in T)$.

Assume then, that $z \in X - \{x\}$, $z \notin V(\mathcal{C} \setminus \{x\})$. Clearly $z \in V(\mathcal{C}^B \setminus \{x\})$ since $z \notin V(\mathcal{C} \setminus \{x\}) \Longrightarrow \exists S \in \mathcal{C}, (x, z \in S\ ) \Longrightarrow \neg\ (\exists T \in \mathcal{C}^B,\ (x, z \in T)) \Longrightarrow z \in V(\mathcal{C}^B \setminus \{x\})$.

Now, for every $w \in X - \{x\}$, if $(w,z) \in E(G)$ it implies that $\exists S \in \mathcal{C}$ such that, $x, w \in S$. (by the assumption on z and the note above). Thus we must have $w \in V(\mathcal{C}^B \setminus \{x\})$, otherwise, for any T of the remark, $w, x \in S \cap T$ which contradicts condition 3. The argument for the case $(w,z) \notin E(G)$ is similar. $\square$

We now complete the proof of lemma 3.5. The lemma is easily verified for $|X| = 2$. We proceed by induction on $|X|$. We have by assumption $\mathcal{C}^A = \mathcal{C}^B$ thus, by claims 3.6 and 3.7, the lemma is true for $G' = G(\mathcal{C})$. Clearly every proper induced subgraph of $G(\mathcal{C})$ is an induced subgraph of $G(\mathcal{C}) - \{x\}$, for some $x \in X$. By claim 3.8 $G' = G(\mathcal{C}) - \{x\}$ is $G(\mathcal{C}')$ for $\mathcal{C}' = \mathcal{C} \setminus \{x\}$ or $\mathcal{C}' = \mathcal{C}/\{x\}$. By remark 3.4, $\mathcal{C}'$ satisfies condition 3. Thus by induction we are done. $\square$

Lemma 3.9 *(4 → 5) Let $\mathcal{C}$ be a clutter on $X = V(\mathcal{C})$ such that $G(\mathcal{C})$ satisfies condition 4, then $G(\mathcal{C})$ is P_4- free.*

Proof: Clearly the graph P_4 does not satisfy condition 4, thus $G(\mathcal{C})$ cannot have a P_4 as an induced subgraph. $\square$

Lemma 3.10 *(5 → 1) Let $\mathcal{C}$ be a clutter on $X = V(\mathcal{C})$ such that $\mathcal{C}$ is the set of maximal cliques of $G(\mathcal{C})$ and $G(\mathcal{C})$ is P_4- free, then $f_{\mathcal{C}}$ is read once.*

Claim 3.11 *If G is a P_4- free graph on more than one vertex, then one of G, G^C is disconnected.*

Proof (claim): Clearly the claim is true for a graph on two vertices. Let G be the minimal counter example, that is, G is P_4- free and both G, G^C are connected. Consider any vertex x; x cannot be connected (in G) to all the vertices because in that case G^C is not connected. By the minimality assumption one of $G - \{x\}$, $(G - \{x\})^C$ is not connected.

Assume that $G - \{x\}$ is not connected, that is, $G - \{x\}$ has at least two components. Therefore there is a vertex u and a vertex t in one of the components of $G - \{x\}$ such that (x, t), (t, u) are edges of $E(G)$ but (x, u) is not an edge. Take some other vertex y such that y is not in the same component with u, t, and (y, x) is an edge of G. The induced graph on the four vertices u, t, x, y is a P_4.

If the case is that $(G - \{x\})^C$ is not connected, observe that $(G - \{x\})^C = G^C - \{x\}$ and apply the above argument to get a P_4 as an induced subgraph of G^C. Since P_4 is self complementary it is an induced subgraph of G too. □

Let $\mathcal{A}$ be the clutter of maximal independent sets of $G = G(\mathcal{C})$.

Claim 3.12 *$\mathcal{C}^B = \mathcal{A}$, and $f_{\mathcal{C}}$ is read-once.*

Proof We prove the claim by induction by induction on the size of $n = |X|$. The claim can be easily checked for $n \leq 3$.

By claim 3.11 one of G, G^C is disconnected. If G is disconnected then $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ where $\mathcal{C}_1$ is the set of maximal cliques of one of the components of G, and $\mathcal{C}_2$ is the set of all other maximal cliques. Clearly $\mathcal{C}_1$, $\mathcal{C}_2$ are clutters on disjoint sets of points, say X_1, X_2 respectively. Define g to be the monotone Boolean function defined by $\mathcal{C}_1$ as its set of minterms, and define h in a similar way with respect to $\mathcal{C}_2$. Clearly $f_{\mathcal{C}} = g \vee h$. Moreover $G(\mathcal{C}_1)$, $G(\mathcal{C}_2)$ are P_4-free, (as induced subgraphs). Thus, by induction, g, h are read-once and then so is $f_{\mathcal{C}}$ (by the fact that X_1 and X_2 are disjoint). Moreover, $\mathcal{C}^B = MAX(f_{\mathcal{C}}) = \{I_1 \cup I_2 | I_1 \in MAX(g),\ I_2 \in MAX(h)\}$. But $MAX(g) = \mathcal{C}_1^B$ and by induction on $\mathcal{C}_1$, that is the set of maximal independent sets of $G(X_1)$. Similarly, $MAX(h)$ is the set of maximal independent sets of $G(X_2)$ and then $MAX(f)$ is indeed $\mathcal{A}$.

In the case where G^C is disconnected, the argument above applies to the components of G^C, shows that $\mathcal{A}^B = \mathcal{C}$ (note that G^C is P_4-free too). By lemma 2.3 that gives $\mathcal{C}^B = \mathcal{A}$. Define g to be the function whose maxtrems are the maximal cliques of a component of G^C and h the function whose maxterms are all other maximal cliques of G^C. It easy to see that $f = g \wedge h$. h, g are read-once, (by induction), on disjoint variable sets, so, f is read-once too. □

4 The nonmonotone case

Theorem 1.1 may be generalized for the non-monotone case using the following definition; Let f be a Boolean function on a set of variables X. A 1-witness of f is a pair (S, T), $S, T \subseteq X$, $S \cap T = \phi$ such that setting the variables in S to 1, and the variables in T to 0, forces the value of f to 1.

We say that a 1-witness (S, T) is a minterm of f if it is 'minimal', that is; there is no other 1-witness (S', T') for which $S' \subseteq S$ and $T' \subseteq T$.

Maxterms of f are defined similarly as the 'minimal' pairs that force the value of f to 0. Denote $MIN(f), (MAX(f))$ the set of all minterms (maxterms) of f. We get

Theorem 4.1 *A Boolean function f that depends on all its variables is read-once if and only if*

$$(S,T) \in MAX(f),\ (P,Q) \in MIN(f) \implies |(S \cup T) \cap (P \cup Q)| = 1 \qquad (**)$$

Proof: The only if part follows directly along the lines of the proof of lemma 3.1. The if part: define $X_1 = \bigcup_{(S,T)\in MIN(f)} S$, $X_2 = \bigcup_{(S,T)\in MIN(f)} T$, $Y_1 = \bigcup_{(S,T)\in MAX(f)} S$, $Y_2 = \bigcup_{(S,T)\in MAX(f)} T$. Condition (**) implies that $X_1 \cap X_2 = \phi$ and $X_1 = Y_1,\ X_2 = Y_2$. Complement the variables in X_2; $x_i \mapsto \neg x_i$. This gives a new monotone function f' that has property (*), therefore f' and hence f are read once. □

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