

## A TRADEOFF BETWEEN SEARCH AND UPDATE TIME FOR THE IMPLICIT DICTIONARY PROBLEM \*

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**Abstract.** This paper proves a tradeoff between the time it takes to search for elements in an implicit dictionary and the time it takes to update the value of elements in specified locations of the dictionary. It essentially shows that if the update time is constant, then the search time is  $\Omega(n^\epsilon)$  for some constant  $\epsilon > 0$ .

### 1. Introduction

A dictionary is a data type that supports two operations: search and update. The former determines whether a given element  $y$  is one of the elements in the dictionary. This is useful, for example, to determine whether a word you have just written is spelled correctly. If the element is in the dictionary, its location may also have to be found.

Updates modify the contents of the dictionary. We may wish to add an element (e.g., a new piece of jargon), delete an element (e.g., an obsolete word), or replace one element by another (e.g., a correction of a spelling mistake). By using dummy values, we may view additions and deletions as instances of replace. This is convenient for purposes of analysis since the number of elements in the dictionary remains fixed.

\* This work began while the authors were at IBM Research Lab, San Jose, CA, and was also supported in part by an IBM Faculty Development Award and National Science Foundation Grant MCS-8402676.

Balanced tree schemes, such as 2-3 trees, can be used to implement dictionaries efficiently. Both search and update can be done in  $O(\log n)$  steps, where  $n$  is the number of elements in the dictionary. However, explicit pointers are used and extra space is needed to store them (i.e., in addition to the space used to store the elements). For large dictionaries, this may be significant.

A natural question to ask is whether this extra space is needed to efficiently implement a dictionary. For example, although heaps are naturally implemented as binary trees using explicit pointers, they can also be represented without extra space. The pointers are implicit—the elements in locations  $2i$  and  $2i+1$  are the left and right children respectively of the element in location  $i$ .

Implicit data structures [11] are those in which only the number of elements and the elements themselves are stored explicitly. The  $n$  elements are stored in an array of length  $n$ , each cell of which is capable of holding exactly one element.

There are a number of ways to implement dictionaries implicitly. If we store the elements in an unordered list, then update can be done in constant time, but search requires linear time. On the other hand, if the dictionary is maintained as an array sorted in increasing order, then search can be done in logarithmic time, but updates may require that all the elements in the array be moved. Rotated lists are arrays that can be sorted into increasing order by performing a cyclic shift (rotation) of the elements. They are not much more difficult to search, but only half the elements have to be moved in the worst case. Notice that, by the information theory lower bound, any comparison-based search algorithm must perform at least a logarithmic number of comparisons in the worst case.

Munro and Suwanda [11] were the first to consider explicitly the implicit dictionary problem. They showed that if the elements are stored partially sorted into a triangular grid, then search and update can be performed in  $O(\sqrt{n})$  steps. Using blocks of rotated lists (sorted relative to one another), they were able to improve the search time to  $O(\log n)$ , keep the number of moves per update at  $O(\sqrt{n})$ , and only increase the number of comparisons per update to  $O(\sqrt{n} \log n)$ . Combining these ideas, they also produced an implicit dictionary that can be searched or updated in  $O(n^{1/3} \log n)$  time. By using rotated lists in a recursive manner, Frederickson [6] was able to achieve  $O(\log n)$  search time and  $O(n^{2/\log n} (\log n)^{3/2})$  update time. Recently, Munro [9, 10] created implicit dictionaries that use  $O((\log n)^2)$  time for both search and update. His basic approach is to have the order of elements within blocks of the array implicitly represent pointers.

Throughout this paper, we employ a comparison-based model with elements drawn from a totally ordered universe. Other models, allowing more general tests or in which elements are drawn from a very small universe [12], will not be considered here.

Munro and Suwanda [11] proved that if the contents of the array locations are constrained to satisfy some fixed partial order, then the product of search and update time is  $\Omega(n)$ . Thus, their triangular grid scheme, which achieves an  $O(n)$  product of search and update time, is optimal among this class of algorithms.

Partial orders naturally arise as the Lynch, and Yao [3] considered the tradeoff between preprocessing and search. The number of comparisons performed to respectively search for an element in an array of length  $n$  then, in the worst case, is  $\Omega(n)$ . Yao proved that this result also holds in the implicit dictionary model.

In general, the information available about the relative order of the elements at the array locations in a dictionary *cannot* be used to improve search. For example, the rotated list. Since even finding the minimum element, the only partial order that is known, is the trivial partial order (containing no information about the relative order of any allowable sequence of elements), the only partial order is by a permutation. An implicit dictionary must store all possible permutations together with an algorithm to search for an element.

Suppose that the  $n$  elements of an array are stored in an array of  $p$  different permutations. If all comparisons involve the element being sought, then Yao [4] prove that  $\Omega(p^{1/n})$  comparisons are required between two elements in the array, which is not allowed.) This lower bound is even stronger than the deterministic algorithms [1]. Notice, however, that the rotated list is, in the worst case, as hard to search as the array.

In this paper, we provide a general tradeoff for the implicit dictionary problem in the rotated list model and then consider a number of special cases. We show that if update is easy, such a search is difficult. In particular, constant update time implies  $\Omega(n^\epsilon)$  for some constant  $\epsilon > 0$ .

## 2. The tradeoff

Associated with an implicit dictionary is the set of possible relative orders of the contents of the array. An algorithm for the dictionary is a comparison-based algorithm for the array  $x$ , provided that the permutation of the contents of the locations of  $x$  belongs to the set of possible orders. The internal nodes of this tree are labelled by  $y: x[j]$ . For convenience, we assume that the root node is the equality branch will not be taken. Thus, the element being sought is not in the array if the root node is at any comparison.

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Partial orders naturally arise as the result of preprocessing. Borodin, Guibas, Lynch, and Yao [3] considered the closely related problem of determining the tradeoff between preprocessing and search time. If  $P(n)$  and  $S(n)$  denote the number of comparisons performed to respectively preprocess and search an initially unsorted array of length  $n$  then, in the worst case,  $P(n) + n \log S(n)$  is  $\Omega(n \log n)$ . Mairson proved that this result also holds in the average case [8].

In general, the information available about the relative order of the contents of the array locations in a dictionary *cannot* be described by a partial order. Consider, for example, the rotated list. Since every location in the array can contain the minimum element, the only partial order guaranteed to be satisfied by a rotated list is the trivial partial order (containing no relations among elements). However, the relative order of any allowable sequence of elements in the array *can* be described by a permutation. An implicit dictionary can be viewed as a set of allowable permutations together with an associated search algorithm and an associated update algorithm.

Suppose that the  $n$  elements of an array can be arranged according to any one of  $p$  different permutations. If all comparisons performed during a search must involve the element being sought, then Alt, Mehlhorn, and Munro [2, 1], and Cook [4] prove that  $\Omega(p^{1/n})$  comparisons are needed in the worst case. (Comparisons between two elements in the array, which may help to identify the permutation, are not allowed.) This lower bound is even true in the average case and for nondeterministic algorithms [1]. Notice, however, that under these assumptions searching a rotated list is, in the worst case, as hard as searching a completely unordered list.

In this paper, we provide a general tradeoff between worst-case search and update time for the implicit dictionary problem. We begin by describing our computational model and then consider a number of situations in which search is difficult. Finally, we show that if update is easy, such a situation must occur and, hence, search is difficult. In particular, constant update time implies a worst case search time of  $\Omega(n^\epsilon)$  for some constant  $\epsilon > 0$ .

## 2. The tradeoff

Associated with an implicit dictionary, there is a set of permutations that describe the possible relative orders of the contents of the locations in the array. A search algorithm for the dictionary is a comparison tree that determines whether  $y$  is in the array  $x$ , provided that the permutation describing the relative order of the contents of the locations of  $x$  belongs to the set of allowable permutations. The internal nodes of this tree are labelled by comparisons of the forms  $x[i]:x[j]$  and  $y:x[j]$ . For convenience, we assume that the elements of the array are distinct and thus the equality branch will not be taken at any comparison of the first type. When the element being sought is not in the array, the equality branch will not be taken at any comparison.

The update algorithm for the dictionary is given an array location  $l$  and a new value  $y$  as input. Depending on the value of  $l$ , it performs a sequence of comparisons involving  $y$  and the elements of the array. It then replaces the element in location  $l$  of the array by the value  $y$ . Based on the outcome of the comparisons, it finally rearranges the elements in at most  $m$  array locations so that the permutation associated with the resulting array is in the set of allowable permutations. We say that such a dictionary performs at most  $m$  moves per update.

Another way to specify an update is to provide two values  $y$  and  $y'$ . Algorithms would be required to replace the value  $y$  by the value  $y'$  and then rearrange the contents of the array as above. Such an approach requires that a search for the value  $y$  be performed as part of the update and, although it is perhaps more natural, it would make the tradeoff result of this paper meaningless.

In certain instances, it is easy to prove lower bounds on the search time. For example, if the set of arrays depicted in Fig. 1 were all allowed by a dictionary, then the search algorithm would have worst case complexity  $n$ . In particular, no information about the relative values of the contents of the other locations in the array can help us to determine if there is a location containing the value 1.

2	4	6	8	...	$2n-2$	$2n$
1	4	6	8	...	$2n-2$	$2n$
2	1	6	8	...	$2n-2$	$2n$
2	4	1	8	...	$2n-2$	$2n$
⋮						
2	4	6	8	...	$2n-2$	1

Fig. 1. A set of arrays that hides the value 1.

In Lemmas 2.1 and 2.2, we shall look at generalizations of the above situation. We obtain our tradeoff between search and update by showing, in Theorem 2.3, that some such situation must occur.

Let  $d \geq 0$  and  $n > 0$  be integers and let  $I \subseteq \{1, \dots, n\}$ . For each  $i \in I \cup \{0\}$ , suppose  $x_i$  is an array of length  $n$  containing distinct numbers. For  $i \in I$ , let

$$D_i = \{j \in \{1, \dots, n\} - \{i\} \mid x_i[j] \neq x_0[j]\}$$

denote the set of locations, excluding  $i$ , where  $x_i$  differs from  $x_0$ . Furthermore, suppose that, for all  $i \in I$ ,  $\#D_i \leq d$  and  $x_i[i] = v$  where  $v$  is a number not appearing in the array  $x_0$ . The sequence of arrays in Fig. 2 satisfies these conditions with  $v = 77$  and  $d = 2$ .

$x_0$	63	12	82
$x_1$	77	12	13
$x_2$	63	77	82
$x_4$	63	12	82
$x_5$	63	12	82
$x_6$	63	59	82

Fig. 2. A set of arrays that

**Lemma 2.1.** Any implicit dictionary that performs at most  $c \geq 1$  comparisons per update and at most  $m$  comparisons per search, in the worst

**Proof.** By induction on  $d$ . If  $d = 0$ , some input, examine all possible locations that there is some location  $l$  such that  $x_0$  in that location. We then consider  $l$  with a new value. For a large fraction of updates performed by the update algorithm,  $d - 1$ .

Let  $T$  be a comparison tree for search. Without loss of generality, we may assume some problem instance  $(x, y)$  where  $y$  is a value whose membership in the set is at depth of  $T$ . Thus  $S$  is the worst-case number of comparisons.

Choose  $u$  to be a number different from all  $i \in I \cup \{0\}$  and all  $j \in \{1, \dots, n\}$  with the same outcome. Notice that  $x_i$  differs from  $x_0$  at  $u$ . Because  $T$  gives different answers for different inputs, they lead to different leaves. However, for two inputs that differ only at  $u$ , their paths to the leaf for  $y$  differ only at  $y : x[i]$ .

In fact, any path from the root to the leaf for  $y : x[j]$ . Otherwise, consider this leaf for some value of  $y$ . Since  $x_i$  differs from  $x_0$  at  $u$ , the path,  $(x, y)$  reaches the same leaf for  $y$  as  $(x_0, y)$ . Thus  $T$  determines whether  $x$  contains  $y$ .

$y$  is given an array location  $l$  and a new value  $v$ . If  $l \in I$ , it performs a sequence of comparisons to find  $v$ . It then replaces the element in location  $l$  with  $v$ . From the outcome of the comparisons, it finally permutes the array locations so that the permutation is in the set of allowable permutations. We say that  $v$  moves per update. Algorithms provide two values  $y$  and  $y'$ . Algorithms search for  $y$  by the value  $y'$  and then rearrange the array locations. This approach requires that a search for the value  $y$  is not possible, and, although it is perhaps more natural, is rather unappealing. The paper is meaningless. Lower bounds on the search time. For the problem in Fig. 1 were all allowed by a dictionary, the worst case complexity  $n$ . In particular, no algorithm can find the contents of the other locations in the array if a location containing the value 1.

...	$2n-2$	$2n$
...	$2n-2$	$2n$
...	$2n-2$	$2n$
...	$2n-2$	$2n$
...	$2n-2$	1

that hides the value 1.

generalizations of the above situation. We can find the update by showing, in Theorem 2.3,

$i \in \{1, \dots, n\}$ . For each  $i \in I \cup \{0\}$ , suppose  $x_i$  is a number. For  $i \in I$ , let

$x_i[j]$

where  $x_i$  differs from  $x_0$ . Furthermore,  $x_i[j] = v$  where  $v$  is a number not appearing in  $x_0$ . Fig. 2 satisfies these conditions with  $v = 77$

$x_0$	63	12	82	18	80	13	23	30
$x_1$	77	12	13	18	80	82	23	30
$x_2$	63	77	82	54	80	13	23	30
$x_4$	63	12	82	77	80	13	54	30
$x_5$	63	12	82	18	77	13	45	30
$x_6$	63	59	82	18	80	77	94	30

Fig. 2. A set of arrays that satisfies the conditions of Lemma 2.1.

**Lemma 2.1.** Any implicit dictionary of size  $n$  that allows the arrays  $x_i$  for all  $i \in I$  and performs at most  $c \geq 1$  comparisons per update must perform at least  $(\#I/2^{(c+1)d})^{1/(d+1)}$  comparisons per search, in the worst case.

**Proof.** By induction on  $d$ . If  $d = 0$ , we show that any search algorithm must, for some input, examine all possible locations in which  $v$  may occur. If  $d > 0$ , we show that there is some location  $l$  such that a large number of the arrays  $x_i$  differ from  $x_0$  in that location. We then consider the effect, on these arrays, of updating location  $l$  with a new value. For a large fraction of these arrays, exactly the same moves are performed by the update algorithm. This results in an instance of the problem for  $d - 1$ .

Let  $T$  be a comparison tree for searching any array allowed by the dictionary. Without loss of generality, we may assume that every leaf of  $T$  can be reached by some problem instance  $(x, y)$  where  $x$  is an array allowed by the dictionary and  $y$  is a value whose membership in the array  $x$  is to be determined. Let  $S$  denote the depth of  $T$ . Thus  $S$  is the worst-case number of comparisons performed in any search.

Choose  $u$  to be a number different from  $v$ , but sufficiently close to  $v$  so that, for all  $i \in I \cup \{0\}$  and all  $j \in \{1, \dots, n\} - \{i\}$ , the comparisons  $v : x_i[j]$  and  $u : x_i[j]$  have the same outcome. Notice that  $x_i$  is an allowable array which contains  $v$ , but not  $u$ . Because  $T$  gives different answers for the inputs  $(x_i, u)$  and  $(x_i, v)$ , it must send them to different leaves. However,  $y : x[i]$  is the only comparison on which these two inputs differ. Therefore, their root-to-leaf paths must split at the comparison  $y : x[i]$ .

In fact, any path from the root to a leaf must contain at least one comparison of the form  $y : x[j]$ . Otherwise, consider any allowable array  $x$  such that  $(x, y)$  reaches this leaf for some value of  $y$ . Since  $y$  is not involved in any comparisons along the path,  $(x, y)$  reaches the same leaf for all values of  $y$ . But this contradicts the fact that  $T$  determines whether  $x$  contains the value  $y$ .

Consider the root-to-leaf path  $P$  taken by the input  $(x_0, u)$ . Note that  $x_0$  is not necessarily an array allowed by the dictionary. Let  $a$  be the number of comparisons of the form  $y : x[j]$  and let  $b$  be the number of comparisons of the form  $x[j] : x[k]$  on the path  $P$ . Then  $S \geq a + b$  and  $a \geq 1$ .

First, suppose  $d = 0$ . Let  $J = \{i \in I \mid \text{the comparison } y : x[i] \text{ does not occur on the path } P\}$ . Then  $\#J \geq \#I - a$ . Since  $i \in J$ , the comparison  $y : x[i]$  does not occur on the path  $P$ . For all  $i \in J$ , the root-to-leaf path taken by the input  $(x_i, u)$  contains the comparison  $y : x[i]$  and thus deviates from the path  $P$  at some point. Now  $x_i$  and  $x_0$  agree everywhere except at location  $i$ . Therefore, the deviation must occur at a comparison involving  $x[i]$  and some other element of  $x$ . For each  $i \in J$ , the point of deviation from the path  $P$  is different. To see this, suppose that  $i, j \in J$  and that the paths taken by inputs  $(x_i, u)$  and  $(x_j, u)$  both deviate from path  $P$  at the comparison  $x[i] : x[j]$ . If  $P$  follows the  $<$  branch at this comparison (i.e.,  $x_0[i] < x_0[j]$ ), then the inputs  $(x_j, u)$  and  $(x_i, u)$  take the other branch (i.e.,  $x_j[i] > x_j[j]$  and  $x_i[i] > x_i[j]$ ). But this is impossible since  $x_0[i] = x_j[i]$ ,  $x_j[j] = v = x_i[i]$ , and  $x_i[j] = x_0[j]$ . Similarly,  $x_0[i] > x_0[j]$  leads to a contradiction. Thus  $b \geq \#J$ ,  $S \geq a + b \geq \#I$ , and the lemma is true for  $d = 0$ .

Now suppose  $d \geq 1$  and assume the lemma is true for  $d - 1$ . Let  $J = \{i \in I \mid i \neq j, k \text{ for all comparisons } y : x[j] \text{ and } x[j] : x[k] \text{ on the path } P\}$ . Then  $\#J \geq \#I - a - 2b$ . As above, all of the inputs  $(x_i, u)$ , with  $i \in J$ , deviate from the path  $P$  at some point. If this occurs at a comparison  $y : x[j]$ , then  $j \in D_i$ . If this occurs at a comparison  $x[j] : x[k]$ , then either  $j$  or  $k$  is an element of  $D_i$ . Therefore, there is a subset  $K \subseteq J$  containing at least  $1/(a + 2b)$  of the elements in  $J$  such that the sets  $D_i$ , for all  $i \in K$ , have an element in common, i.e.,  $\bigcap_{i \in K} D_i \neq \emptyset$ . Since  $a + b \leq S$  and  $a \geq 1$ , it follows that

$$\#K \geq \frac{\#J}{a + 2b} \geq \frac{\#I}{2S - 1} - 1.$$

Let  $l \in \bigcap_{i \in K} D_i$ . Consider the effect on the arrays  $x_i$ ,  $i \in K$ , of performing an update that replaces the element in location  $l$  with a new value  $w$ . Here  $w$  is chosen so that it is not an element of array  $x_i$  for any  $i \in K \cup \{0\}$ . The moves that are performed in response to this update depend only on the sequence of outcomes of the  $c$  comparisons performed. Because all of the elements being compared are distinct, the outcome of any comparison cannot be equality. There are  $2^c$  different sequences of outcomes possible and, hence, there is a subset  $L \subseteq K$ , with  $\#L \geq \#K/2^c$ , such that the same moves are performed on all arrays  $x_i$  with  $i \in L$ . Let  $\tau : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  be the permutation describing this set of moves.

Viewed properly, we now have an instance of the original problem for  $d - 1$ . Specifically, let  $I' = \tau(L)$  and, for  $j \in \{1, \dots, n\}$  and  $i \in I'$ , let

$$x'_0[j] = \begin{cases} w & \text{if } j = \tau(l), \\ x_0[\tau^{-1}(j)] & \text{if } j \neq \tau(l) \end{cases}$$

and

$$x'_i[j] = \begin{cases} w & \text{if } j = \tau(l), \\ x_{\tau^{-1}(i)}[\tau^{-1}(j)] & \text{if } j \neq \tau(l) \end{cases}$$

Because  $v$  does not appear in the array  $x_0$ , it does not appear in any  $x_i$ . If  $i \in I'$ , then  $\tau^{-1}(i) \in L \subseteq K$ . Note that  $l \notin D_i$ . Hence,  $l \neq \tau^{-1}(i)$  and  $x'_i[l] = x_{\tau^{-1}(i)}[\tau^{-1}(l)]$ . Thus  $x'_i$  does not contain duplicate entries. Since  $i \in I' \cup \{0\}$ , the elements of the array

$$\begin{aligned} D'_i &= \{j \in \{1, \dots, n\} - \{i\} \mid x'_i[j] = x'_i[l]\} \\ &= \{j \in \{1, \dots, n\} - \{i, \tau(l)\} \mid x_{\tau^{-1}(i)}[\tau^{-1}(j)] = x_{\tau^{-1}(i)}[\tau^{-1}(l)]\} \\ &= \{k \in \{1, \dots, n\} - \{\tau^{-1}(i), \tau^{-1}(l)\} \mid x_{\tau^{-1}(i)}[k] = x_{\tau^{-1}(i)}[\tau^{-1}(l)]\} \\ &= D_{\tau^{-1}(i)} - \{l\} \end{aligned}$$

and, because  $l \in D_{\tau^{-1}(i)}$  for all  $\tau^{-1}(i) \in L$ ,

By the induction hypothesis,  $\#D_{\tau^{-1}(i)} \leq (\#I/(2S - 1) - 1)/2^c$ ; therefore,

$$\begin{aligned} \#I &\leq (2^c \#I' + 1)(2S - 1) \\ &\leq (2^c 2^{(c+1)(d-1)} S^d + 1)(2S - 1) \\ &= 2^{(c+1)d} S^{d+1} - 2^c 2^{(c+1)(d-1)} S^d \\ &\leq 2^{(c+1)d} S^{d+1} - 1 \\ &\leq 2^{(c+1)d} S^{d+1}. \end{aligned}$$

Hence, the lemma is true for  $d$ .  $\square$

We now consider another situation. Let  $x_0$  be an array of length  $n$  containing the same value  $v_0$  in all locations. Let  $v_1, \dots, v_{d-1}$  be a different value. For  $i \in I \subseteq \{1, \dots, d-1\}$ , let  $x_i$  be an array of length  $n$  containing the same value  $v_0$  in all locations except at location  $i$ , where it contains  $v_i$ .

$$D_i = \{j \in \{1, \dots, n\} - \{i\} \mid x_i[j] = v_0\}$$

Furthermore, suppose  $\#D_i \leq d$ .

These conditions are similar to those of the previous situation. They require  $v_0$  to be in different locations in each array  $x_i$  is compared. Figure 3 shows a sequence of arrays in Fig. 3, with

y the input  $(x_0, u)$ . Note that  $x_0$  is not  
 ry. Let  $a$  be the number of comparisons  
 of comparisons of the form  $x[i]:x[k]$

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 to see this, suppose that  $i, j \in J$  and that  
 $(u)$  both deviate from path  $P$  at the  
 branch at this comparison (i.e.,  $x_0[i] <$   
 take the other branch (i.e.,  $x_j[i] > x_j[j]$   
 since  $x_0[i] = x_j[i]$ ,  $x_j[j] = v = x_i[i]$ , and  
 to a contradiction. Thus  $b \geq \#J, S \geq$

a is true for  $d-1$ . Let  $J = \{i \in I \mid i \neq j, k$   
 on the path  $P\}$ . Then  $\#J \geq \#I - a - 2b$ .  
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 $j \in D_i$ . If this occurs at a comparison  
 of  $D_i$ . Therefore, there is a subset  $K \subseteq J$   
 in  $J$  such that the sets  $D_i$ , for all  $i \in K$ ,  
 $\emptyset$ . Since  $a + b \leq S$  and  $a \geq 1$ , it follows

the arrays  $x_i, i \in K$ , of performing an  
 with a new value  $w$ . Here  $w$  is chosen  
 any  $i \in K \cup \{0\}$ . The moves that are  
 only on the sequence of outcomes of  
 of the elements being compared are  
 not be equality. There are  $2^c$  different  
 there is a subset  $L \subseteq K$ , with  $\#L \geq$   
 rmed on all arrays  $x_i$  with  $i \in L$ . Let  
 describing this set of moves.

ce of the original problem for  $d-1$ .  
 $n\}$  and  $i \in I'$ , let

and

$$x'_i[j] = \begin{cases} w & \text{if } j = \tau(l), \\ x_{\tau^{-1}(i)}[\tau^{-1}(j)] & \text{if } j \neq \tau(l). \end{cases}$$

Because  $v$  does not appear in the array  $x_0$ , it does not appear in the array  $x'_0$  either.  
 If  $i \in I'$ , then  $\tau^{-1}(i) \in L \subseteq K$ . Note that  $l \notin K$  since the definition of  $D_l$  implies that  
 $l \notin D_l$ . Hence,  $l \neq \tau^{-1}(i)$  and  $x'_i[i] = x_{\tau^{-1}(i)}[\tau^{-1}(i)] = v$ . For  $i \in I \cup \{0\}$ , the array  $x_i$   
 does not contain duplicate entries and  $w$  is not contained in  $x_i$ . Therefore, for  
 $i \in I' \cup \{0\}$ , the elements of the array  $x'_i$  are distinct. Finally,

$$\begin{aligned} D'_i &= \{j \in \{1, \dots, n\} - \{i\} \mid x'_i[j] \neq x'_0[j]\} \\ &= \{j \in \{1, \dots, n\} - \{i, \tau(l)\} \mid x_{\tau^{-1}(i)}[\tau^{-1}(j)] \neq x_0[\tau^{-1}(j)]\} \\ &= \{k \in \{1, \dots, n\} - \{\tau^{-1}(i), l\} \mid x_{\tau^{-1}(i)}[k] \neq x_0[k]\} \\ &= D_{\tau^{-1}(i)} - \{l\} \end{aligned}$$

and, because  $l \in D_{\tau^{-1}(i)}$ , for all  $\tau^{-1}(i) \in L = \tau^{-1}(I')$ , it follows that  $\#D'_i \leq d-1$ .

By the induction hypothesis,  $S \geq (\#I'/2^{(c+1)(d-1)})^{1/d}$ . Now  $\#I' = \#L \geq$   
 $(\#I/(2S-1) - 1)/2^c$ ; therefore,

$$\begin{aligned} \#I &\leq (2^c \#I' + 1)(2S - 1) \\ &\leq (2^c 2^{(c+1)(d-1)} S^d + 1)(2S - 1) \\ &= 2^{(c+1)d} S^{d+1} - 2^c 2^{(c+1)(d-1)} S^d + 2S - 1 \\ &\leq 2^{(c+1)d} S^{d+1} - 1 && \text{(since } c, d \geq 1) \\ &\leq 2^{(c+1)d} S^{d+1}. \end{aligned}$$

Hence, the lemma is true for  $d$ . By induction, the lemma is true for all integers  
 $d \geq 0$ .  $\square$

We now consider another situation in which searching is difficult. Let  $z_0$  be an  
 array of length  $n$  containing the sequence  $v_1, \dots, v_n$  of distinct values and let  $v_0$   
 be a different value. For  $i \in I \subseteq \{1, \dots, n\}$ , let  $z_i$  be an array of length  $n$  containing  
 all  $n$  values in the set  $\{v_0, v_1, \dots, v_n\} - \{v_i\}$  and let

$$D_i = \{j \in \{1, \dots, n\} - \{i\} \mid z_0[j] \neq z_i[j]\}.$$

Furthermore, suppose  $\#D_i \leq d$ .

These conditions are similar to the conditions for Lemma 2.1. Here we do not  
 require  $v_0$  to be in different locations for each of the  $z_i$  arrays. However, the set of  
 elements in each array  $z_i$  is completely determined. For example, consider the  
 sequence of arrays in Fig. 3, with  $v_0 = 77$  and  $d = 2$ .

$z_0$	63	12	82	18	80	13	23	30
$z_1$	77	12	13	18	80	82	23	30
$z_2$	63	80	82	18	77	13	23	30
$z_4$	63	12	82	80	77	13	23	30
$z_5$	63	12	82	18	23	13	77	30
$z_6$	63	80	82	18	77	12	23	30

Fig. 3. A set of arrays that satisfies the conditions of Lemma 2.2.

**Lemma 2.2.** Any implicit dictionary that allows the arrays  $z_i$  for all  $i \in I$  and performs at most  $c \geq 1$  comparisons per update must perform at least

$$\left\lceil \left( \frac{\#I^{2/(d+1)}}{2^{(c+1)d}} \right)^{1/(d+2)} \right\rceil$$

comparisons per search, in the worst case.

**Proof.** By induction on  $d$ . Let  $T$  be a comparison tree for searching any array allowed by the dictionary and let  $S$  denote the depth of  $T$ . Essentially, we shall show that there is some value which occurs at many different locations in the  $z_i$  arrays, and this will enable us to apply Lemma 2.1.

If  $d = 0$ , then  $z_i[i] = v_0$  for all  $i \in I$ . By Lemma 2.1,  $S \geq \#I$ . Now let  $d \geq 1$  and assume the lemma is true for  $d - 1$ .

Let  $L = \{l \in \{1, \dots, n\} \mid z_i[l] = v_0 \text{ for some } i \in I\}$  denote the set of locations where  $v_0$  occurs and, for each  $l \in L$ , let  $I_l = \{i \in I \mid z_i[l] = v_0\}$  be the indices of those arrays in which  $v_0$  occurs in location  $l$ .

We first consider the case when

$$\#I_l \leq \left\lceil \left( \frac{\#I}{2^{(c+1)}} \right)^{d/(d+2)} \right\rceil \text{ for all } l \in L.$$

Then

$$\#L \geq \#I / \left\lceil \left( \frac{\#I}{2^{(c+1)}} \right)^{d/(d+2)} \right\rceil \geq (\#I^2 2^{(c+1)d})^{1/(d+2)}.$$

In this case, the value  $v_0$  occurs in a large number of different locations and Lemma 2.1 can be applied directly.

More formally, let  $z'_0 = z_0$  and, for each  $i \in I$ , let  $D'_i = \{j \in \{1, \dots, n\} - \{l\} \mid z'_i[j] \neq z'_0[j]\}$ . Then we know that  $l \in D_i \cup \{i\}$ . Thus  $\#D'_i \leq \#I - 1$ .

$$\begin{aligned} S &\geq \left( \frac{\#L}{2^{(c+1)d}} \right)^{1/(d+1)} \geq \left( \frac{\#I^2 2^{(c+1)d}}{2^{(c+1)d(d+1)}} \right)^{1/(d+1)} \\ &= \left( \frac{\#I^{2/(d+1)}}{2^{(c+1)d}} \right)^{1/(d+2)} \geq \left\lceil \left( \frac{\#I}{2^{(c+1)}} \right)^{d/(d+2)} \right\rceil. \end{aligned}$$

Now, we suppose there is a location  $l$  such that

$$\#I_l \geq \left\lceil \left( \frac{\#I}{2^{(c+1)}} \right)^{d/(d+2)} \right\rceil + 1.$$

Thus location  $l$  contains the value  $v_0$  for many different arrays.

Viewed properly, we have an instance of the problem. Specifically, define

$$z'_0[j] = \begin{cases} z_0[j] & \text{for } j \neq l, \\ v_0 & \text{for } j = l \end{cases}$$

and

$$v'_i = \begin{cases} v_i & \text{for } i = 0, \\ v_0 & \text{for } i = l, \\ v_i & \text{for } i \in \{1, \dots, n\} - \{l\}. \end{cases}$$

Let  $I' = I_l - \{l\}$  and let  $z'_i = z_i$  for  $i \in \{1, \dots, n\} - \{l\}$ . Then  $D'_i = D_i - \{l\}$ .

By the induction hypothesis,

$$S \geq \left\lceil \left( \frac{(\#I')^{2/d}}{2^{(c+1)(d-1)}} \right)^{1/(d+1)} \right\rceil.$$

Since

$$\#I' \geq \left\lceil \left( \frac{\#I}{2^{(c+1)}} \right)^{d/(d+2)} \right\rceil,$$

we have

$$S \geq \left\lceil \left( \frac{\left\lceil \left( \frac{\#I}{2^{(c+1)}} \right)^{d/(d+2)} \right\rceil^{2/d}}{2^{(c+1)(d-1)}} \right)^{1/(d+1)} \right\rceil.$$

We use the facts that, for all positive integers  $r$  and  $b$ ,  $\lceil r \rceil^{1/b} \geq \lceil r^{1/b} \rceil$ . Since  $2^{(c+1)(d-1)d/2}$

$$S \geq \left\lceil \left( \frac{\left( \frac{\#I}{2^{(c+1)}} \right)^{d/(d+2)}}{2^{(c+1)(d-1)d/2}} \right)^{2/d(d+1)} \right\rceil.$$



80	13	23	30
80	82	23	30
77	13	23	30
77	13	23	30
23	13	77	30
77	12	23	30

the conditions of Lemma 2.2.

ws the arrays  $z_i$  for all  $i \in I$  and performs perform at least

comparison tree for searching any array e the depth of  $T$ . Essentially, we shall rs at many different locations in the  $z_i$  nma 2.1.

Lemma 2.1,  $S \geq \#I$ . Now let  $d \geq 1$  and

$i \in I$  denote the set of locations where  $z_i[l] = v_0$  be the indices of those arrays

$\in L$ .

$I^{2^{(c+1)d} / (d+2)}$ .

number of different locations and Lemma

More formally, let  $z'_0 = z_0$  and, for each  $l \in L$ , choose  $z'_i = z_i$  for some  $i \in I_l$ . Then  $D'_i = \{j \in \{1, \dots, n\} - \{l\} \mid z'_i[j] \neq z'_0[j]\} = D_i \cup \{i\} - \{l\}$ . Since  $z_i[l] = v_0 \neq z_0[l] = z'_0[l]$ , we know that  $l \in D_i \cup \{i\}$ . Thus  $\#D'_i \leq \#D_i \leq d$ . By Lemma 2.1,

$$S \geq \left( \frac{\#L}{2^{(c+1)d}} \right)^{1/(d+1)} \geq \left( \frac{(\#I^2 2^{(c+1)d})^{1/(d+2)}}{2^{(c+1)d}} \right)^{1/(d+1)}$$

$$= \left( \frac{\#I^{2/(d+1)}}{2^{(c+1)d}} \right)^{1/(d+2)} \geq \left[ \left( \frac{\#I^{2/(d+1)}}{2^{(c+1)d}} \right)^{1/(d+2)} \right].$$

Now, we suppose there is a location  $l \in L$  such that

$$\#I_l \geq \left\lceil \left( \frac{\#I}{2^{(c+1)}} \right)^{d/(d+2)} \right\rceil + 1.$$

Thus location  $l$  contains the value  $v_0$  for a large number of the  $z_i$  arrays.

Viewed properly, we have an instance of the original problem for  $d-1$ . Specifically, define

$$z'_0[j] = \begin{cases} z_0[j] & \text{for } j \neq l, \\ v_0 & \text{for } j = l \end{cases}$$

and

$$v'_i = \begin{cases} v_l & \text{for } i = 0, \\ v_0 & \text{for } i = l, \\ v_i & \text{for } i \in \{1, \dots, n\} - \{l\}. \end{cases}$$

Let  $I' = I_l - \{l\}$  and let  $z'_i = z_i$  for all  $i \in I'$ . Then  $D'_i = \{j \in \{1, \dots, n\} - \{i\} \mid z'_i[j] \neq z'_0[j]\} = D_i - \{l\}$  for all  $i \in I'$ ; hence,  $\#D'_i \leq d-1$ .

By the induction hypothesis,

$$S \geq \left[ \left( \frac{(\#I')^{2/d}}{2^{(c+1)(d-1)}} \right)^{1/(d+1)} \right].$$

Since

$$\#I' \geq \left\lceil \left( \frac{\#I}{2^{(c+1)}} \right)^{d/(d+2)} \right\rceil,$$

we have

$$S \geq \left[ \left( \frac{\left\lceil \left( \frac{\#I}{2^{(c+1)}} \right)^{d/(d+2)} \right\rceil^{2/d}}{2^{(c+1)(d-1)}} \right)^{1/(d+1)} \right] = \left[ \left( \frac{\left\lceil \left( \frac{\#I}{2^{(c+1)}} \right)^{d/(d+2)} \right\rceil^{2/d(d+1)}}{2^{(c+1)(d-1)d/2}} \right) \right].$$

We use the facts that, for all positive integers  $b$  and real numbers  $r$ ,  $\lceil r \rceil / b \geq \lfloor r/b \rfloor$  and  $\lceil r \rceil^{1/b} \geq \lfloor r^{1/b} \rfloor$ . Since  $2^{(c+1)(d-1)d/2}$  and  $d(d+1)/2$  are integers,

$$S \geq \left[ \left( \frac{\left\lceil \left( \frac{\#I}{2^{(c+1)}} \right)^{d/(d+2)} \right\rceil^{2/d(d+1)}}{2^{(c+1)(d-1)d/2}} \right) \right] \geq \left[ \left( \frac{\left\lceil \left( \frac{\#I}{2^{(c+1)}} \right)^{d/(d+2)} \right\rceil^{2/d(d+1)}}{2^{(c+1)(d-1)d/2}} \right) \right].$$

$$= \left\lceil \left( \frac{\# I^{2/(d+1)}}{2^{(c+1)d}} \right)^{1/(d+2)} \right\rceil.$$

Thus the claim is true for  $d$  and, hence, by induction, for all integers  $d \geq 0$ .  $\square$

Finally, we prove the desired tradeoff between update and search time.

**Theorem 2.3.** *Any implicit dictionary for length- $n$  arrays that performs at most  $m$  moves and  $c \geq 1$  comparisons per update must perform at least  $\lfloor (n^{2/(m+1)} / 2^{(c+1)m})^{1/(m+2)} \rfloor$  comparisons per search, in the worst case.*

**Proof.** Let  $I = \{1, \dots, n\}$ , let  $z_0$  be any allowable array in the dictionary, and let  $v$  be a value not in  $z_0$ . For  $i \in I$ , let  $z_i$  be the array that is obtained from  $z_0$  using the update algorithm to replace the element in location  $i$  with the value  $v$ .

For  $i \in I$ , define  $D_i = \{j \in \{1, \dots, n\} - \{i\} \mid z_i[j] \neq z_0[j]\}$ . Since at most  $m$  moves per update are performed,  $\#D_i \leq m$ . The result follows from the application of Lemma 2.2.  $\square$

**Corollary 2.4.** *Any implicit dictionary for length  $n$  arrays that has constant update time has  $\Omega(n^\epsilon)$  search time for some constant  $\epsilon > 0$ .*

### 3. Conclusions

The technique used to obtain the tradeoff between search and update time may consider only a small fraction of the allowable permutations. Furthermore, it allows the search and update algorithms to change after any update has been performed. Improvements to this tradeoff could therefore be possible. It remains unclear whether logarithmic update time must imply nonpolylogarithmic search time for implicit dictionaries.

A tradeoff between the number of moves per update and search time would also be interesting. For this result, it would not matter whether the element to be replaced during an update is specified by its value or its location. We believe that any dictionary in which only a constant number of moves per update are performed should very quickly get disorganized and, hence, be difficult to search.

One of the motivations for studying the implicit dictionary problem is to understand the relationship between the amount of time it takes to perform a search within an array and easily described properties of the set of allowable permutations. Specifically, we would like to characterize those sets of permutations for which searching is easy (i.e.,  $O(\log n)$  or  $(\log n)^{O(1)}$  comparisons, in the worst case) and those for which searching is hard.

This is analogous to work done by Li [1]. They showed that the logarithm of the bound for the worst-case number of comparisons is only information known about the comparisons that satisfy the partial order.

We conjecture that if the set of all allowable permutations of worst-case search time must be large. More precisely, the search time is not logarithmic or perhaps not even polylogarithmic, it is  $\Omega(n!/c^n)$  for some constant  $c > 0$ .

An interesting example of a large set of permutations was devised by Feldman [5]. He considered permutations of  $\{1, \dots, n\}$  that are involutions (i.e.,  $\pi^2 = \text{id}$ ). He fixes all the even-numbered locations (i.e.,  $2, 4, \dots, n$ ) and permutes more than  $(\frac{n}{2})!$  such permutations. Such a set of elements in the array  $x$  can be described by a permutation. Searching for an element  $y$  in the array  $x$  can be described by the following procedure. First, binary search for  $y$  in the even-numbered locations of  $x$ . Suppose  $y$  is not found. Then search for  $y$  in the odd-numbered locations of  $x$ . Suppose  $y$  is not found. Then search for  $y$  in the even-numbered locations of  $x$ . (If no such  $i$  exists, which happens if  $y$  is not in the array  $x$ .) Next, the rank  $j$  of element  $y$  is determined by a search on the even-numbered locations of  $x$ . Since the permutation associated with  $x$  changes locations  $i$  and  $j$ . Thus, if  $y$  is not found, it is in location  $j$ .

Studying tradeoffs for the implicit dictionary problem is rather a stepping stone towards a general theory of search problems. In particular, for dictionaries with constant update time imply slow search time.

### References

- [1] H. Alt and K. Mehlhorn, Searching semisorted arrays, *Proc. 19th Ann. Symp. on Foundations of Comp. Sci.*, pp. 100-110, 1978.
- [2] H. Alt, K. Mehlhorn and J.I. Munro, Partially sorted arrays, *Process. Lett.* **19** (1984) 61-65.
- [3] A. Borodin, L. Guibas, N. Lynch and A. Meyer, Searching sorted arrays, *Process. Lett.* **12** (1981) 71-75.
- [4] S. Cook, Personal communication, 1982.
- [5] P. Feldman, Personal communication, 1982.
- [6] G.N. Frederickson, Implicit data structures, *Proc. 19th Ann. Symp. on Foundations of Comp. Sci.*, pp. 111-120, 1978.
- [7] N. Linial and M. Saks, Information bounds for searching, in: *Proc. 24th Ann. Symp. on Foundations of Comp. Sci.*, pp. 100-110, 1983.
- [8] H. Mairson, Average case lower bounds for searching, in: *Proc. 26th Ann. Symp. on Foundations of Comp. Sci.*, pp. 100-110, 1985.

This is analogous to work done by Linial and Saks [7] for searching partial orders. They showed that the logarithm of the number of ideals in a partial order is a lower bound for the worst-case number of comparisons needed to search an array if the only information known about the contents of the array locations are that they satisfy the partial order.

We conjecture that if the set of allowable permutations is very large, then the worst-case search time must be large. More specifically, the search time is probably not logarithmic or perhaps not even polylogarithmic if the number of permutations is  $\Omega(n!/c^n)$  for some constant  $c > 0$ .

An interesting example of a large set of permutations that can be searched quickly was devised by Feldman [5]. He considered the set of permutations  $\pi: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  that are involutions (i.e.,  $\pi^2$  is the identity permutation) and, moreover, fix all the even-numbered locations (i.e.,  $\pi(2i) = 2i$  for  $i = 1, \dots, \lfloor \frac{1}{2}n \rfloor$ ). There are more than  $(\frac{1}{2}n)!$  such permutations. Suppose that the relative order of the sequence of elements in the array  $x$  can be described by an unknown one of these permutations. Searching for an element  $y$  in the array  $x$  can be accomplished in logarithmic time by the following procedure. First, binary search is performed on the even-numbered locations of  $x$ . Suppose  $y$  is not found. Let  $i$  denote the unique odd-numbered array location such that all even-numbered locations less than  $i$  contain elements less than  $y$  and all even-numbered locations greater than  $i$  contain elements greater than  $y$ . (If no such  $i$  exists, which happens when  $n$  is even and  $x[n] < y$ , then  $y$  is not in the array  $x$ .) Next, the rank  $j$  of element  $x[i]$  is determined by performing binary search on the even-numbered locations of  $x$  a second time. Finally,  $y$  is compared with  $x[j]$ . Since the permutation associated with  $x$  is an involution, it must interchange locations  $i$  and  $j$ . Thus, if  $y$  is in the array  $x$ , it is in location  $j$ .

Studying tradeoffs for the implicit dictionary problem is not an end in itself, but rather a stepping stone towards a general understanding of tradeoffs for data structure problems. In particular, for dictionaries that may have explicit pointers, does constant update time imply slow search?

## References

- [1] H. Alt and K. Mehlhorn, Searching semisorted tables, *SICOMP* **14** (1985) 840-848.
- [2] H. Alt, K. Mehlhorn and J.I. Munro, Partial match retrieval in implicit data structures, *Inform. Process. Lett.* **19** (1984) 61-65.
- [3] A. Borodin, L. Guibas, N. Lynch and A. Yao, Efficient searching using partial ordering, *Inform. Process. Lett.* **12** (1981) 71-75.
- [4] S. Cook, Personal communication, 1982.
- [5] P. Feldman, Personal communication, 1985.
- [6] G.N. Frederickson, Implicit data structures for the dictionary problem, *J. ACM* **30** (1983) 80-94.
- [7] N. Linial and M. Saks, Information bounds are good for search problems on ordered data structures in: *Proc. 24th Ann. Symp. on Foundations of Computer Science* (1983) 473-475.
- [8] H. Mairson, Average case lower bounds on the construction and searching of partial orders, in: *Proc. 26th Ann. Symp. on Foundations of Computer Science* (1985) 303-311.

- [9] J.I. Munro, An implicit data structure for the dictionary problem that runs in polylog time, in: *Proc. 25th Ann. Symp. on Foundations of Computer Science* (1984) 369-374.
- [10] J.I. Munro, An implicit data structure supporting insertion, deletion and search in  $O(\log^2 n)$  time, Manuscript, University of Waterloo, 1985.
- [11] J.I. Munro and H. Suwanda, Implicit data structures, *J. Comput. System Sci.* **21** (1980) 236-250.
- [12] A.C. Yao, Should tables be sorted?, *J. ACM* **3** (1981) 615-628.

## ON THE INTERSECTION

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**Abstract.** What do a pushdown stack and a counter? Is it a counter? If we add a retrieval register and a queue, or, by symmetry, of a pushdown and a one-reset queue? Is the intersection of a pushdown and a one-reset queue? Is it conjectured by Autebert et al. (1979),

We approach these problems in terms of real-time machines whose storage tapes are pushdowns and queues. We disprove all conjectures from the intersection of pushdowns and queues and one-reset. In fact, there is a complete characterization of the intersection of languages and the context-free languages that are defined by machines with a pushdown and a counter respectively.

## 1. Introduction

The cover of Jean Berstel's book "Context-Free Languages" illustrates the relationship between context-free languages and one-counter languages. A "?" at the intersection of the one-counter languages,  $\text{Roc1}$ , and the context-free languages,  $\text{CFL}$ , is whether this intersection is exactly the context-free languages. In terms of machines the question is whether machines that are at the intersection of one-reversal machines and one-counter machines have the restrictions on a pushdown to one-counter machines and do they

Rephrased in terms of least trios of machines the question holds or not? Autebert et al. [1, p. 208, 266] indicates the opposite. Autebert et al. conjecture that  $\mathcal{M}(\text{PAL}) \cap \mathcal{M}(\text{CFL}) = \mathcal{M}(\text{CFL})$  and Autebert et al. conjecture that  $\mathcal{M}(\text{PAL}) \cap \mathcal{M}(\text{CFL}) = \mathcal{M}(\text{CFL})$  which in turn is a special case of a

We disprove all these conjectures. We disprove all these conjectures for families of languages over PAL, CFL and one-counter languages. We do not have an answer on the