

On linear-algebraic notions of expansion

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January 16, 2023

Abstract

A fundamental fact about bounded-degree graph expanders is that three notions of expansion—vertex expansion, edge expansion, and spectral expansion—are all equivalent. In this paper, we study to what extent such a statement is true for linear-algebraic notions of expansion.

There are two well-studied notions of linear-algebraic expansion, namely dimension expansion (defined in analogy to graph vertex expansion) and quantum expansion (defined in analogy to graph spectral expansion). Lubotzky and Zelmanov proved that the latter implies the former. We prove that the converse is false: there are dimension expanders which are not quantum expanders.

Moreover, this asymmetry is explained by the fact that there are two *distinct* linear-algebraic analogues of graph edge expansion. The first of these is *quantum edge expansion*, which was introduced by Hastings, and which he proved to be equivalent to quantum expansion. We introduce a new notion, termed *dimension edge expansion*, which we prove is equivalent to dimension expansion and which is implied by quantum edge expansion. Thus, the separation above is implied by a finer one: dimension edge expansion is strictly weaker than quantum edge expansion. This new notion also leads to a new, more modular proof of the Lubotzky–Zelmanov result that quantum expanders are dimension expanders.

Yinan Li would like to dedicate this paper to the memory of Keding Ma, a beloved grandfather who passed away at 88.

1 Introduction

1.1 Graph-theoretic and linear-algebraic notions of expansion

Expansion is a fundamental graph-theoretic notion, with applications in and connections to combinatorics, geometry, group theory, number theory, probability, theoretical computer science, and

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many other fields. For an in-depth introduction to expanders and their applications, we refer the reader to the monograph [HLW06].

One of the reasons why expanders are so ubiquitous is that there are three different notions of expansion in graphs, which are all equivalent. These equivalences naturally yield connections between different perspectives on expansion, and allow expanders to be utilized and studied in many different contexts. We briefly recall the three notions of expansion.

Let $G = ([n], E)$ be a d -regular graph. The *edge expansion* of G , $h(G)$, is defined as¹

$$h(G) := \min_{\substack{W \subseteq [n] \\ 1 \leq |W| \leq \frac{n}{2}}} \frac{|\partial W|}{d|W|}, \quad (1.1)$$

where $\partial W := \{\{i, j\} \in E : i \in W, j \in [n] \setminus W\}$. The *vertex expansion* of G , $\mu(G)$, is defined as

$$\mu(G) := \min_{\substack{W \subseteq [n] \\ 1 \leq |W| \leq \frac{n}{2}}} \frac{|\partial_{out}(W)|}{|W|}, \quad (1.2)$$

where $\partial_{out}(W) := \{j \in [n] \setminus W : \exists i \in W, \text{ s.t. } \{i, j\} \in E\}$. The *spectral expansion*² of G , $\lambda(G)$, is defined as

$$\lambda(G) := \text{the second-smallest eigenvalue of } L \quad (1.3)$$

where L is the normalized Laplacian matrix of G , which is the matrix with $L_{i,i} = 1$, $L_{i,j} = -1/d$ if $\{i, j\} \in E$ and $L_{i,j} = 0$ otherwise. Equivalently, $L = I_n - A$, where I_n is the $n \times n$ identity matrix and A is the normalized adjacency matrix of G , defined by $A_{i,j} = 1/d$ if $\{i, j\} \in E$ and $A_{i,j} = 0$ otherwise.

Note that for any d -regular graph G , all three quantities $h(G)$, $\mu(G)$, $\lambda(G)$ are non-negative. We say that a sequence of d -regular graphs $(G_n)_{n \in \mathbb{N}}$ is an *edge expander* (resp. *vertex expander*, *spectral expander*) if the relevant parameter is *uniformly* bounded away from zero for the whole family, namely if $\inf_n h(G_n) > 0$ (resp. $\inf_n \mu(G_n) > 0$, $\inf_n \lambda(G_n) > 0$). As discussed above, these three notions are equivalent; the precise quantitative relationships between them are given in the following proposition. Note that in this proposition, the size of the graph is irrelevant and does not affect any of the bounds.

Proposition 1.1. *Let G be a d -regular graph. Then we have*

- (1) $\frac{\mu(G)}{d} \leq h(G) \leq \mu(G)$;
- (2) $\frac{\lambda(G)}{2} \leq h(G) \leq \sqrt{2\lambda(G)}$.

The proof of Proposition 1.1(1) is straightforward. Proposition 1.1(2) is a discrete analogue of the celebrated *Cheeger's inequality* [Che70], proved by Dodziuk [Dod84], and independently by Alon–Milman [AM85] and Alon [Alo86]. Note that we think of the degree d as a constant, so that we only lose a constant factor when moving between the notions of vertex and edge expansion, and only lose a quadratic factor when moving between these and the notion of spectral expansion.

In this paper, we are interested in studying linear-algebraic notions of expansion and their relationships. There are several well-studied notions of linear-algebraic expansion, including dimension expanders (introduced by Barak, Impagliazzo, Shpilka, and Wigderson [BISW04]), quantum expanders (introduced independently by Hastings [Has07] and by Ben-Aroya and Ta-Shma

¹Some authors define edge expansion without the factor of d , but we use this normalization to match the definition of quantum edge expansion introduced by [Has07].

²This notion is also sometimes called the *spectral gap* of G .

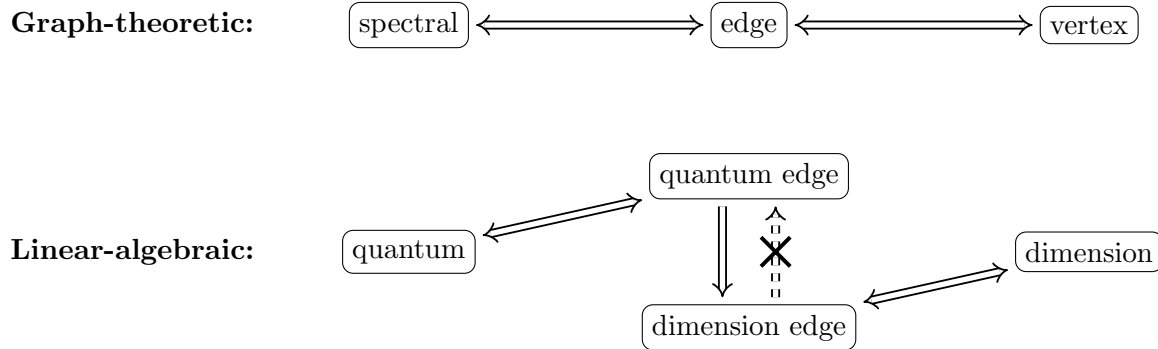


Figure 1: A schematic depiction of the relationships between different notions of expansion.

[BATS07]), and quantum edge expanders (introduced by Hastings [Has07]). We will not discuss here the motivations for these definitions (besides being very natural extensions of the related graph-theoretic parameters), but note that they have led to much further exploration. For dimension expansion see e.g. [DS11, Bou09, BY13, DW10, FG15, LZ08] and for quantum expansion see e.g. [Har08, BASTS10, HH09, Sen18, FM20, KLR21]. Many of these papers and others deal with the important problem of explicitly constructing quantum and dimension expanders; some make use of connections between different notions of expansion, and have led to the introduction of new notions of expansion³. The linear-algebraic notions of expansion will be formally defined momentarily, but first we wish to make some high-level remarks about them.

Dimension expansion is defined in natural analogy to the graph-theoretic definition of vertex expansion, quantum expansion is defined in natural analogy to spectral expansion, and quantum edge expansion is defined in natural analogy to edge expansion. Because of these analogies, it is natural to wonder whether the three notions are equivalent. Hastings [Has07] proved an analogue of Proposition 1.1(2), showing that quantum expansion and quantum edge expansion are equivalent. Additionally, it is implicit in work of Lubotzky and Zelmanov [LZ08] that (under mild assumptions) quantum expansion implies dimension expansion. However, no reverse implication was known, nor any analogue of Proposition 1.1(1) relating dimension expansion and quantum edge expansion.

Our first result is that such statements, showing that dimension expansion implies quantum expansion or quantum edge expansion, are *false*. Indeed, we show the existence of dimension expanders that are arbitrarily poor quantum expanders (and thus arbitrarily poor quantum edge expanders). Moreover, we are able to explain “why” no such equivalence holds: it is because there is a “missing” fourth notion of linear-algebraic expansion, which we term *dimension edge expansion*. This is yet another natural linear-algebraic analogue of edge expansion, which had not been previously defined. For this notion, it is straightforward to show an analogue of Proposition 1.1(1), proving that dimension expansion and dimension edge expansion are equivalent. Additionally, we prove that quantum edge expansion implies dimension edge expansion.

To understand what all these implications mean, consider Figure 1 above, which clarifies the conceptual value of the new definition. We stress, as the figure suggests, that both linear-algebraic notions of edge expansion specialize to the same graph-theoretic one. Additionally, the figure shows the two equivalences discussed above, namely that both “quantum” notions are equivalent, and both “dimension” notions are equivalent. Moreover, it depicts the fact that quantum edge expansion implies dimension edge expansion. This connection yields a new proof of the Lubotzky–Zelmanov

³e.g. monotone expanders [DS11, BY13, DW10].

result discussed above; moreover, our new proof is more modular, and gives a result that is both qualitatively and quantitatively stronger.

Finally, as depicted in the figure, our negative result shows that there is no reverse implication, as there exist dimension expanders which are not quantum expanders. Said differently, in the linear-algebraic setting, the “quantum” notions of expansion are strictly stronger than the “dimension” notions. We stress again the surprising consequence: although both linear-algebraic notions of edge expansion generalize the *same* graph-theoretic notion, they are not equivalent.

We now turn to the formal definitions of these linear-algebraic notions. Once the definitions are in place, we can state our main theorems.

1.2 Definitions of linear-algebraic expansion

Throughout, our main object of study will be a *matrix tuple* $\mathbf{B} = (B_1, \dots, B_d) \in M(n, \mathbb{F})^d$, where \mathbb{F} is some field and $M(n, \mathbb{F})$ denotes the space of $n \times n$ matrices over \mathbb{F} . The above notions of linear-algebraic expansion will all be properties of such matrix tuples \mathbf{B} . The analogies to graphs will be apparent when considering \mathbf{B} as a tuple of *permutation* matrices, which naturally define a d -regular graph⁴.

1.2.1 Quantum expansion and quantum edge expansion

When working with quantum expanders and quantum edge expanders, we work with the field $\mathbb{F} = \mathbb{C}$. Additionally, rather than working with arbitrary matrix tuples, we work with *doubly stochastic matrix tuples*, which are those tuples $\mathbf{B} = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d$ with $\sum_{i=1}^d B_i B_i^* = \sum_{i=1}^d B_i^* B_i = dI_n$, where I_n is the $n \times n$ identity matrix. An important special case is that of *unitary matrix tuples*, where each B_i is a unitary matrix.

Definition 1.2. Given a doubly stochastic matrix tuple $\mathbf{B} = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d$, the *associated quantum operator* is the linear map $\Phi_{\mathbf{B}} : M(n, \mathbb{C}) \rightarrow M(n, \mathbb{C})$ defined by

$$\Phi_{\mathbf{B}}(X) = \frac{1}{d} \sum_{i=1}^d B_i X B_i^*.$$

The quantum operator $\Phi_{\mathbf{B}}$ should be thought of as an analogue of the normalized adjacency matrix A of a graph G . Indeed, it is straightforward to check that if each B_i is a permutation matrix, and if X is a diagonal matrix, then $\Phi_{\mathbf{B}}(X)$ is also diagonal, and the diagonal entries are precisely the entries of Ax , where x is the vector of diagonal entries of X . Similarly, it is easy to verify that the largest eigenvalue of $\Phi_{\mathbf{B}}$ is 1, with eigenvector I_n . Continuing the analogy, we define the *Laplacian operator* by $\Lambda_{\mathbf{B}} := \mathcal{I} - \Phi_{\mathbf{B}}$, where \mathcal{I} is the identity map $M(n, \mathbb{C}) \rightarrow M(n, \mathbb{C})$.

Based on this analogy between quantum operators and adjacency matrices, the following definition⁵ from [Has07, BATS07] is a natural analogue of the definition of spectral expansion of a graph.

⁴For it to define an undirected graph, it must be a symmetric set of permutations, but this is essentially without loss of generality. Additionally, it is well-known [Gro77] that any d -regular graph can be decomposed as a union of d permutations (at least when d is even), so we may always view a d -regular graph as a tuple of permutation matrices.

⁵We remark that often, quantum expansion is defined for a quantum operator, rather than for a tuple of matrices, where a *quantum operator* is defined abstractly as a linear map $M(n, \mathbb{C}) \rightarrow M(n, \mathbb{C})$ satisfying certain properties. However, it is well-known [NC00, Theorem 8.2] that any quantum operator arises in an essentially unique way from a doubly stochastic matrix tuple, so the two perspectives are equivalent.

Definition 1.3. Given a doubly stochastic matrix tuple $\mathbf{B} = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d$, its *quantum expansion*, $\lambda(\mathbf{B})$, is defined to be the second-smallest singular value⁶ of $\Lambda_{\mathbf{B}}$.

For fixed d and $\lambda > 0$, we say that a family of doubly stochastic matrix tuples $\{\mathbf{B}_n = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d \mid n \in \mathbb{N}\}$ is an (n, d, λ) -*quantum expander* if $\lambda(\mathbf{B}_n) \geq \lambda$ for all $n \in \mathbb{N}$.

Similarly, the following definition from [Has07] is a linear-algebraic analogue of the edge expansion of a graph.

Definition 1.4. Given a doubly stochastic matrix tuple $\mathbf{B} = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d$, its *quantum edge expansion* is defined as

$$h_Q(\mathbf{B}) := \min_{\substack{V \leq \mathbb{C}^n \\ 1 \leq \dim(V) \leq \frac{n}{2}}} \frac{\langle I_n - P_V, \Phi_{\mathbf{B}}(P_V) \rangle}{\dim(V)}, \quad (1.4)$$

where P_V is the orthogonal projection onto the subspace $V \leq \mathbb{C}^n$, and where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $M(n, \mathbb{C})$.

For fixed d and $h > 0$, we say that a family of doubly stochastic matrix tuples $\{\mathbf{B}_n = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d \mid n \in \mathbb{N}\}$ is an (n, d, h) -*quantum edge expander* if $h_Q(\mathbf{B}_n) \geq h$ for all $n \in \mathbb{N}$.

Suppose again that each B_i is a permutation matrix. Let $W \subseteq [n]$, and suppose that $V = \langle e_j \rangle_{j \in W}$ is a *coordinate subspace*, spanned by a subset of the standard basis $\{e_1, \dots, e_n\}$ of \mathbb{C}^n . Then P_V is simply a diagonal matrix with a 1 in positions indexed by W and a 0 elsewhere. Similarly, as discussed above, $\Phi_{\mathbf{B}}(P_V)$ is another diagonal matrix, whose diagonal entries are precisely the entries of $A\mathbf{1}_W$, where A is the normalized adjacency matrix of G and $\mathbf{1}_W$ is the indicator vector of W . Therefore, $\langle I_n - P_V, \Phi_{\mathbf{B}}(P_V) \rangle$ is equal to $\mathbf{1}_{[n] \setminus W} A \mathbf{1}_W$, which in turn equals $\frac{1}{d} |\partial W|$. Thus, when \mathbf{B} comprises permutation matrices and when we restrict the minimum to coordinate subspaces $V \leq \mathbb{C}^n$, the definition of quantum edge expansion precisely recovers the definition of edge expansion of a graph.

1.2.2 Dimension expanders and dimension edge expanders

Dimension expansion and dimension edge expansion are well-defined over any field, but for simplicity, we continue working with $\mathbb{F} = \mathbb{C}$ for the moment.

Given a tuple of matrices $\mathbf{B} = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d$, the image of $V \leq \mathbb{C}^n$ under \mathbf{B} is $\mathbf{B}(V) := \langle \cup_{i \in [d]} B_i(V) \rangle$, where $B_i(V) := \{B_i v : v \in V\}$ and $\langle \cdot \rangle$ denotes linear span over \mathbb{C} .

Definition 1.5. The *dimension expansion* of a matrix tuple $\mathbf{B} = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d$ is defined as

$$\mu(\mathbf{B}) := \min_{\substack{V \leq \mathbb{C}^n \\ 1 \leq \dim(V) \leq \frac{n}{2}}} \frac{\dim(V + \mathbf{B}(V)) - \dim(V)}{\dim(V)}. \quad (1.5)$$

For fixed d and $\mu > 0$, we say that a family of matrix tuples $\{\mathbf{B}_n = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d \mid n \in \mathbb{N}\}$ is an (n, d, μ) -*dimension expander* if $\mu(\mathbf{B}_n) \geq \mu$ for all $n \in \mathbb{N}$.

⁶In general, $\Phi_{\mathbf{B}}$ (and thus also $\Lambda_{\mathbf{B}}$) may not be a self-adjoint operator on $M(n, \mathbb{C})$, and hence its eigenvalues may not be real; this is why we restrict our attention to singular values. In many special cases (e.g. if all the matrices B_i are Hermitian, or if the tuple \mathbf{B} is a symmetric set of permutation matrices), $\Phi_{\mathbf{B}}$ and $\Lambda_{\mathbf{B}}$ are self-adjoint, and then we may equivalently define $\lambda(\mathbf{B})$ as the second-smallest eigenvalue of $\Lambda_{\mathbf{B}}$.

To see the analogy to vertex expansion in graphs, suppose that each B_i is a permutation matrix, let $W \subseteq [n]$, and suppose that $V = \langle e_j \rangle_{j \in W}$ is a coordinate subspace. Then $B_i(V)$ is another coordinate subspace, spanned by the images of W under the permutation B_i . Therefore, $\dim(V + \mathbf{B}(V)) - \dim(V)$ precisely counts how many elements in the complement of W are in the image of some B_i . In other words, if we restrict the definition of dimension expansion to tuples of permutation matrices, and restrict the minimum to coordinate subspaces $V \leq \mathbb{C}^n$, we precisely recover the definition of vertex expansion in graphs.

The same perspective motivates our new definition of dimension edge expanders. To define them, we first need to define the restriction of a matrix to a subspace. Let $V \leq \mathbb{C}^n$ be a subspace of dimension r , and let $T \in M(n \times r, \mathbb{C})$ be a matrix whose columns form an orthonormal basis of V . Similarly, let $R \in M((n-r) \times n, \mathbb{C})$ be a matrix whose rows form an orthonormal basis of V^\perp , the orthogonal complement of V . Given $B \in M(n, \mathbb{C})$, its restriction to (V^\perp, V) is defined by $B|_{V^\perp, V} := RBT$. Note that while T and R are not unique, a different choice of T and R would give rise to a matrix that is equivalent to $B|_{V^\perp, V}$ up to multiplication by unitary matrices (and thus in particular has the same rank).

With this setup, we can define dimension edge expansion.

Definition 1.6. For a matrix tuple $\mathbf{B} = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d$, the *dimension edge expansion* of \mathbf{B} is defined as

$$h_D(\mathbf{B}) = \min_{\substack{V \leq \mathbb{C}^n \\ 1 \leq \dim(V) \leq \frac{n}{2}}} \frac{\sum_{i=1}^d \text{rank}(B_i|_{V^\perp, V})}{d \cdot \dim(V)}. \quad (1.6)$$

For fixed d and $h > 0$, we say that a family of matrix tuples $\{\mathbf{B}_n = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d \mid n \in \mathbb{N}\}$ is an (n, d, h) -*dimension edge expander* if $h_D(\mathbf{B}_n) \geq h$ for all $n \in \mathbb{N}$.

Again suppose that $V = \langle e_i \rangle_{i \in W}$ is a coordinate subspace corresponding to some $W \subseteq [n]$. Then $B_i|_{V^\perp, V}$ is simply the submatrix of B_i with columns indexed by W and rows indexed by $[n] \setminus W$. In case B_i is a permutation matrix, the rank of this submatrix is precisely the number of non-zero entries in it, which is equal to the number of elements of W mapped to $[n] \setminus W$ by the permutation B_i . In other words, if we restrict our attention to permutation matrices and to coordinate subspaces, we again recover the definition of edge expansion of graphs.

Remark 1.7. To extend the definition of dimension edge expansion to any field \mathbb{F} , we denote by V^\perp the *annihilator* of $V \leq \mathbb{F}^n$ with respect to the standard dot product, namely $V^\perp := \{u \in \mathbb{F}^n \mid u^t v = 0 \forall v \in V\}$. Then let $T \in M(n \times r, \mathbb{F})$ be a matrix whose columns form a basis of V , and $R \in M((n-r) \times n, \mathbb{F})$ whose rows form a basis of V^\perp . Given $B \in M(n, \mathbb{F})$, its restriction to (V^\perp, V) is defined by $B|_{V^\perp, V} := RBT$. Dimension edge expansion is now defined identically to Definition 1.6.

1.3 Main results

As discussed in Section 1.1, our results demonstrate that in the linear-algebraic setting, the relationship between the various notions of expansion is substantially subtler than it is in the graph-theoretic setting. Our first main result is negative: it says that in general, a matrix tuple can be a good dimension expander while having arbitrarily poor quantum expansion.

Theorem 1.8. *A dimension expander may be an arbitrarily poor quantum expander. More precisely, there are constants $\mu > 0$ and $d \in \mathbb{N}$ so that for all $\varepsilon > 0$ and all sufficiently large n , there exists a unitary matrix tuple $\mathbf{B} = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d$ such that $\mu(\mathbf{B}) \geq \mu$ but $\lambda(\mathbf{B}) < \varepsilon$.*

In fact, we prove something stronger: given any unitary matrix tuple $\mathbf{B}_0 \in M(n, \mathbb{C})^d$, we can find another unitary matrix tuple $\mathbf{B} \in M(n, \mathbb{C})^d$ so that $\mu(\mathbf{B}) \geq \mu(\mathbf{B}_0)/d$ and $\lambda(\mathbf{B}) < \varepsilon$. Moreover \mathbf{B} is obtained from \mathbf{B}_0 in a very simple way: each matrix in \mathbf{B} is simply a very small fractional power of the corresponding matrix in \mathbf{B}_0 . Perhaps surprisingly, our proof uses a number of compactness arguments, and thus gives no quantitative information on how small this fractional power must be.

In the other direction, we have a number of relationships between the various notions of linear-algebraic expansion, in analogy to Proposition 1.1.

Theorem 1.9. *Let $\mathbf{B} = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d$ be a doubly stochastic matrix tuple. Then we have*

- (1) $\frac{\mu(\mathbf{B})}{d} \leq h_D(\mathbf{B}) \leq \mu(\mathbf{B});$
- (2) $\frac{\lambda(\mathbf{B})}{2} \leq h_Q(\mathbf{B}) \leq \sqrt{2\lambda(\mathbf{B})};$
- (3) $h_Q(\mathbf{B}) \leq d \cdot h_D(\mathbf{B}).$

In case \mathbf{B} is a unitary matrix tuple, we may replace (3) by the stronger bound

- (4) $h_Q(\mathbf{B}) \leq h_D(\mathbf{B}).$

In fact, Theorem 1.9(1) holds for any matrix tuple over any field; the assumptions that we are working over \mathbb{C} and have a doubly stochastic matrix tuple are only necessary so that $\lambda(\mathbf{B})$ and $h_Q(\mathbf{B})$ are well-defined. As mentioned above, Theorem 1.9(2) is not new, and was proved by Hastings [Has07, Appendix A]. So we do not prove Theorem 1.9(2) in this paper, but we include it in the statement of Theorem 1.9 in order to make the analogy to Proposition 1.1 as transparent as possible.

Speaking of Proposition 1.1, the key thing to stress about the difference between the graph-theoretic and linear-algebraic settings is that the *single* graph-theoretic notion of edge expansion, which appears in both Propositions 1.1(1) and 1.1(2), corresponds to *two distinct* linear-algebraic notions, namely dimension edge expansion and quantum edge expansion. That these two quantities are not equivalent is implied by Theorem 1.8, which gives a separation between μ and λ . However, the fact that there is an inequality relating them—Theorem 1.9(3)—shows that quantum expansion implies dimension expansion, as stated in the next corollary.

Corollary 1.10. *Every quantum expander is a dimension expander. More precisely, if $\mathbf{B} \in M(n, \mathbb{C})^d$ is a doubly stochastic matrix tuple, then*

$$\mu(\mathbf{B}) \geq \frac{\lambda(\mathbf{B})}{2d}.$$

In case \mathbf{B} is a unitary matrix tuple, we have the stronger bound

$$\mu(\mathbf{B}) \geq \frac{\lambda(\mathbf{B})}{2}.$$

Proof. We have that

$$\mu(\mathbf{B}) \geq h_D(\mathbf{B}) \geq \frac{h_Q(\mathbf{B})}{d} \geq \frac{\lambda(\mathbf{B})}{2d},$$

where the three inequalities follow from the first three parts of Theorem 1.9. In case \mathbf{B} is unitary, we may replace the second inequality above by $h_D(\mathbf{B}) \geq h_Q(\mathbf{B})$, thanks to Theorem 1.9(4). \square

As discussed above, a similar result was proved by Lubotzky and Zelmanov [LZ08]⁷.

Theorem 1.11 ([LZ08, Proposition 2.1]). *Let $\mathbf{B} = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d$ be a unitary matrix tuple. Then*

$$\mu(\mathbf{B}) \geq \frac{\lambda(\mathbf{B})}{6}.$$

Lubotzky and Zelmanov only stated this result for matrix tuples arising from irreducible unitary representations of finite groups, but their proof actually works in the generality of arbitrary unitary matrix tuples.

Note that our result (Corollary 1.10) is both quantitatively stronger (in that we obtain a better constant factor) and qualitatively stronger (as we obtain a result for all doubly stochastic matrix tuples, not only unitary tuples). Additionally, we believe that our proof is conceptually simpler than that of [LZ08], since the implication from quantum to dimension expansion naturally breaks into three simpler implications (quantum implies quantum edge, which implies dimension edge, which implies dimension).

Lubotzky and Zelmanov [LZ08] proved Theorem 1.11 in order to explicitly construct dimension expanders over \mathbb{C} . Indeed, this motivation is a natural reason to study the relationships between linear-algebraic notions of expansion: Corollary 1.10 implies that any construction of a quantum expander also yields a construction of a dimension expander, whereas Theorem 1.8 shows that the converse does not hold. We remark that explicit constructions of quantum and dimension expanders is an important and highly active area of research, which we do not discuss further, except to stress that this is a very natural reason to study connections between different notions of expansion.

1.4 Connections between graphs and matrix spaces

There is a natural way of associating a matrix tuple to a graph. Namely, given a d -regular graph $G = ([n], E)$, we define the *graphical matrix tuple* to be the matrix tuple⁸ $\mathbf{B}_G = (\sqrt{n}E_{i,j} : \{i, j\} \in E)$, where $E_{i,j}$ is the elementary matrix with a 1 in position (i, j) and zeros in all other entries. Note that \mathbf{B}_G is a tuple of $2|E| = dn$ matrices, and thus is not particularly natural from the perspective of expansion: we are usually interested in families of matrix tuples where the length of the tuple stays constant as the dimension of the matrices grows. Here, even if the degree d of G is fixed, the number of matrices in \mathbf{B}_G tends to infinity with n .

Nonetheless, the construction of \mathbf{B}_G is very natural from other perspectives. Indeed, the matrices in \mathbf{B}_G form the standard basis for the *graphical matrix space* associated to G . In [LQW⁺22], we proved that many important graph-theoretic properties of G are equivalent to linear-algebraic properties of \mathbf{B}_G . In particular, we proved a number of results that we term *inherited correspondences*, which say that the value of a certain optimum associated to G is equal to a related optimum associated to \mathbf{B}_G , even though the feasible region in the latter optimum is generally much larger; for example, an optimum over all subgraphs of G may equal an optimum over all subspaces of $\langle \mathbf{B}_G \rangle$, even though there are many more subspaces than subgraphs.

As it turns out, the various expansion parameters give us a number of other results of this type. The first of these is due to Bannink, Briët, Labib, and Maassen [BBLM20], who proved that the quantum expansion of \mathbf{B}_G equals the spectral expansion of G .

⁷Lubotzky and Zelmanov were unaware of the notion of quantum expansion, and thus did not state this result in this language. Harrow [Har08] was the first to observe that their approach passes through quantum expansion as an intermediate step.

⁸Note that we scale $E_{i,j}$ by a factor of \sqrt{n} in order to ensure that \mathbf{B}_G is a doubly stochastic matrix tuple, as $\sum_{\{i,j\} \in E} (\sqrt{n}E_{i,j})(\sqrt{n}E_{i,j})^* = \sum_{\{i,j\} \in E} (\sqrt{n}E_{i,j})^*(\sqrt{n}E_{i,j}) = dnI_n$.

Theorem 1.12 ([BBLM20, Proposition 3.7]). *For every d -regular graph G , $\lambda(G) = \lambda(\mathbf{B}_G)$.*

We prove analogous results for dimension expansion and dimension edge expansion.

Theorem 1.13. *For every d -regular graph G , $\mu(G) = \mu(\mathbf{B}_G)$ and $h(G) = h_D(\mathbf{B}_G)$.*

In contrast, no such equality holds for quantum edge expansion. In fact, it fails already for the simplest possible graph, consisting of two vertices and a single edge.

Proposition 1.14. *$h(K_2) = 1$, but $h_Q(\mathbf{B}_{K_2}) \leq \frac{1}{2}$.*

Using the same proof technique, one can check that $h_Q(\mathbf{B}_{K_n}) < h(K_n)$ for all $n \geq 2$. The fact that $h(G)$ and $h_Q(\mathbf{B}_G)$ are different in general may shed further light on our main results in Theorems 1.8 and 1.9, that dimension edge expansion and quantum edge expansion are different. However, we stress that these simple examples have no direct bearing on Theorems 1.8 and 1.9, as \mathbf{B}_G is not a matrix tuple of constant length, which is the setting in which Theorems 1.8 and 1.9 are interesting.

Remark 1.15. We include the \sqrt{n} normalization in the definition of \mathbf{B}_G in order to obtain a doubly stochastic tuple. When working with a non- d -regular graph or over a field other than \mathbb{C} , one should omit this normalization; with this definition, Theorem 1.13 holds for any graph G and over any field.

1.5 Notation and basic definitions

For $n \in \mathbb{N}$, $[n] := \{1, \dots, n\}$. Throughout, we work with finite-dimensional vector spaces over a field \mathbb{F} . The elements of \mathbb{F}^n are length- n column vectors over \mathbb{F} . The linear space of $n' \times n$ matrices over \mathbb{F} is denoted $M(n' \times n, \mathbb{F})$. For simplicity we shall write $M(n, \mathbb{F})$ for $M(n \times n, \mathbb{F})$. We use $E_{i,j} \in M(n' \times n, \mathbb{F})$ to denote the $n' \times n$ elementary matrix where the (i, j) th entry is 1, and other entries are 0.

We use $\langle x, y \rangle$ to denote inner products. In particular, for $x = (x_1, \dots, x_n)^t, y = (y_1, \dots, y_n)^t \in \mathbb{C}^n$, their inner product is $\langle x, y \rangle := \sum_{i=1}^n \overline{x_i} y_i$. For every $p \in [1, \infty)$, the L_p -norm of $x \in \mathbb{C}^n$ is defined as

$$\|x\|_{L_p} := \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$

Define $\|x\|_{L_\infty} := \max_i |x_i|$.

We use $\text{Tr}(\cdot)$ to denote the usual trace function on $M(n, \mathbb{F})$. For $X, Y \in M(n, \mathbb{C})$, their inner product is $\langle X, Y \rangle := \text{Tr}(X^* Y)$. For every $p \in [1, \infty)$, the Schatten- p norm of $X \in M(n, \mathbb{C})$ is defined as

$$\|X\|_{S_p} := \left(\text{Tr} \left[(X^* X)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}}.$$

Define $\|X\|_{S_\infty} := \sup\{|\langle x, Xy \rangle| : \|x\|_{L_2}, \|y\|_{L_2} \leq 1\}$, which is the operator norm of X .

1.6 Organization

In Section 2, we prove Theorem 1.8, showing that a small fractional power of a dimension expander yields another dimension expander which is an arbitrarily poor quantum expander. In Section 3, we prove Theorem 1.9, showing that dimension and dimension edge expansion are equivalent, and that quantum edge expansion implies dimension edge expansion. In Section 4, we prove Theorem 1.13 and Proposition 1.14, relating graph expansion and linear-algebraic expansion of the associated graphical matrix tuple. We conclude in Section 5 with some open problems and concluding remarks.

2 Dimension expanders that are not quantum expanders

In this section, we prove Theorem 1.8. Our main technical result is the following, which says that given an arbitrary unitary matrix tuple, any sufficiently small fractional power of it has equal or greater dimension expansion. As it is easy to show that a small fractional power of any unitary matrix tuple has poor quantum expansion, it becomes straightforward to deduce Theorem 1.8, as we show below.

Recall that given a unitary matrix U , there exists a Hermitian matrix H such that $U = e^{iH}$. Using this, we can define arbitrary powers of U : for any $s > 0$, we define $U^s := e^{isH}$. Note that this notion depends on the choice of H , and a different choice of H may yield a different outcome, but this will not matter for us. So for any unitary matrix U , we fix a choice of Hermitian H such that $U = e^{iH}$, and then define powers of U with respect to this choice.

Theorem 2.1. *Let $\mathbf{U} = (U_1, \dots, U_d) \in \mathbf{U}(n)^d$ be a unitary tuple of matrices. There exists some $s_0 > 0$ such that for all $s \in (0, s_0)$, the unitary matrix tuple $\mathbf{U}^s = (U_1^s, \dots, U_d^s) \in \mathbf{U}(n)^d$ satisfies $\mu(\mathbf{U}^s) \geq \mu(\mathbf{U})/d$.*

Assuming Theorem 2.1, we can prove Theorem 1.8.

Proof of Theorem 1.8. We recall the variational definition of quantum expansion: for any doubly stochastic matrix tuple \mathbf{B} , we have that

$$\lambda(\mathbf{B}) = 1 - \max_{\substack{0 \neq X \in \mathbf{M}(n, \mathbb{C}) \\ \text{Tr}(X) = 0}} \frac{\|\Phi_{\mathbf{B}}(X)\|_{S_2}}{\|X\|_{S_2}}. \quad (2.1)$$

Indeed, we first note that the second-smallest singular value of $\Lambda_{\mathbf{B}}$ equals one minus the second-largest singular value of $\Phi_{\mathbf{B}}$. Since the largest eigenvalue of $\Phi_{\mathbf{B}}$ is 1, with I_n as both a left and a right eigenvector, we know that the second-largest singular value of $\Phi_{\mathbf{B}}$ is simply the operator norm of $\Phi_{\mathbf{B}}$ when restricted to the orthogonal complement of I_n . The orthogonal complement of I_n is the set of matrices X with $\text{Tr}(X) = 0$, and thus the maximum in (2.1) precisely defines the second-largest singular value of $\Phi_{\mathbf{B}}$. In particular, by the triangle inequality, we see that

$$\lambda(\mathbf{B}) = \max_{\substack{0 \neq X \in \mathbf{M}(n, \mathbb{C}) \\ \text{Tr}(X) = 0}} \frac{\|X\|_{S_2} - \|\Phi_{\mathbf{B}}(X)\|_{S_2}}{\|X\|_{S_2}} \leq \max_{\substack{0 \neq X \in \mathbf{M}(n, \mathbb{C}) \\ \text{Tr}(X) = 0}} \frac{\|X - \Phi_{\mathbf{B}}(X)\|_{S_2}}{\|X\|_{S_2}}. \quad (2.2)$$

It is well-known that for constant $d \in \mathbb{N}$ and $\mu > 0$, there exist unitary $(n, d, d\mu)$ -dimension expanders for all sufficiently large n (e.g. by taking 100 random $n \times n$ unitary matrices). So fix some $\mathbf{U} = (U_1, \dots, U_d) \in \mathbf{U}(n)^d$ with $\mu(\mathbf{U}) \geq d\mu$. By Theorem 2.1, we know that \mathbf{U}^s is still an (n, d, μ) -dimension expander for any sufficiently small s . Given $\varepsilon > 0$, we claim that $\lambda(\mathbf{U}^s) < \varepsilon$ for sufficiently small s .

Note that as s tends to 0, \mathbf{U}^s approaches the identity tuple (I_n, \dots, I_n) . More precisely, for any $\varepsilon > 0$ and any $i \in [d]$, there exists a sufficiently small s such that

$$\|I_n - U_i^s\|_{S_2} < \frac{\varepsilon}{2} \quad \text{and} \quad \|I_n - (U_i^s)^*\|_{S_2} < \frac{\varepsilon}{2}. \quad (2.3)$$

Note that the two conditions are actually equivalent, as the Schatten norm is invariant under taking conjugate transpose. Now fix s sufficiently small so that (2.3) holds for all $i \in [d]$, and so that $\mu(\mathbf{U}^s) \geq \mu(\mathbf{U})/d \geq \mu$. It remains to prove that for this choice of s , we have $\lambda(\mathbf{U}^s) < \varepsilon$.

Fix some $0 \neq X \in M(n, \mathbb{C})$ with $\text{Tr}(X) \neq 0$. By our choice of s , we have that

$$\begin{aligned} \frac{\|X - \Phi_{\mathbf{U}^s}(X)\|_{S_2}}{\|X\|_{S_2}} &= \frac{\|\sum_{i=1}^d (X - U_i^s X (U_i^s)^*)\|_{S_2}}{d\|X\|_{S_2}} \\ &\leq \sum_{i=1}^d \frac{\|X(I_n - (U_i^s)^*) + (I_n - U_i^s)X(U_i^s)^*\|_{S_2}}{d\|X\|_{S_2}} \end{aligned} \quad (2.4)$$

$$\leq \sum_{i=1}^d \frac{\|X(I_n - (U_i^s)^*)\|_{S_2} + \|(I_n - U_i^s)X(U_i^s)^*\|_{S_2}}{d\|X\|_{S_2}} \quad (2.5)$$

$$= \sum_{i=1}^d \frac{\|X(I_n - (U_i^s)^*)\|_{S_2} + \|(I_n - U_i^s)X\|_{S_2}}{d\|X\|_{S_2}} \quad (2.6)$$

$$\leq \sum_{i=1}^d \frac{\|I_n - (U_i^s)^*\|_{S_2} + \|I_n - U_i^s\|_{S_2}}{d} \quad (2.7)$$

$$< \varepsilon, \quad (2.8)$$

where (2.4) and (2.5) use the triangle inequality, (2.6) uses the fact that Schatten norms are invariant under unitary multiplications, (2.7) uses the submultiplicativity of the Schatten norm, and (2.8) uses our assumption that (2.3) holds for all $i \in [d]$. By taking the maximum over all such X , we conclude that for sufficiently small s , the tuple \mathbf{U}^s is an (n, d, μ) -dimension expander, but $\lambda(\Phi_{\mathbf{U}^s}) < \varepsilon$. \square

In order to prove Theorem 2.1, we will need the following lemma. It says that for a *fixed* subspace $W \leq \mathbb{C}^n$, taking a small power of a matrix U does not “ruin the dimension expansion”. Namely, if there is a subspace $V \leq W$ so that $UV \cap W = \{0\}$, then $U^s V \cap W = \{0\}$ for all sufficiently small s . This almost suffices to prove Theorem 2.1, except that the “sufficiently small” condition depends on V and W ; in order to obtain an absolute bound that is independent of these choices, we will use a compactness argument.

Lemma 2.2. *Let U be a unitary matrix and let $V \leq W \leq \mathbb{C}^n$ be subspaces. Suppose that $Uv \notin W$ for all non-zero $v \in V$. There exists some $\delta > 0$ so that $U^s v \notin W$ for all $0 < s < \delta$.*

Here, as before, the matrix power U^s is defined by fixing a Hermitian matrix H so that $U = e^{iH}$, and then defining $U^s := e^{isH}$. At a high level, Lemma 2.2 is proved as follows. For a fixed non-zero $v \in V$, we know that $Uv \notin W$. Consider the “trajectory” $U^s v$, viewed as a function of s . We know that this trajectory eventually leaves W (as $Uv \notin W$); let us assume for the moment that in fact, the first derivative of the trajectory has a non-zero component in W^\perp . This means that for sufficiently small s , the trajectory $U^s v$ is well-approximated by a line which does not lie in W , and thus for some small but positive amount of time, this trajectory stays outside W . By bounding the second derivative of the trajectory, this argument can be made rigorous.

Unfortunately, it need not be the case that the first derivative of the trajectory lies outside W , so one cannot just apply the argument as sketched above. However, since $Uv \notin W$ for all $v \in V$, we see that for each $v \in V$, *some* derivative of the trajectory lies outside W . By performing a backwards induction on the order of the first derivative which lies outside W (and at each step repeating the argument above), one can prove Lemma 2.2. Here are the details.

Proof of Lemma 2.2. Fix a non-zero vector $v \in V$. For $s \geq 0$, we write $U^s v$ in terms of the Taylor

expansion of e^{isH} , i.e.

$$U^s v = e^{isH} v = \sum_{j=0}^{\infty} \frac{1}{j!} (isH)^j v = \sum_{j=0}^{\infty} \frac{(is)^j}{j!} H^j v.$$

Note that if $H^k v \in W$ for all $k \geq 0$, then $U^s v \in W$ for all $s \geq 0$, and in particular $Uv \in W$, which contradicts our assumption. Therefore, we see that for every $v \in V$, there is some integer $k \geq 0$ so that $H^j v \in W$ for all $0 \leq j \leq k$, but $H^{k+1} v \notin W$.

For $k \geq 0$, let $V_k \leq V$ be the set of $v \in V$ so that $H^j v \in W$ for all $0 \leq j \leq k$. Note that V_k is a subspace of V , and they are nested as $V = V_0 \geq V_1 \geq V_2 \geq \dots$. By the discussion above, we know that $\bigcap_{k \geq 0} V_k = \{0\}$. Additionally, since V is finite-dimensional, we see that this chain eventually stabilizes at 0, i.e., there is some K so that $V_{K-1} \neq \{0\}$ but $V_K = \{0\}$. We will now prove the following claim by induction on k , starting at $k = K$ and working down to $k = 1$.

Claim 2.3. *For every $1 \leq k \leq K$, there exists some $\delta_k > 0$ so that the following holds. For every $v \in V_{k-1}$ with $\|v\|_{L_2} = 1$, we have that $U^s v \notin W$ for all $s \in (0, \delta_k)$.*

Note that the $k = 1$ case of this claim is simply the desired lemma statement, since $V_0 = V$ and since we lose nothing by restricting to vectors of norm 1.

Proof of Claim 2.3. We prove the claim by (reverse) induction on k . For the base case of $k = K$, let $v \in V_{K-1}$ be a vector of norm 1. Let $P \in M(n, \mathbb{C})$ denote the orthogonal projection onto the orthogonal complement of W . Observe that for any $s > 0$, we have

$$PU^s v = \sum_{j=0}^{\infty} \frac{(is)^j}{j!} PH^j v = \sum_{j=K}^{\infty} \frac{(is)^j}{j!} PH^j v,$$

since $H^j v \in W$ for all $j < K$, and thus $PH^j v = 0$ for all $j < K$. For any $s \in (0, 1)$, we have that

$$\left\| \frac{(is)^{K+1}}{(K+1)!} PH^{K+1} v \right\|_{L_2} = \frac{s^{K+1}}{(K+1)!} \|PH^{K+1} v\|_{L_2} \leq \frac{1}{(K+1)!} \|H^{K+1} v\|_{L_2} \leq \frac{\|H\|_{S_\infty}^{K+1}}{(K+1)!} =: C_K,$$

where we use that $s \leq 1$ and that P is a contraction in the first inequality, and the definition of the Schatten- ∞ norm and the assumption $\|v\|_{L_2} = 1$ in the second inequality. Therefore, by Taylor's theorem, we find that for all $s \in (0, 1)$ and all $v \in V_{K-1}$ with $\|v\|_{L_2} = 1$, we have that

$$\|PU^s v\|_{L_2} \geq \left\| \frac{(is)^K}{K!} PH^K v \right\|_{L_2} - C_K s^{K+1} = \frac{s^K}{K!} \|PH^K v\|_{L_2} - C_K s^{K+1}.$$

Note that the function $v \mapsto \|PH^K v\|_{L_2}$ is a continuous real-valued function on the unit sphere in V_{K-1} , which is compact. Moreover, since $V_K = \{0\}$, we know that $H^K v \notin W$ for all non-zero $v \in V_{K-1}$, and thus $\|PH^K v\|_{L_2}$ is strictly positive for all $v \in V_{K-1}$ with $\|v\|_{L_2} = 1$. Therefore, there exists some $c_K > 0$ so that $\|PH^K v\|_{L_2} \geq K! c_K$ for all $v \in V_{K-1}$ with $\|v\|_{L_2} = 1$. Continuing our computation above, we conclude that

$$\|PU^s v\|_{L_2} \geq c_K s^K - C_K s^{K+1}$$

for all $s \in (0, 1)$ and all $v \in V_{K-1}$ with $\|v\|_{L_2} = 1$. If we let $\delta_K = \min\{c_K/C_K, 1\}$, then this implies that $\|PU^s v\|_{L_2} > 0$ for all $s \in (0, \delta_K)$. This is equivalent to saying that $U^s v \notin W$ for all $s \in (0, \delta_K)$, which proves the claim for $k = K$.

We now move to the inductive step. There is nothing to prove if $V_{k-1} = V_k$, so we may assume that V_k is a proper subspace of V_{k-1} . Inductively, suppose we know the claim holds for $k+1$, namely that there exists some $\delta_{k+1} > 0$ so that $U^s v \notin W$ for all $v \in V_k$ with $\|v\|_{L_2} = 1$ and all $s \in (0, \delta_{k+1})$. As $v \mapsto U^s v$ is a continuous map, and as W is closed, we conclude that $U^s v \notin W$ for all $s \in (0, \delta_{k+1})$ and all v which is in a sufficiently small open neighborhood of the unit sphere in V_k . More precisely, there exists some $\varepsilon_k > 0$ so that the following holds for all $v \in V_{k-1}$ with $\|v\|_{L_2} = 1$: Suppose we write $v = u + w$ where $u \in V_k^\perp$ and $w \in V_k$, and suppose that $\|u\|_{L_2} < \varepsilon_k$. Then $U^s v \notin W$ for all $s \in (0, \delta_{k+1})$.

So it suffices to now only consider such v with $\|u\|_{L_2} \geq \varepsilon_k$. Note that for any such v , we have that

$$\|PH^k v\|_{L_2} = \|PH^k(u+w)\|_{L_2} = \|PH^k u\|_{L_2},$$

since $H^k w \in W$, as $w \in V_k$. Now, the set of unit vectors $v \in V_{k-1}$ for which $\|u\|_{L_2} \geq \varepsilon_k$ is compact, and the function $v \mapsto \|PH^k v\|_{L_2}$ is continuous and strictly positive on it. So there exists some $c_k > 0$ so that $\|PH^k v\|_{L_2} \geq k!c_k$ for all such v . The rest of the proof is very similar to the base case. For any $s > 0$, we have that

$$PU^s v = \sum_{j=0}^{\infty} \frac{(is)^j}{j!} PH^j v = \sum_{j=k}^{\infty} \frac{(is)^j}{j!} PH^j v.$$

For any $s \in (0, 1)$, we have that

$$\left\| \frac{(is)^{k+1}}{(k+1)!} PH^{k+1} v \right\|_{L_2} = \frac{s^{k+1}}{(k+1)!} \|PH^{k+1} v\|_{L_2} \leq \frac{1}{(k+1)!} \|H^{k+1} v\|_{L_2} \leq \frac{\|H\|_{S_\infty}^{k+1}}{(k+1)!} =: C_k.$$

By Taylor's theorem, we conclude that if $v = u + w$ is such that $\|u\|_{L_2} \geq \varepsilon_k$, then for any $s \in (0, 1)$,

$$\|PU^s v\|_{L_2} \geq \left\| \frac{(is)^k}{k!} PH^k v \right\|_{L_2} - C_k s^{k+1} = \frac{s^k}{k!} \|PH^k v\|_{L_2} - C_k s^{k+1} \geq c_k s^k - C_k s^{k+1}.$$

Thus, for such v , we see that $U^s v \notin W$ for all $s \in (0, c_k/C_k)$. On the other hand, for those v with $\|u\| < \varepsilon_k$, we know that $U^s v \notin W$ for all $s \in (0, \delta_{k+1})$. Thus, we get the desired result by setting $\delta_k = \min\{c_k/C_k, \delta_{k+1}, 1\}$. \square

As discussed above, the $k = 1$ case of the claim is equivalent to the lemma statement, so this concludes the proof. \square

For $1 \leq r \leq n$, let $\text{Gr}(n, r)$ denote the Grassmannian of r -dimensional subspaces of \mathbb{C}^n . We make the following definition, which will be useful in the proof of Theorem 2.1.

Definition 2.4. Let $\mathbf{U} = (U_1, \dots, U_d) \in \text{U}(n)^d$ be a tuple of $n \times n$ unitary matrices. Given a real number $\mu > 0$ and a subspace $W \in \text{Gr}(n, r)$, let us say that a tuple (V, i, δ) is μ -expansive for W if it satisfies the following conditions.

- (i) V is a subspace of W , $i \in [d]$ is an integer, and $\delta > 0$ is a strictly positive real number.
- (ii) We have $\dim V \geq \mu r/d$.
- (iii) For every $s \in (0, \delta)$, we have that $U_i^s V \cap W = \{0\}$.

Our next simple lemma shows that if \mathbf{U} is a (n, d, μ) -dimension expander, then every subspace has an expansive tuple. The implication is a straightforward consequence of Lemma 2.2, but the language of expansive tuples will be more convenient for the compactness argument we use in the proof of Theorem 2.1.

Lemma 2.5. *Let $\mathbf{U} = (U_1, \dots, U_d) \in \mathbf{U}(n)^d$ be a unitary matrix tuple, and let $\mu > 0$ be a real number. If \mathbf{U} is a (n, d, μ) -dimension expander, then for all $1 \leq r \leq n/2$ and all $W \in \text{Gr}(n, r)$, there is a μ -expansive tuple for W .*

Proof. By the definition of dimension expansion, we know that $\dim(W + U_1W + \dots + U_dW) - \dim(W) \geq \mu r$. Therefore, there exists some i so that $\dim(W + U_iW) - \dim(W) \geq \mu r/d$. Let V be a maximum-dimensional subspace of W with the property that $U_iV \cap W = \{0\}$; then the above implies that $\dim(V) \geq \mu r/d$. Finally, by Lemma 2.2, we see that there exists some $\delta > 0$ so that $U^sV \cap W = \{0\}$ for all $0 < s < \delta$, implying that (V, i, δ) is μ -expansive for W . \square

Now suppose we are given a unitary matrix $U \in \mathbf{U}(n)$ and a subspace $V \leq W$ with $U^sV \cap W = \{0\}$ for all $0 < s < \delta$. Intuitively, the continuity of the map $V \mapsto U^sV$ implies that if we perturb W to a “nearby” subspace W' , we can similarly perturb V to $V' \leq W'$ with the property that $U^sV' \cap W' = \{0\}$ for all $0 < s < \delta$. The following lemma makes this precise, for which it is best to use the language of fiber bundles. Let $\text{Gr}(n, \leq r)$ denote the disjoint union of $\text{Gr}(n, \ell)$ over $0 \leq \ell \leq r$. There is a fiber bundle π over $\text{Gr}(n, r)$ whose fibers are $\text{Gr}(r, \leq r)$, namely above $W \in \text{Gr}(n, r)$ we simply put all possible subspaces of W . More precisely, the total space of the bundle is

$$E = \{(W, V) \in \text{Gr}(n, r) \times \text{Gr}(n, \leq r) : V \text{ is a subspace of } W\},$$

and the bundle map $\pi : E \rightarrow \text{Gr}(n, r)$ is given by $\pi(W, V) = W$.

Lemma 2.6. *Let $W \in \text{Gr}(n, r)$, and let U be an $n \times n$ unitary matrix. Suppose that there exist $\delta > 0$ and a subspace $V \leq W$ so that $U^sV \cap W = \{0\}$ for all $0 < s < \delta$. Then there exists an open set $O \subseteq \text{Gr}(n, r)$ with $W \in O$ and a continuous section $\sigma : O \rightarrow E$ of the fiber bundle π so that $\sigma(W) = V$ and for all $W' \in O$, we have that $U^s\sigma(W') \cap W' = \{0\}$.*

Proof. Let $\ell = \dim V$. For $X \in \text{Gr}(n, r)$, let $\mathcal{V}(X)$ be the collection of ℓ -dimensional subspaces Y of X with the property that $U^sY \cap X = \{0\}$ for all $0 < s < \delta$. As $(s, Y) \mapsto U^sY$ is continuous in both variables, we see that $\mathcal{V}(X)$ is open for all X . Again by continuity, $\mathcal{V}(X)$ also varies continuously as we vary $X \in \text{Gr}(n, r)$. Since $V \in \mathcal{V}(W)$, these properties imply that we can find an open neighborhood O of W and a section σ as claimed. \square

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\mathbf{U} = (U_1, \dots, U_d) \in \mathbf{U}(n)^d$ be a (n, d, μ) -dimension expander. By Lemma 2.5, for every $1 \leq r \leq n/2$ and every $W \in \text{Gr}(n, r)$, we may find an expansive tuple (V, i, δ) for W . By Lemma 2.6, there exists an open neighborhood O_W of W as well as a section $\sigma : O_W \rightarrow E$ of the bundle π so that $U_i^s\sigma_i(W') \cap W' = \{0\}$ for all $W' \in O_W$ and all $0 < s < \delta$.

Now, the collection $\{O_W\}_{W \in \text{Gr}(n, r)}$ forms an open cover of $\text{Gr}(n, r)$, so by compactness, we can find a finite subcover, say O_1, \dots, O_T . By the way we constructed these O_W , we see that there are $\delta_1, \dots, \delta_T > 0$ so that for each $W \in N_j$, there is an expansive tuple for W with $\delta = \delta_j$. By letting $s_0 = \min_j \delta_j$ we conclude that for every $W \in \text{Gr}(n, r)$, there is an expansive tuple for W with $\delta \geq s_0$. In other words, for every W , there exist $i \in [d]$ and $V \leq W$ with $\dim V \geq \mu r/d$ so that $U_i^sV \cap W = \{0\}$ for all $0 < s < s_0$. This implies that $\dim(W + \mathbf{U}^s(W)) - \dim(W) \geq \mu r/d$ for all W . In other words, we see that $\mu(\mathbf{U}^s) \geq \mu/d$ for all $0 < s < s_0$, as claimed. \square

To summarize, we have proven that given *any* dimension expander $\mathbf{U} = (U_1, \dots, U_d) \in \mathbf{U}(n)^d$, two things are simultaneously true. On the one hand, all sufficiently small powers \mathbf{U}^s remain dimension expanders. On the other hand, as $s \rightarrow 0$, the tuple \mathbf{U}^s converges to the identity tuple, and thus a sufficiently small power is an arbitrarily bad quantum expander. It is natural to hope that one can “reverse” this process; namely, that by taking a *large* power s , we can convert any dimension expander into one that is also a quantum expander. Sadly, this is also not true, as shown by the following simple counterexample.

Proposition 2.7. *There exists some dimension expander $\mathbf{U} \in \mathbf{U}(n)^d$ such that for any $s > 0$, \mathbf{U}^s is not a quantum expander.*

More precisely, there exists an absolute constant $\mu > 0$ such that the following holds for all $\varepsilon > 0$ and all sufficiently large n . There exists a $(n, 100, \mu)$ -dimension expander $\mathbf{U} = (U_1, \dots, U_{100}) \in \mathbf{U}(n)^{100}$ such that for all $s > 0$, we have $\lambda(\mathbf{U}^s) < \varepsilon$.

Proof sketch. Fix some $\varepsilon > 0$. Let e_1, \dots, e_n be the standard basis of \mathbb{C}^n . Let $E_\varepsilon \subseteq \mathbf{U}(n)$ denote the set of $n \times n$ unitary matrices M with the property that for all $j \in [n]$, we have $|\langle m_j, e_j \rangle| > 1 - \varepsilon$, where m_j is the j th column of M . Then E_ε is a non-empty open subset of $\mathbf{U}(n)$, which means that we can sample according to the induced Haar measure on E_ε . Let M_1, \dots, M_{100} be 100 independently random samples from this measure. Additionally, let D_1, \dots, D_{100} be independent random diagonal matrices whose diagonal entries are drawn uniformly at random from the unit circle. Finally, let $U_i = M_i D_i M_i^*$, so that each U_i is a random unitary matrix whose eigenvectors are the columns of M_i and whose eigenvalues are the diagonal entries of D_i . Note that for any $s > 0$, we have $U_i^s = M_i D_i^s M_i^*$, and D_i^s is a diagonal matrix whose diagonal entries are the s th powers of the diagonal entries of D_i .

It is well-known that 100 random unitary matrices form a $(n, 100, \mu)$ -dimension expander for some fixed $\mu > 0$ when n is large [DS11]. For the same reason, it is straightforward to check that $\mathbf{U} = (U_1, \dots, U_{100})$ forms an $(n, 100, \mu)$ -dimension expander for some fixed $\mu > 0$. The point is that while U_1, \dots, U_d are not uniformly random unitary matrices, they are “generic” in an appropriate sense, which suffices for them to form a dimension expander. However, we claim that for any $s > 0$, we have $\lambda(\mathbf{U}^s) \leq 10\varepsilon$. Since ε was arbitrary, this yields an example of a dimension expander none of whose powers is a quantum expander.

To see this, we first observe that by the definition of E_ε , the $(1, 1)$ entry of M_i has absolute value at least $1 - \varepsilon$, and the first row of M_i is a unit vector. Since D_i^s is a diagonal matrix whose diagonal entries have absolute value 1, this implies that both $M_i^* e_1$ and $D_i^s M_i^* e_1$ are unit vectors whose first coordinate has norm at least $1 - \varepsilon$. Let $v, w \in \mathbb{C}^{n-1}$ be the last $n - 1$ coordinates of $M_i^* e_1$ and $D_i^s M_i^* e_1$, respectively, so that $\|v\|_{L_2}, \|w\|_{L_2} \leq \sqrt{1 - (1 - \varepsilon)^2} \leq \sqrt{2\varepsilon}$. The Cauchy–Schwarz inequality then gives $|\langle v, w \rangle| \leq 2\varepsilon$, which implies

$$|\langle e_1, U_i^s e_1 \rangle| = |\langle e_1, M_i D_i^s M_i^* e_1 \rangle| = |\langle M_i^* e_1, D_i^s M_i^* e_1 \rangle| \geq (1 - \varepsilon)^2 - |\langle v, w \rangle| \geq 1 - 4\varepsilon.$$

Let $P = e_1 e_1^*$ be the projection on to the subspace spanned by e_1 . Then the entry of $U_i^s P (U_i^s)^*$ in the $(1, 1)$ position is

$$e_1^* (U_i^s P (U_i^s)^*) e_1 = (e_1^* U_i^s e_1) (e_1^* (U_i^s)^* e_1) = |\langle e_1, U_i^s e_1 \rangle|^2 \geq 1 - 8\varepsilon.$$

Let $X = P - \frac{1}{n} I_n$, so that X is a traceless matrix with S_2 -norm $1 - O(\frac{1}{n})$. The computation above implies that the $(1, 1)$ entry of $U_i^s X (U_i^s)^*$ is at least $1 - 8\varepsilon - \frac{1}{n}$. As this holds for all i , we conclude that it also holds for $\Phi_{\mathbf{U}^s}(X)$, which in turn implies that $\|\Phi_{\mathbf{U}^s}(X)\|_{S_2} \geq 1 - 8\varepsilon - \frac{1}{n}$. As $\|X\|_{S_2} \geq 1 - O(\frac{1}{n})$, we conclude that $\lambda(\mathbf{U}^s) \leq 10\varepsilon$ for all sufficiently large n . \square

3 Relations between linear-algebraic notions of expansion

In this section, we prove Theorem 1.9. Theorem 1.9(2) was proved by Hastings [Has07, Appendix A], so it remains to prove Theorems 1.9(1) and 1.9(3).

We begin with Theorem 1.9(1). As remarked in the introduction, the result actually holds for arbitrary matrix tuples over arbitrary fields, as stated in the following result.

Proposition 3.1. *For $\mathbf{B} := (B_1, \dots, B_d) \in M(n, \mathbb{F})^d$, it holds that $\frac{\mu(\mathbf{B})}{d} \leq h_D(\mathbf{B}) \leq \mu(\mathbf{B})$.*

Proof. We first show that $\mu(\mathbf{B}) \leq d \cdot h_D(\mathbf{B})$. For any $\mathbf{B} \in M(n, \mathbb{F})^d$, let $\mathbf{B}' = (\mathbf{B}, I_n) \in M(n, \mathbb{F})^{d+1}$. From the definitions, it is clear that $\mu(\mathbf{B}') = \mu(\mathbf{B})$ and $h_D(\mathbf{B}') = \frac{d}{d+1} h_D(\mathbf{B})$. We shall prove that $\mu(\mathbf{B}') \leq (d+1)h_D(\mathbf{B}')$.

Fix some $V \leq \mathbb{F}^n$ of dimension $1 \leq r \leq n/2$ and let $T \in M(n \times r, \mathbb{F})$ be a matrix whose columns form a basis of V . Let $R \in M((n-r) \times n, \mathbb{F})$ be a matrix whose rows form a basis of V^\perp , as defined in Remark 1.7. Let $B_i|_{V^\perp, V} = RB_iT \in M((n-r) \times r, \mathbb{F})$ for each $i \in [d]$. Note that we have $\text{rank}(T) = r$ and $\text{rank}(R) = n-r$.

Let $t = \dim(V + \mathbf{B}'(V))$, and notice that $t = \dim(\mathbf{B}'(V))$ since $B_{d+1} = I_n$. We have that $t \geq (1 + \mu(\mathbf{B}')) \cdot \dim(V) = (1 + \mu(\mathbf{B}')) \cdot r$. Let $W \in M(n \times t, \mathbb{F})$ be a matrix whose columns form a basis of $\mathbf{B}'(V)$. Then we have $\text{rank}(RW) = \dim(\mathbf{B}'(V)) - \dim(\ker(R) \cap \mathbf{B}'(V)) \geq t - r \geq \mu(\mathbf{B}') \cdot r$ and

$$\begin{aligned} \sum_{i=1}^{d+1} \text{rank}(B_i|_{V^\perp, V}) &= \sum_{i=1}^{d+1} \dim(\text{colspan}(B_i|_{V^\perp, V})) \\ &\geq \dim(\langle \cup_{i \in [d+1]} \text{colspan}(RB_iT) \rangle) \\ &= \dim(R \langle \cup_{i \in [d+1]} \text{colspan}(B_iT) \rangle) \tag{3.1} \\ &= \text{rank}(RW) \tag{3.2} \\ &\geq \mu(\mathbf{B}') \cdot r, \end{aligned}$$

where (3.1) holds since R has full row rank and (3.2) holds since $\langle \cup_{i \in [d+1]} \text{colspan}(B_iT) \rangle = \mathbf{B}'(V)$. Ranging over all subspaces $V \leq \mathbb{F}^n$ of dimension at most $n/2$, we find that $\mu(\mathbf{B}') \leq (d+1)h_D(\mathbf{B}')$, and $\mu(\mathbf{B}) \leq d \cdot h_D(\mathbf{B})$ follows.

Now we show that $h_D(\mathbf{B}) \leq \mu(\mathbf{B})$. Fix some $V \leq \mathbb{F}^n$ of dimension $1 \leq r \leq n/2$, let T, R be as above, and note that $V = \ker(R)$. Let W be a matrix whose columns are a basis of $\mathbf{B}(V)$. We have that

$$\begin{aligned} d \cdot h_D(\mathbf{B}) \cdot \dim(V) &\leq \sum_{i=1}^d \text{rank}(B_i|_{V^\perp, V}) \\ &= \sum_{i=1}^d \dim(\text{colspan}(RB_iT)) \\ &\leq d \cdot \dim(\cup_{i \in [d]} \text{colspan}(RB_iT)) && \text{[since } \sum_{i \in [d]} \dim(W_i) \leq d \cdot \dim(\cup_{i \in [d]} W_i)\text{]} \\ &= d \cdot \dim(R \langle \cup_{i \in [d]} \text{colspan}(B_iT) \rangle) && \text{[since } R \text{ has full row rank]} \\ &= d \cdot \text{rank}(RW) && \text{[since } \cup_{i \in [d]} \text{colspan}(B_iT) = \mathbf{B}(V)\text{]} \\ &= d \cdot (\dim(\mathbf{B}(V)) - \dim(\mathbf{B}(V) \cap \ker(R))) && \text{[since } \mathbf{B}(V) = \text{colspan}(W)\text{]} \\ &= d \cdot (\dim(\mathbf{B}(V)) - \dim(V \cap \mathbf{B}(V))) && \text{[since } V = \ker(R)\text{]} \\ &= d \cdot (\dim(V + \mathbf{B}(V)) - \dim(V)). \end{aligned}$$

This implies that $\dim(V + \mathbf{B}(V)) - \dim(V) \geq h_D(\mathbf{B}) \cdot \dim(V)$. Ranging over all subspace $V \leq \mathbb{F}^n$ of dimension at most $n/2$, we conclude that $h_D(\mathbf{B}) \leq \mu(\mathbf{B})$. \square

We now turn to Theorems 1.9(3) and 1.9(4), which lower-bound the dimension edge expansion in terms of the quantum edge expansion. As explained in the introduction, it is this inequality which allows us to prove that quantum expanders are dimension expanders.

In order to prove Theorems 1.9(3) and 1.9(4), we will use the following equivalent formulation of quantum edge expansion.

Lemma 3.2. *For any doubly stochastic matrix tuple $\mathbf{B} = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d$, we have*

$$h_Q(\mathbf{B}) = \min_{\substack{V \leq \mathbb{C}^n \\ 1 \leq \dim(V) \leq \frac{n}{2}}} \frac{\sum_{i=1}^d \|B_i|_{V^\perp, V}\|_{\mathcal{S}_2}^2}{d \cdot \dim(V)}. \quad (3.3)$$

Proof. Fix some subspace $V \leq \mathbb{C}^n$ of dimension $1 \leq r \leq n/2$. Let T_V be an $n \times r$ matrix whose columns form an orthonormal basis of V . Then $P_V = T_V T_V^*$. Let V^\perp be the orthogonal complement of V , and T_{V^\perp} be an $n \times (n-r)$ matrix whose columns form an orthonormal basis of V^\perp . Then $P_{V^\perp} = I_n - P_V = T_{V^\perp} T_{V^\perp}^*$.

We then have

$$\begin{aligned} \langle I_n - P_V, \Phi(P_V) \rangle &= \text{Tr}((I_n - P_V)^* \Phi(P_V)) \\ &= \frac{1}{d} \sum_{i=1}^d \text{Tr}(T_{V^\perp} T_{V^\perp}^* B_i T_V T_V^* B_i^*) \\ &= \frac{1}{d} \sum_{i=1}^d \text{Tr}(T_{V^\perp}^* B_i T_V T_V^* B_i^* T_{V^\perp}) \\ &= \frac{1}{d} \sum_{i=1}^d \|T_{V^\perp}^* B_i T_V\|_{\mathcal{S}_2}^2 \\ &= \frac{1}{d} \sum_{i=1}^d \|B_i|_{V^\perp, V}\|_{\mathcal{S}_2}^2. \end{aligned}$$

This implies that the objective functions in (3.3) and (1.4) are identical. The feasible regions are also the same, which concludes the proof. \square

We are now ready to prove Theorems 1.9(3) and 1.9(4).

Proof of Theorems 1.9(3) and 1.9(4). Fix a doubly stochastic matrix tuple $\mathbf{B} = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d$. By (3.3) and (1.6), we wish to prove that

$$\min_{\substack{V \leq \mathbb{C}^n \\ 1 \leq \dim(V) \leq \frac{n}{2}}} \frac{\sum_{i=1}^d \|B_i|_{V^\perp, V}\|_{\mathcal{S}_2}^2}{d \cdot \dim(V)} \leq d \cdot \min_{\substack{V \leq \mathbb{C}^n \\ 1 \leq \dim(V) \leq \frac{n}{2}}} \frac{\sum_{i=1}^d \text{rank}(B_i|_{V^\perp, V})}{d \cdot \dim(V)}.$$

So it suffices to prove that $\|B_i|_{V^\perp, V}\|_{\mathcal{S}_2}^2 \leq d \cdot \text{rank}(B_i|_{V^\perp, V})$ for any subspace $V \leq \mathbb{C}^n$ and all $i \in [d]$. We first claim that the operator norm of $B_i|_{V^\perp, V} = T_{V^\perp}^* B_i T_V$ is at most \sqrt{d} . Indeed, recall that $\sum_{i=1}^d B_i^* B_i = dI_n$, so $dI_n - B_i^* B_i$ is positive semidefinite for any $i \in [d]$. Thus, the operator norm of B_i is upper bounded by \sqrt{d} for each $i \in [d]$. Moreover, the operator norm of any isometry is at

most 1, thus $\|T_{V^\perp}\|_{S_\infty}, \|T_V\|_{S_\infty} \leq 1$. Using the submultiplicativity of the operator norm, we have $\|T_{V^\perp}^* B_i T_V\|_{S_\infty} \leq \sqrt{d}$.

Recall that $\|B_i|_{V^\perp, V}\|_{S_2}^2 = \|T_{V^\perp}^* B_i T_V\|_{S_2}^2$ is the sum of the squares of the singular values of $T_{V^\perp}^* B_i T_V$. As there are precisely $\text{rank}(T_{V^\perp}^* B_i T_V)$ non-zero singular values, and each one is upper-bounded by $\|T_{V^\perp}^* B_i T_V\|_{S_\infty} \leq \sqrt{d}$, we conclude that

$$\|B_i|_{V^\perp, V}\|_{S_2}^2 \leq \text{rank}(T_{V^\perp}^* B_i T_V) \|T_{V^\perp}^* B_i T_V\|_{S_\infty}^2 \leq d \cdot \text{rank}(B_i|_{V^\perp, V}),$$

as claimed. This proves Theorem 1.9(3).

In order to prove Theorem 1.9(4), note that if B_i is a unitary matrix, then $\|B_i\|_{S_\infty} = 1$. This implies that $\|T_{V^\perp}^* B_i T_V\|_{S_\infty}^2 \leq 1$, so the argument above shows that $\|B_i|_{V^\perp, V}\|_{S_2}^2 \leq \text{rank}(B_i|_{V^\perp, V})$, which yields Theorem 1.9(4). \square

4 Connections between graphs and matrix spaces

In this section, we study the graphical matrix tuple \mathbf{B}_G associated to a d -regular graph G . We begin by proving that $h(\mathbf{B}_G)$ is in general different from $h(G)$, as stated in Proposition 1.14.

Proof of Proposition 1.14. We certainly have that $h(K_2) = 1$. Note that $\mathbf{B}_{K_2} = \left(\begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix} \right)$.

Consider the subspace $V \leq \mathbb{C}^2$ spanned by the vector $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. The orthogonal projection onto V is given by the matrix $P_V = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and

$$\Phi_{\mathbf{B}_{K_2}}(P_V) = \frac{1}{2} \left(\begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}^* + \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix}^* \right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Therefore,

$$\langle I_2 - P_V, \Phi_{\mathbf{B}_{K_2}}(P_V) \rangle = \text{Tr} \left(\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^* \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) = \frac{1}{2}.$$

Hence, as $h_Q(\mathbf{B}_{K_2})$ is defined as a minimum over all one-dimensional subspaces, we find that $h_Q(\mathbf{B}_{K_2}) \leq \langle I_2 - P_V, \Phi_{\mathbf{B}_{K_2}}(P_V) \rangle = \frac{1}{2}$, as claimed. \square

Remark 4.1. It is not hard to show that in fact, $h_Q(\mathbf{B}_{K_2}) = \frac{1}{2}$. More generally, one can show that $h_Q(\mathbf{B}_{K_n}) \leq \frac{1}{n}$, which is smaller than $h(K_n)$ for all $n \geq 2$.

We now turn to the proof of Theorem 1.13. As the proofs that $h_D(\mathbf{B}_G) = h(G)$ and $\mu(\mathbf{B}_G) = \mu(G)$ are disjoint, we separate the Theorem 1.13 into two statements, Propositions 4.2 and 4.4. Note that Theorem 1.13 holds over any field, so we will work with \mathbb{F} instead of \mathbb{C} in the rest of this section.

Proposition 4.2. *For any d -regular graph $G = ([n], E)$, we have $h_D(\mathbf{B}_G) = h(G)$.*

Proof. For $v \in \mathbb{F}^n$, denote by $\text{supp}(v) \subseteq [n]$ the set of indices of the non-zero coordinates of v . For $V \leq \mathbb{F}^n$, we let $\text{supp}(V) = \cup_{v \in V} \text{supp}(v)$. Define V^\perp as in Remark 1.7. Now we claim the following.

Claim 4.3. *We have that*

$$\text{rank}(E_{i,j}|_{V^\perp, V}) = \begin{cases} 1 & \text{if } i \in \text{supp}(V^\perp) \text{ and } j \in \text{supp}(V) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose $\dim(V) = r$ and $\dim(V^\perp) = n - r$. Let T_V (resp. T_{V^\perp}) be an $n \times r$ (resp. $n \times (n - r)$) matrix whose columns form a basis of V (resp. V^\perp). Denote by $v_1, \dots, v_n \in \mathbb{F}^r$ and $v'_1, \dots, v'_n \in \mathbb{F}^{n-r}$ the vectors corresponding to the rows of T_V and T_{V^\perp} , respectively. Then for any $i, j \in [n]$,

$$\mathbf{E}_{i,j}|_{V^\perp, V} = T_{V^\perp}^t \mathbf{E}_{i,j} T_V = v'_i v_j^t.$$

Note that $\text{rank}(\mathbf{E}_{i,j}|_{V^\perp, V}) = 1$ if and only if $\mathbf{E}_{i,j}|_{V^\perp, V} \neq 0$ (and otherwise $\text{rank}(\mathbf{E}_{i,j}|_{V^\perp, V}) = 0$). This, in turn, happens if and only if $v'_i \neq 0$ and $v_j \neq 0$, which is equivalent to $i \in \text{supp}(V^\perp)$ and $j \in \text{supp}(V)$. \square

For any fixed $V \leq \mathbb{F}^n$ of dimension $1 \leq r \leq n/2$, we shall construct a vertex subset $W \subseteq [n]$ of size r such that

$$\sum_{\{i,j\} \in E} \text{rank}(\mathbf{E}_{i,j}|_{V^\perp, V}) \geq |\partial W|,$$

where here and throughout the sum is over all *ordered* pairs of vertices which are adjacent in G . We use the same notation of T_V and T_{V^\perp} as in Claim 4.3. Let $T = \begin{bmatrix} T_V & T_{V^\perp} \end{bmatrix} \in M(n, \mathbb{F})$. Since T_V is an $n \times r$ matrix of rank r , we can find a permutation matrix $P \in \text{GL}(n, \mathbb{F})$ such that the first r rows and columns of PT_V form an invertible matrix. We can extend any basis of $P(V)$ with $(n - r)$ linearly independent vectors to span \mathbb{F}^n . Specifically, the full basis can be represented as an invertible matrix $M \in \text{GL}(n, \mathbb{F})$ of which the first r columns form PT_V . Now break M into blocks:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A \in M(r, \mathbb{F})$, $B \in M(r \times (n - r), \mathbb{F})$, $C \in M((n - r) \times r, \mathbb{F})$ and $D \in M(n - r, \mathbb{F})$. Note that $PT_V = \begin{bmatrix} A \\ C \end{bmatrix}$. It follows that A is invertible. Similarly, break M^{-1} into blocks:

$$M^{-1} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix},$$

where $A' \in M(r, \mathbb{F})$, $B' \in M(r \times (n - r), \mathbb{F})$, $C' \in M((n - r) \times r, \mathbb{F})$ and $D' \in M(n - r, \mathbb{F})$. Since D' is the Schur complement of A , D' is also invertible. Note that

$$\begin{bmatrix} C' & D' \end{bmatrix} PT_V = \begin{bmatrix} C' & D' \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = 0.$$

Since $\text{rank}(\begin{bmatrix} C' & D' \end{bmatrix} P) = n - r$, it follows that the rows of $\begin{bmatrix} C' & D' \end{bmatrix} P$ form a basis of V^\perp . So we let $T_{V^\perp} = (\begin{bmatrix} C' & D' \end{bmatrix} P)^t = P^{-1} \begin{bmatrix} C'^t \\ D'^t \end{bmatrix}$ and thus,

$$T = \begin{bmatrix} T_V & T_{V^\perp} \end{bmatrix} = P^{-1} \begin{bmatrix} A & C'^t \\ C & D'^t \end{bmatrix}.$$

Let $W = \{P^{-1}(i) : i \in [r]\}$. Then $W \subseteq \text{supp}(V)$ and $[n] \setminus W = \{P^{-1}(i) : i \in [n] \setminus [r]\} \subseteq \text{supp}(V^\perp)$. By Claim 4.3, $\text{rank}(\mathbf{E}_{i,j}|_{V^\perp, V}) = 1$ if and only if $i \in \text{supp}(V^\perp)$ and $j \in \text{supp}(V)$. On the other hand, $\partial W = \{\{i, j\} \in E : i \in W, j \in [n] \setminus W\}$. Thus $\sum_{\{i,j\} \in E} \text{rank}(\mathbf{E}_{i,j}|_{V^\perp, V}) \geq |\partial W|$.

In short, for every subspace $V \leq \mathbb{F}^n$ of dimension $1 \leq r \leq n/2$, we can find a set W of r vertices such that

$$\frac{\sum_{\{i,j\} \in E} \text{rank}(\mathbf{E}_{i,j}|_{V^\perp, V})}{d \cdot \dim(V)} \geq \frac{|\partial W|}{d|W|}. \quad (4.1)$$

Ranging over all subspace V of dimension at most $n/2$ implies that $h_D(\mathbf{B}_G) \geq h(G)$. The reverse inequality $h(\mathbf{B}_G) \leq h(G)$ follows by simply choosing V to be the coordinate subspace $\langle e_i \rangle_{i \in W}$, for which it is clear that $\sum_{\{i,j\} \in E} \text{rank}(\mathbf{E}_{i,j}|_{V^\perp, V}) = |\partial W|$. \square

We now turn to vertex and dimension expansion. The following proof is based on the ideas of [DS11, DW10]:

Proposition 4.4. *For any d -regular graph $G = ([n], E)$, we have $\mu(\mathbf{B}_G) = \mu(G)$.*

Proof. For a non-zero vector $v \in \mathbb{F}^n$, denote by $\pi(v) \in [n]$ the largest index of a non-zero coordinate of v . Similarly, let $\pi(V) = \{\pi(v) \mid v \in V \setminus \{0\}\}$. For a set $S \subseteq [n]$ and $i, j \in [n]$, we define $f_{i,j}(S) = \{i\}$ if $j \in S$ and $f_{i,j}(S) = \emptyset$ otherwise. For any $V \leq \mathbb{F}^n$, we claim that

$$\pi(\mathbf{E}_{i,j}(V)) \supseteq f_{i,j}(\pi(V)), \quad (4.2)$$

where equality holds if V is a coordinate subspace. To see this, it suffices to consider the following two cases:

- If there exists $v \in V$ such that the j th coordinate of v is non-zero, then $\mathbf{E}_{i,j}(V) = \langle e_i \rangle$, which implies $\pi(\mathbf{E}_{i,j}(V)) = \{i\} \supseteq f_{i,j}(\pi(V))$. Moreover, if V is a coordinate subspace, we can also conclude that $e_j \in V$ and thus $j \in \pi(V)$, which implies that $\pi(\mathbf{E}_{i,j}(V)) = \{i\} = f_{i,j}(\pi(V))$.
- If there doesn't exist $v \in V$ such that the j th coordinate of v is non-zero, then $\mathbf{E}_{i,j}(V) = \{0\}$ and $j \notin \pi(V)$, which implies $\pi(\mathbf{E}_{i,j}(V)) = \emptyset$ and $f_{i,j}(\pi(V)) = \emptyset$.

Furthermore, for any $V_1, V_2 \leq \mathbb{F}^n$, we have that

$$\pi(V_1 + V_2) \supseteq \pi(V_1) \cup \pi(V_2), \quad (4.3)$$

where equality holds if V_1 and V_2 are coordinate subspaces or if one of them only consists of the zero vector. Thus, for any subspace $V \leq \mathbb{F}^n$ of dimension $\leq n/2$, we see that

$$\begin{aligned} |\pi(V + \mathbf{B}_G(V))| &\geq |\pi(V) \cup \pi(\mathbf{B}_G(V))| && \text{[by (4.3)]} \\ &= \left| \pi(V) \cup \pi \left(\sum_{\{i,j\} \in E} \mathbf{E}_{i,j}(V) \right) \right| \\ &= \left| \pi(V) \cup \left(\bigcup_{\{i,j\} \in E} \pi(\mathbf{E}_{i,j}(V)) \right) \right| && \text{[by (4.3) and } \mathbf{E}_{i,j}(V) = \langle e_i \rangle \text{ or } \{0\}] \\ &\geq \left| \pi(V) \cup \left(\bigcup_{\{i,j\} \in E} f_{i,j}(\pi(V)) \right) \right| && \text{[by (4.2)].} \end{aligned}$$

Observe that $|\pi(V)| = \dim(V) \leq n/2$, as we can always find a basis of V with distinct last non-zero coordinates. Additionally, for each vertex $j \in \pi(V)$, we have that $\cup_{\{i,j\} \in E} f_{i,j}(j)$ is the set of neighbors of j in G . It follows that

$$\pi(V) \cup \left(\bigcup_{\{i,j\} \in E} f_{i,j}(\pi(V)) \right) = \pi(V) \cup (\partial_{out}(\pi(V))).$$

Therefore, by the definition of vertex expansion,

$$|\pi(V + \mathbf{B}_G(V))| \geq \left| \pi(V) \cup \left(\bigcup_{\{i,j\} \in E} f_{i,j}(\pi(V)) \right) \right| = |\pi(V) \cup (\partial_{out}(\pi(V)))| \geq (1 + \mu(G)) \cdot |\pi(V)|.$$

Therefore, we conclude that

$$\mu(\mathbf{B}_G) = \min_{\substack{V \leq \mathbb{F}^n \\ 1 \leq \dim(V) \leq n/2}} \frac{\dim(V + \mathbf{B}_G(V)) - \dim(V)}{\dim(V)} \geq \mu(G).$$

The reverse inequality follows by picking V to be a coordinate subspace, which turns all the inequalities above into equalities. \square

Remark 4.5. The proof of Proposition 4.4 actually works for any graph, and Proposition 4.2 also holds for any graph after removing the d -normalization from the definitions of edge expansion and dimension edge expansion.

5 Conclusion and open problems

Recall (3.3), which gives an equivalent definition of the quantum edge expansion of a doubly stochastic matrix tuple in terms of the Schatten-2 norm, namely

$$h_Q(\mathbf{B}) = \min_{\substack{V \leq \mathbb{C}^n \\ 1 \leq \dim(V) \leq \frac{n}{2}}} \frac{\sum_{i=1}^d \|B_i|_{V^\perp, V}\|_{S_2}^2}{d \cdot \dim(V)}.$$

Given this formulation, the following definition is natural.

Definition 5.1. Given $p \in [1, \infty)$ and a doubly stochastic matrix tuple $\mathbf{B} = (B_1, \dots, B_d) \in M(n, \mathbb{C})^d$, the *Schatten- p edge expansion* of \mathbf{B} is defined as

$$h_{S_p}(\mathbf{B}) := \min_{\substack{V \leq \mathbb{C}^n \\ 1 \leq \dim(V) \leq \frac{n}{2}}} \frac{\sum_{i=1}^d \|B_i|_{V^\perp, V}\|_{S_p}^p}{d \cdot \dim(V)}.$$

The proof of Theorem 1.9(3) immediately shows that for any doubly stochastic matrix tuple $\mathbf{B} \in M(n, \mathbb{C})^d$ and for any $p \in [1, \infty)$, we have

$$h_{S_p}(\mathbf{B}) \leq d^{\frac{p}{2}} \cdot h_D(\mathbf{B}).$$

In case \mathbf{B} is a unitary matrix tuple, we have the stronger inequality

$$h_{S_p}(\mathbf{B}) \leq h_D(\mathbf{B}).$$

Indeed, to prove both of these, we simply recall that $\|B_i|_{V^\perp, V}\|_{S_p}^p$ is the sum of the p th powers of the singular values of $B_i|_{V^\perp, V}$. There are $\text{rank}(B_i|_{V^\perp, V})$ non-zero singular values, and each of them is upper-bounded by the operator norm of $B_i|_{V^\perp, V}$. This operator norm, in turn, is upper-bounded by \sqrt{d} , and by 1 in case B_i is unitary.

Therefore, for any $p \in [1, \infty)$, Schatten- p edge expansion implies dimension edge expansion, and thus dimension expansion. On the other hand, one can modify the proof of Theorem 1.8 to show that the converse does not hold for any $p \in [1, \infty)$. Indeed, if $\mathbf{U} \in U(n)^d$ is a unitary matrix tuple, then it is easy to see that $h_{S_p}(\mathbf{U}^s) \rightarrow 0$ as $s \rightarrow 0$, since the tuple \mathbf{U}^s converges to the identity tuple (I_n, \dots, I_n) as $s \rightarrow 0$. However, Theorem 2.1 states that $\mu(\mathbf{U}^s) \geq \mu(\mathbf{U})/d$ for all sufficiently small s , and thus $h_D(\mathbf{U}^s)$ stays bounded away from zero as $s \rightarrow 0$.

Given this, it is very natural to ask whether the notions of Schatten- p edge expansion are all equivalent.

Open Question 5.2. Fix $p, q \in [1, \infty)$ and $d \in \mathbb{N}$. Do there exist increasing functions $f, g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that

$$f(h_{S_p}(\mathbf{B})) \leq h_{S_q}(\mathbf{B}) \leq g(h_{S_p}(\mathbf{B}))$$

holds for all doubly stochastic matrix tuples $\mathbf{B} \in M(n, \mathbb{C})^d$?

If the answer is positive, this could be viewed as a linear-algebraic analogue of a theorem of Matoušek [Mat97], who proved that a certain L^p notion of graph expansion is equivalent to spectral expansion (i.e. the L^2 notion) for all $p \in [1, \infty)$.

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