ON THE PERIODIC TOPOLOGICAL CYCLIC HOMOLOGY OF DG CATEGORIES IN CHARACTERISTIC P

ALEXANDER PETROV AND VADIM VOLOGODSKY

ABSTRACT. We prove that the p-adically completed periodic topological cyclic homology of a DG category over a perfect field k of characteristic $p > 2$ is isomorphic to the (p-adically completed) periodic cyclic homology of a lifting of the DG category over the Witt vectors $W(k)$.

CONTENTS

1. INTRODUCTION

Let $\mathcal{C}$ be a DG category over a perfect field k of characteristic p, and let $\mathrm{TP}(\mathcal{C})$ be the periodic topological cyclic homology spectrum introduced by L. Hesselholt in ([H]). Denote by $\widehat{\mathrm{TP}}(\mathcal{C})$ the p-adic completion of the spectrum $\mathrm{TP}(\mathcal{C})$. A lifting of $\mathcal{C}$ over the ring $W(k)$ of Witt vectors is a DG category $\tilde{\mathcal{C}}$ over $W(k)$ whose objects are those of $\mathcal{C}$ and morphisms are complexes of flat $W(k)$-modules, together with quasi-isomorphisms

$$Mor_{\tilde{\mathcal{C}}}(X, Y) \otimes_{W(k)} k \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{C}}(X, Y),$$

for every pair of objects X and Y, compatible with the compositions. Denote by $\widehat{\mathrm{HP}}(\tilde{\mathcal{C}}/W(k))$ the p-adic completion of the periodic cyclic homology complex $\mathrm{HP}(\tilde{\mathcal{C}}/W(k))$. By definition, $\widehat{\mathrm{HP}}(\tilde{\mathcal{C}}/W(k))$ is an object of the derived category of $W(k)$-modules; abusing notations we shall also denote by $\widehat{\mathrm{HP}}(\tilde{\mathcal{C}}/W(k))$ the underlying spectrum. The main result of this paper is the following.

Theorem 1. *For any DG category $\mathcal{C}$ over a perfect field k of characteristic $p > 2$ and a lifting $\tilde{\mathcal{C}}$ of $\mathcal{C}$ over $W(k)$, one has a natural isomorphism of spectra*

$$(1.1) \qquad\qquad \widehat{\mathrm{TP}}(\mathcal{C}) \xrightarrow{\sim} \widehat{\mathrm{HP}}(\tilde{\mathcal{C}}/W(k)).$$

Recall that the periodic topological cyclic homology spectrum of a DG category over k is a module over the topological cyclic homology ring spectrum $\mathrm{TC}(k)$. Using a computation from ([NS]), for every perfect field k, the connective cover of $\mathrm{TC}(k)$ is identified with the Eilenberg-MacLane ring spectrum $HW(k)$. This makes $\widehat{\mathrm{TP}}(\mathcal{C})$ into a $HW(k)$-module. Note that the right-hand side of (1.1) has a $HW(k)$-module

structure by construction. We do not know if isomorphism (1.1) can be promoted to an isomorphism of $HW(k)$-modules. However, it does induce an isomorphism of $W(k)$-modules:

$$\pi_i \widehat{\mathrm{TP}}(\mathcal{C}) \xrightarrow{\sim} \widehat{\mathrm{HP}}_i(\tilde{\mathcal{C}}/W(k)).$$

To explain the key idea in the proof of Theorem 1 we need to introduce a bit of notation. Recall that for a DG category $\mathcal{C}$ over a commutative ring W the Hochschild homology $\mathrm{HH}(\mathcal{C}/W)$ is naturally an object of the symmetric monoidal stable ∞-category $\mathrm{Mod}_{W[S^1]}$ of W-modules equipped with an action of the circle. The periodic cyclic homology $\mathrm{HP}(\mathcal{C}/W)$ is constructed from $\mathrm{HH}(\mathcal{C}/W)$ by applying the Tate invariants functor $\mathrm{Mod}_{W[S^1]} \to \mathrm{Mod}_{W^{tS^1}}$, $M \mapsto M^{tS^1}$, with respect to the circle action. We consider a certain quotient of $\mathrm{Mod}_{W[S^1]}$ by a monoidal ideal the Tate invariants functor factors through:

$$(1.2) \qquad\qquad \mathrm{Mod}_{W[S^1]} \to \mathrm{Mod}^t_{W[S^1]} \to \mathrm{Mod}_{W^{tS^1}} .$$

The second arrow in (1.2) is fully faithful when restricted to the subcategory of $\mathrm{Mod}^t_{W[S^1]}$ spanned by W (with the trivial circle action). Moreover, we have that, for every $M \in \mathrm{Mod}_{W[S^1]}$

$$M^{tS^1} \xrightarrow{\sim} Mor_{\mathrm{Mod}^t_{W[S^1]}}(W, M).$$

The advantage of $\mathrm{Mod}^t_{W[S^1]}$ is that the functor $\mathrm{Mod}_{W[S^1]} \to \mathrm{Mod}^t_{W[S^1]}$ is symmetric monoidal, whereas $\mathrm{Mod}_{W[S^1]} \to \mathrm{Mod}_{W^{tS^1}}$ is not.

We also consider the p-completed version of (1.2). For a fixed prime number p and a stable ∞-category $\mathcal{M}$ we denote by $\widehat{\mathcal{M}}$ the stable ∞-category obtained from $\mathcal{M}$ by first taking the p-completions of the spaces of morphisms in $\mathcal{M}$ and then applying the stabilization. We apply this construction to (1.2).

$$(1.3) \qquad\qquad \widehat{\mathrm{Mod}}_{W[S^1]} \to \widehat{\mathrm{Mod}}^t_{W[S^1]} \to \widehat{\mathrm{Mod}}_{W^{tS^1}} .$$

Denote by $\widehat{\overline{\mathrm{HH}}}(\mathcal{C}/W)$ the image of $\mathrm{HH}(\mathcal{C}/W)$ in the category $\widehat{\mathrm{Mod}}^t_{W[S^1]}$. For any E_∞-algebra $\mathcal{A}$ over W, the object $\widehat{\overline{\mathrm{HH}}}(\mathcal{A}/W) \in \widehat{\mathrm{Mod}}^t_{W[S^1]}$ has a natural structure of an E_∞-algebra over $\widehat{\overline{\mathrm{HH}}}(W/W)$.

The key step in the proof of Theorem 1 is a construction of a homomorphism of E_∞-algebras over $\widehat{\overline{\mathrm{HH}}}(W(k)/W(k))$

$$(1.4) \qquad\qquad \widehat{\overline{\mathrm{HH}}}(k/W(k)) \longrightarrow \widehat{\overline{\mathrm{HH}}}(W(k)/W(k)),$$

that reduces to $\overline{\mathrm{HH}}(k \otimes_{W(k)} k/k) \to \overline{\mathrm{HH}}(k/k)$ modulo p[1]. Our construction of (1.4) uses the Gauss-Manin connection on the periodic cyclic homology. Namely, given any E_∞-algebra $\mathcal{A}$ over the polynomial algebra $W(k)[x]$ we show the existence of a natural isomorphism of E_∞-algebras

$$(1.5) \qquad\qquad \widehat{\overline{\mathrm{HH}}}(\mathcal{A}_{x=0}/W(k)) \xrightarrow{\sim} \widehat{\overline{\mathrm{HH}}}(\mathcal{A}_{x=p}/W(k)).$$

Here $\mathcal{A}_{x=t}$ stands for the (derived) fiber $\mathcal{A} \otimes_{W(k)[x]} W(k)[x]/(x-t)$ of $\mathcal{A}$ over the point $x = t$. We then apply (1.5) to the DG algebra $\mathcal{A} = W(k)[x, \epsilon]$ generated over $W(k)[x]$ by ϵ with $\deg \epsilon = -1$, $\epsilon^2 = 0$, and $d\epsilon = x$. Observing that $\mathcal{A}_{x=p} \xrightarrow{\sim} k$ and that the

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[1] Throughout this paper $\mathrm{HH}(\mathcal{A}/W)$ stands for the "derived" version of the Hochschild homology functor (see, for example, [BMS, §2.2]). In particular, $\mathrm{HH}(k/W(k))$ is computed via the cyclic bar construction applied to a $W(k)$-flat replacement for k.

fiber $\mathcal{A}_{x=0} \xrightarrow{\sim} W(k)[\epsilon]$ admits an algebra map to $W(k)$ we get the augmentation (1.4). Note that the classical construction of the Gauss-Manin connection on the periodic cyclic homology due to Daletsky, I. Gelfand, Tsygan, and Getzler (see [DGT], [G]) is not quite sufficient for our purposes as it does not, at least on the nose, give an isomorphism of E_∞-algebras (1.5); in §2 we propose a different construction of the connection where its multiplicative property is obvious. This is the technical heart of the paper.

Next, using the homomorphism of E_∞-algebras (1.4) we define a lax symmetric monoidal functor

$$(1.6) \qquad \mathrm{HP}^{cris}(-, W(k)) : \mathrm{DG~categories~over~} k \longrightarrow \mathrm{Mod}_{W(k)^{tS^1}}$$

as follows. Given a DG category $\mathcal{C}$ over k we consider the $\widehat{\widehat{\mathrm{HH}}}(k/W(k))$-module $\widehat{\widehat{\mathrm{HH}}}(\mathcal{C}/W(k))$. Using (1.4) we form the "tensor product"

$$(1.7) \qquad \widehat{\widehat{\mathrm{HH}}}(\mathcal{C}/W(k)) \otimes_{\widehat{\widehat{\mathrm{HH}}}(k/W(k))} \widehat{\widehat{\mathrm{HH}}}(W(k)/W(k)) \in \mathrm{Fun}(\Delta^{op}, \widehat{\mathrm{Mod}}^t_{W(k)[S^1]})$$

considered as a simplicial object in $\widehat{\mathrm{Mod}}^t_{W(k)[S^1]}$ given by the two-sided bar construction. Projecting this object to $\mathrm{Fun}(\Delta^{op}, \widehat{\mathrm{Mod}}_{W(k)^{tS^1}})$ and taking the colimit over Δ^{op} we get $\mathrm{HP}^{cris}(\mathcal{C}, W(k))$. We remark that, by construction, the functor $\mathrm{HP}^{cris}(-, W(k))$ is a direct summand of $\widehat{\mathrm{HP}}(-/W(k))$. We show that, for any lifting $\tilde{\mathcal{C}}$ of $\mathcal{C}$, the compositions

$$\mathrm{HP}(\tilde{\mathcal{C}}/W(k)) \to \mathrm{HP}(\mathcal{C}/W(k)) \to \mathrm{HP}^{cris}(\mathcal{C}, W(k))$$

$$\mathrm{TP}(\mathcal{C}) \to \mathrm{HP}(\mathcal{C}/W(k)) \to \mathrm{HP}^{cris}(\mathcal{C}, W(k))$$

are isomorphisms after the p-completion.

Our definition of the functor $\mathrm{HP}^{cris}(-, W(k))$ is inspired by Bhatt's construction of crystalline cohomology from the derived de Rham cohomology ([B], Corollary 8.6).

We were informed that Peter Scholze has also obtained a proof of Theorem 1 in case when $\mathcal{C}$ is a smooth and proper DG category over k. We are grateful to Alexander Beilinson, Bhargav Bhatt, Alexander Efimov, Jacob Lurie, and Nick Rozenblyum for useful discussions. Special thanks go to Dmitry Kaledin for his constant attention to this work and many useful suggestions and to Akhil Mathew for drawing our attention to Bhatt's paper [B]. The work of the second author was supported in part by RNF grant № $18 - 11 - 00141$.

2. The Gauss-Manin connection revisited

We start by introducing a bit of notations. For a commutative ring W we denote by $\mathrm{Mod}_{W[S^1]}$ the symmetric monoidal stable ∞-category of W-modules equipped with an action of the circle, that is the category of functors from BS^1 to Mod_W. Let $s : * \to BS^1$ be the inclusion of the unique 0-cell and consider the induction functor $s_! : \mathrm{Mod}_W \to \mathrm{Mod}_{W[S^1]}$. Let $T \subset \mathrm{Mod}_{W[S^1]}$ be the smallest stable subcategory $T \subset \mathrm{Mod}_{W[S^1]}$ that contains all objects of the form $s_! N$, where N is a W-module. Note that, for any $M \in \mathrm{Mod}_{W[S^1]}$, we have that

$$s_!(N \otimes_W s^* M) = s_! N \otimes_W M$$

In particular, we have $s_! N \otimes_W M \in T$. It follows that T is a tensor ideal in $\mathrm{Mod}_{W[S^1]}$, that is, for every $M \in \mathrm{Mod}_{W[S^1]}$ and $M' \in T$, the tensor product $M \otimes M'$ is T. We can

thus form the symmetric monoidal quotient category $\mathrm{Mod}_{W[S^1]}/T$, that we denote by $\mathrm{Mod}^t_{W[S^1]}$. It is equipped with a symmetric monoidal projection functor

$$\mathrm{Mod}_{W[S^1]} \to \mathrm{Mod}^t_{W[S^1]}, \quad M \mapsto \overline{M}$$

We observe that the Tate invariants functor $\mathrm{Mod}_{W[S^1]} \to \mathrm{Mod}_{W^{tS^1}}$ factors uniquely through

$$(2.1) \qquad\qquad\qquad \mathrm{Mod}^t_{W[S^1]} \to \mathrm{Mod}_{W^{tS^1}} .$$

Abusing notation, we shall write $\overline{M}^{tS^1}$ for the latter applied to $\overline{M} \in \mathrm{Mod}^t_{W[S^1]}$. By functoriality, for every $\overline{M} \in \mathrm{Mod}^t_{W[S^1]}$, we have a morphism

$$\mathrm{Hom}_{\mathrm{Mod}^t_{W[S^1]}}(\overline{W}, \overline{M}) \to \mathrm{Hom}_{\mathrm{Mod}_{W^{tS^1}}}(W^{tS^1}, \overline{M}^{tS^1}) = \overline{M}^{tS^1} .$$

Lemma 2.1. *The above morphism is quasi-isomorphism.*

It follows from the Lemma that functor (2.1) is fully faithfull when restricted to the subcategory of $\mathrm{Mod}^t_{W[S^1]}$, whose objects are perfect as W-modules.

Let R be the divided power envelope of the ideal (x) in the polynomial algebra $W[x]$, $I \subset R$ the divided power ideal generated by x. For a DG algebra $\mathcal{A} \in \mathrm{DGA}_R$ and an integer $n \geq 0$, we set

$$\mathcal{A}_n := \mathcal{A} \otimes_R R/I^{[n+1]} \in \mathrm{DGA}_{R/I^{[n]}} .$$

The main result of this section is the following.

Theorem 2. *For any $\mathcal{A} \in \mathrm{DGA}_R$, the projection*

$$(2.2) \qquad\qquad \overline{\mathrm{HH}}(\mathcal{A}_n/R/I^{[n+1]}) \to \overline{\mathrm{HH}}(\mathcal{A}_0/W)$$

admits a right inverse:

$$\alpha_n : \overline{\mathrm{HH}}(\mathcal{A}_0/W) \to \overline{\mathrm{HH}}(\mathcal{A}_n/R/I^{[n+1]}),$$

that induces a qusi-isomorphism

$$\overline{\mathrm{HH}}(\mathcal{A}_0/W) \otimes_W R/I^{[n+1]} \xrightarrow{\sim} \overline{\mathrm{HH}}(\mathcal{A}_n/R/I^{[n+1]}).$$

Moreover, the maps α_n, $n = 0, 1, \cdots$, can be lifted to a morphism of symmetric monoidal functors from the category DGA_R to the category of pro-objects in $\mathrm{Mod}^t_{W[S^1]}$:

$$\alpha : \overline{\mathrm{HH}}(\mathcal{A}_0/W) \to \lim_n \overline{\mathrm{HH}}(\mathcal{A}_n/R/I^{[n+1]}),$$

Remark 2.2. Merely the existence of a right inverse to (2.2) can be easily derived from the existence of the Gauss-Manin connection on the periodic cyclic homology. Our main observation in Theorem 2 is that morphisms α_n behave well with respect to the tensor product of algebras. In particular, if $\mathcal{A}$ is an E_m algebra over R then (2.2) admits a right inverse as a morphism of E_{m-1} algebras.

Proof. For the construction of the inverse we will work in the additive symmetric monoidal category of cyclic W-complexes. That is, the category of functors from the Connes' cyclic category Λ to the category of chain complexes of W-modules. The objects of Λ are indexed by positive integers and are denoted by $[1], [2], \ldots$. See Appendix B in [NS] or Chapter 6 of [L] for a detailed discussion of the cyclic category.

Recall the construction of a particular cyclic W-module whose corresponding S^1-module will be free. Let Q be the cyclic W-module spanned by the cyclic set represented by $[1]$:

$$Q([k]) = W \cdot \mathrm{Hom}_\Lambda([1], [k])$$

Given a DG algebra $\mathcal{B}$ over a commutative ring S we will denote by $\mathcal{B}^{\#/S}$ the cyclic bar construction given by $[k] \mapsto \mathcal{B}^{\otimes_S k}$ (with the face maps defined using multiplication on $\mathcal{B}$). If the base ring S is W we will denote this cyclic object just by $\mathcal{B}^\#$. The constant cyclic algebra $S^{\#/S}$ will be denoted by $\underline{S}$. Given two cyclic modules X_1, X_2 we define their tensor product object-wise: $(X_1 \otimes_W X_2)([k]) = X_1([k]) \otimes_W X_2([k])$.

There is a functor of geometric realization from the category of cyclic modules to the category of modules with S^1-action:

Proposition 2.3. *There is a symmetric monoidal functor*

$$| - | : Func(\Lambda, \mathrm{Mod}_W) \to \mathrm{Mod}_{W[S^1]}$$

such that

(i) *For any DG algebra $\mathcal{B}$ over a commutative W-algebra S that is h-flat as an S-module there is a canonical quasi-isomorphism $|\mathcal{B}^{\#/S}| \simeq \mathrm{HH}(\mathcal{B}/S)$.*

(ii) *The object $|Q|$ is quasi-isomorphic to $s_! M$ for a certain W-module M.*

Proof. The functor is defined in Proposition B.5 of [NS] and (i) is the definition of Hochschild homology. Denote by $\widetilde{Q}$ the functor from the paracyclic category Λ_∞ to the category of W-modules spanned by the representable functor $\mathrm{Hom}_{\Lambda_\infty}([1]_{\Lambda_\infty}, -)$. The projection $j : \Lambda_\infty \to \Lambda$ provides a functor $j_! : Func(\Lambda_\infty, \mathrm{Mod}_W) \to Func(\Lambda, \mathrm{Mod}_W)$ that preserves colimits and takes $\widetilde{Q}$ to Q. Hence, $|Q| = \mathrm{colim}_{\Lambda_\infty^{op}} j^* j_! \widetilde{Q} = s_! (\mathrm{colim}_{\Lambda_\infty^{op}} \widetilde{Q})$ so the assertion (ii) is proven. $\square$

The cyclic commutative algebra $R^\#$ contains the ideal $J := \ker(R^{\#/W} \to R^{\#/R} = \underline{R})$. It is equipped with the divided power structure coming from the inclusions $J([k]) \subset \sum_{i=1}^{k} R^{\otimes(k-i)} \otimes I \otimes R^{\otimes(i-1)}$. Explicitly, $R^\#([k])$ is isomorphic to $W[x_0^{[\cdot]}, \ldots x_{k-1}^{[\cdot]}]$ with $J([k])$ generated by elements of the form $x_i^{[l]} - x_j^{[l]}$ and the divided powers $(x_i^{[l]} - x_j^{[l]})^{[m]}$ are defined according to the usual binomial formula.

Taking the quotient by the divided power square of the ideal J we obtain the cyclic algebra F given by

$$F = R^\# / J^{[2]}$$

Note that the map $R^\# \to \underline{W}$ induced by the quotient $R \to R/I = W$ factors through a surjection $F \to \underline{W}$. For brevity, denote the quotient $R/I^{[n+1]}$ by R_n.

Lemma 2.4. (i) *For every k the elements 1 and $x_i^{[m]}$ for $i \in \{0, 1, \ldots k-1\}, m \in \mathbb{Z}_{\geq 1}$ form a W-basis of the module $F([k])$.*

(ii) *There exists a filtration on F by $R^\#$-submodules $F = Fil^0 \supset \ker(F \to \underline{W}) = Fil^1 \supset Fil^2 \supset \ldots$ such that for every $i \geq 1$ the quotient Fil^i/Fil^{i+1} is isomorphic to Q as an $R^\#$-module (where $R^\#$ acts on Q through the map $R^\# \to \underline{W}$). Moreover, for every n the map $F \to \underline{R_n}$ factors through F/Fil^{n+1}.*

(iii) *Denote F/Fil^{n+1} by F_n. Consider the tensor product $F_n \otimes_W \underline{R_n}$ as a module over $\underline{R_n^\#}$ through the action of $R_n^\#$ on F_n. The kernel of the multiplication map*

$F_n \otimes_W \underline{R}_n \to \underline{R}_n$ admits a finite $R_n^\#$-linear filtration $\widetilde{Fil}^\bullet$ such that all quotients $\widetilde{Fil}^i/\widetilde{Fil}^{i+1}$ are isomorphic to Q as $R_n^\#$-modules.

Proof. (i) Firstly, we show that these elements indeed generate all of $F([k]) = R^{\otimes k}/J([k])^{[2]}$ over W. For any $i, j \in \{0, \ldots, k-1\}$ we have $x_j^{[m+1]} = (x_i + x_j - x_i)^{[m+1]} = x_i^{[m+1]} + x_i^{[m]}(x_j - x_i)$ where the last equality holds because all other terms in the binomial formula are divisible by $(x_j - x_i)^{[l]}$ for $l \geq 2$ and thus lie in the ideal $J([k])^{[2]}$. It follows that $x_i^{[m]}x_j = mx_i^{[m+1]} + x_j^{[m+1]}$. Next, for any numbers l, r we get $x_i^{[l]}x_j^{[r]} = x_i^{[l]}(x_i + x_j - x_i)^{[r]} = \binom{r+l}{l}x_i^{[r+l]} + x_i^{[l]}x_i^{[r-1]}(x_j - x_i) = (1-r)\binom{r+l}{r}x_i^{[r+l]} + \binom{r+l-1}{l}x_i^{[r+l-1]}x_j$. Combining it with the computation in the previous sentence, we obtain

$$x_i^{[l]}x_j^{[r]} = \binom{r+l}{l}x_i^{[r+l]} + \binom{r+l-1}{l}x_j^{[r+l]}$$

Applying this formula repeatedly, we can express any monomial as a linear combination of the separate powers $x_i^{[m]}$ so these powers indeed generate the module $F([k])$.

Suppose that there is a non-trivial linear relation between these powers. For every $i \in \{0, \ldots, k-1\}$ there is a map $R^{\otimes k}/J([k])^{[2]} \to R/I^{[2]}$ induced by $x_j \mapsto \delta_{ij}x$, so the constant and degree 1 terms of a relation must vanish. Next, the differential operator $\Delta = \sum_{i=0}^{k-1} \partial_{x_i}$ on $R^{\otimes k}$ kills all the elements of the form $(x_i - x_j)^{[r]}$, in particular preserves the ideal $J([k])^{[2]}$ and thus acts on $F([k])$. On the other hand, applying Δ to a non-trivial linear combination of the powers $x_i^{[m]}$ an appropriate number of times gives a relation with a non-trivial linear term which we have seen to be impossible.

(ii) Having constructed an explicit basis in $F([k])$ we define the desired filtration by $Fil^i([k]) = \langle x_j^{[m]} | m \geq i, j \in \{0, 1, \ldots, k-1\}\rangle$. The quotient Fil^i/FIl^{i+1} is isomorphic to Q as a cyclic W-module. Since the $R^\#$-module structure on this quotient factors through $R^\# \to \underline{W}$ we have constructed a filtration with the desired properties.

(iii)The elements $t_j^{[m]} \otimes t^{[l]} - 1 \otimes t^{[m]} \cdot t^{[l]}$ for $1 \leq m \leq n, 0 \leq l \leq n, 0 \leq j \leq k-1$ form a basis in the kernel of the multiplication map $m : F_n \otimes_W \underline{R}_n \to \underline{R}_n$. For an index i define the i-th step of the filtration $\widetilde{Fil}^i$ as
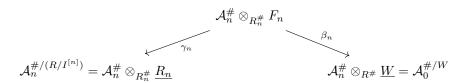
$$(2.3) \quad \widetilde{Fil}^i([k]) = \langle t_j^{[m]} \otimes t^{[l]} - 1 \otimes t^{[m]} \cdot t^{[l]} | l \geq \lfloor \frac{i}{n} \rfloor + 1 \text{ or}$$

$$l = \lfloor \frac{i}{n} \rfloor \text{ and } m \geq (i \bmod n) + 1, j \in \{0, 1, \ldots, k-1\}\rangle$$

These are $R_n^\#$-submodules and the quotients $\widetilde{Fil}^i([k])/\widetilde{Fil}^{i+1}([k])$ admit a basis $t_j^{[(i \bmod n)+1]} \otimes t^{\lfloor \frac{i}{n}\rfloor} - 1 \otimes t^{[(i \bmod n)+1]} \cdot t^{[\lfloor \frac{i}{n}\rfloor]}, j \in \{0, 1, \ldots, k-1\}$ so the cyclic modules $\widetilde{Fil}^i/\widetilde{Fil}^{i+1}$ are all isomorphic to Q and we get the filtration $\ker(F_n \otimes_W \underline{R}_n \xrightarrow{m} \underline{R}_n) = \widetilde{Fil}^0 \supset \widetilde{Fil}^1 \supset \cdots \supset \widetilde{Fil}^{n(n+1)} = 0$ with the desired property.
$\square$

We will use the auxiliary objects $F_n = F/Fil^{n+1}$ to produce the required maps in the category $\mathrm{Mod}_{W[S^1]}^t$. Replace $\mathcal{A}$ by a semi-free resolution(according to Proposition

13.5 of [D]) over R so that $\mathcal{A}$ is h-flat as an R-module. From the explicit description of F provided by the proof of the lemma we also see that $F_n = F \otimes_{R^\#} R_n^\#$. These cyclic objects come with the maps $F_n \to \underline{R_n} \xrightarrow{\text{ev}_0} \underline{W}$. For every n we have a diagram of cyclic W-modules

$$\mathcal{A}_n^\# \otimes_{R_n^\#} F_n$$

$$\gamma_n \qquad\qquad\qquad \beta_n$$

$$\mathcal{A}_n^{\#/(R/I^{[n]})} = \mathcal{A}_n^\# \otimes_{R_n^\#} \underline{R_n} \qquad\qquad \mathcal{A}_n^\# \otimes_{R^\#} \underline{W} = \mathcal{A}_0^{\#/W}$$

The map β_n is surjective with the kernel given by $\mathcal{A}_n^\# \otimes_{R_n^\#} Fil^1 F_n$. The filtration $Fil^\bullet F_n$ constructed in Lemma 2.4(ii) induces a finite filtration on this tensor product with quotients $gr^i(\mathcal{A}_n^\# \otimes_{R_n^\#} Fil^1 F_n)$ isomorphic to $\mathcal{A}_0^\# \otimes_{W^\#} Q$. According to Proposition 2.3, under the functor from cyclic W-modules to $\text{Mod}_{W[S^1]}$ any module admitting a finite filtration with quotients of the form $M \otimes_{\underline{W}} Q$ gets sent to an object in the subcategory T. Hence, the map β_n turns into an equivalence in $\text{Mod}_{W[S^1]}^t$ and the desired splitting is given by

$$\alpha_n = \gamma_n \circ \beta_n^{-1} : \overline{\text{HH}}(\mathcal{A}_0/W) \to \overline{\text{HH}}(\mathcal{A}_n/R/I^{[n]})$$

To prove that these maps induce equivalences $\overline{\text{HH}}(\mathcal{A}_0/W) \otimes_W R_n \simeq \overline{\text{HH}}(\mathcal{A}_n/R_n)$ consider the following diagram where β_n' is obtained from β_n by taking the tensor product with $\underline{R_n}$ and γ_n' is induced by the map $F_n \otimes_{\underline{W}} \underline{R_n} \to \underline{R_n} \otimes_{\underline{W}} \underline{R_n} \xrightarrow{m} \underline{R_n}$

$$\mathcal{A}_n^\# \otimes_{R_n^\#} F_n \otimes_{\underline{W}} \underline{R_n}$$

$$\gamma_n' \qquad\qquad\qquad \beta_n'$$

$$\mathcal{A}_n^{\#/(R/I^{[n]})} = \mathcal{A}_n^\# \otimes_{R_n^\#} \underline{R_n} \qquad\qquad \mathcal{A}_n^\# \otimes_{R_n^\#} \underline{W} \otimes_{\underline{W}} \underline{R_n} = \mathcal{A}_0^{\#/W} \otimes_{\underline{W}} \underline{R_n}$$

The maps β_n', γ_n' become equivalences in the category $\text{Mod}_{W[S^1]}^t$. Indeed, the filtration on F_n induces a finite filtration on $\ker(\beta_n')$ with quotients isomorphic to $Q \otimes_W \mathcal{A}_0^\# \otimes_W \underline{R_n}$ so β_n' becomes an equivalence in $\text{Mod}_{W[S^1]}^t$. Similarly, the filtration constructed in 2.4(iii) induces a filtration on $\ker(\gamma_n')$ with quotients isomorphic to $\mathcal{A}_0^\# \otimes_W Q$.

Hence, the maps α_n induce equivalences

$$\gamma_n' \circ (\beta_n')^{-1} : \overline{\text{HH}}(\mathcal{A}_0/W) \otimes_W \underline{R_n} \simeq \overline{|\mathcal{A}_0^\# \otimes_{\underline{W}} \underline{R_n}|} \xrightarrow{\sim} \overline{\text{HH}}(\mathcal{A}_n/R_n)$$

$\square$

Remark 2.5. (i) This theorem and its proof are analogous to the Poincare lemma for the de Rham cohomology of commutative algebras. It asserts that given a split pd-nilpotent thickening of W-algebras $A \to A/I$ for any smooth scheme X over A the relative de Rham cohomology complex $\Omega_{X/A}^\bullet$ is canonically quasi-isomorphic to the constant module $\Omega_{X \times_A A/I/A/I}^\bullet \otimes_{A/I} A$. The object F is analogous to the absolute de Rham complex $\Omega_{X/A/I}^{\bullet,pd}$ modded out by the divided power relations of the form $d(x^{[n]}) = x^{[n-1]} dx$. Note, however, that in the theorem above we are restricting to

the case of a particular pd-thickening $W[x^{[\cdot]}] \to W$. Even though one can define the cyclic algebra F for an arbitrary thickening, the Lemma 2.4 is false in general.

(ii)A simpler form of the above computation is used in Kaledin's construction [K1] of the Gauss-Manin connection on periodic cyclic homology.

Let k be a perfect field of characteristic $p > 0$. Denote by $\widehat{\mathrm{Mod}}^t_{W(k)[S^1]}$ the category whose objects are those of $\mathrm{Mod}^t_{W(k)[S^1]}$ and whose space of morphisms is obtained by the p-completion of that in $\mathrm{Mod}^t_{W(k)[S^1]}$. The image of an objects $\overline{\mathrm{HH}}(-)$ in the completed category is denoted by $\widehat{\overline{\mathrm{HH}}}(-)$.

Corollary 2.6. *Assume that p is an odd prime. Then, for any E_m algebra $\mathcal{A}$ over $W(k)[x]$, $\infty \geq m > 0$, we have an isomorphism*

(2.4) $$\widehat{\overline{\mathrm{HH}}}(\mathcal{A}_{x=0}/W(k)) \xrightarrow{\sim} \widehat{\overline{\mathrm{HH}}}(\mathcal{A}_{x=p}/W(k))$$

of E_{m-1} algebras in $\widehat{\mathrm{Mod}}^t_{W(k)[S^1]}$ that is equal to identity modulo p.

Proof. For an integer $n \geq 0$ applying the above theorem to the base ring $W = W_n(k)$ and the algebra $\mathcal{A} \otimes_{W(k)[x]} R$ (here R is the divided power envelope of (x) in $W_n(k)[x]$) gives an equivalence:

$$\overline{\mathrm{HH}}(\mathcal{A}_{x=0} \otimes_{W(k)} W_n(k)/W_n(k)) \otimes_{W_n(k)} R \simeq \overline{\mathrm{HH}}(\mathcal{A}_k/R/I^{[m+1]})$$

for any m. For a large enough m the map $\mathrm{ev}_p : R \to W_n(k)$ induced by $W_n(k)[x] \xrightarrow{x \mapsto p} W_n(k)$ factors through $R/I^{[m+1]}$(namely, take m such that $v_p(\frac{p^{m+1}}{(m+1)!}) \geq n$: it exists by the assumption $p > 2$). Since $\overline{\mathrm{HH}}(\mathcal{A}_k/R/I^{[m+1]}) \otimes_{R/I^{[m+1]},\mathrm{ev}_p} W_n(k) \simeq \overline{\mathrm{HH}}(\mathcal{A}_{x=p} \otimes_{W(k)} W_n(k)/W_n(k))$ the above equivalence yields an equivalence

$$\overline{\mathrm{HH}}(\mathcal{A}_{x=0}/W(k)) \otimes_{W(k)} W_n(k) \simeq \overline{\mathrm{HH}}(\mathcal{A}_{x=p}/W(k)) \otimes_{W(k)} W_n(k)$$

These isomorphisms form an inverse system for varying n and passing to the inverse limit proves the corollary. $\square$

Corollary 2.7. *Assume that p is an odd prime. Then there exists a homomorphism of E_∞-algebras over $\widehat{\overline{\mathrm{HH}}}(W(k)/W(k))$*

(2.5) $$\widehat{\overline{\mathrm{HH}}}(k/W(k)) \longrightarrow \widehat{\overline{\mathrm{HH}}}(W(k)/W(k)),$$

that reduces to $\overline{\mathrm{HH}}(k \otimes_{W(k)} k/k) \to \overline{\mathrm{HH}}(k/k)$ modulo p.

Proof. Consider the commutative DG algebra $\mathcal{A} = W(k)[x][\epsilon]$ over $W(k)[x]$ with $\deg \epsilon = -1$ and $d\epsilon = x$. The fibers $\mathcal{A}_{x=0}$ and $\mathcal{A}_{x=p}$ are isomorphic to $W(k)[\epsilon]$ with $d\epsilon = 0$ and k, respectively. The previous corollary gives an isomorphism $\widehat{\overline{\mathrm{HH}}}(k/W(k)) \simeq \widehat{\overline{\mathrm{HH}}}(W(k)[\epsilon]/W(k))$ and the desired map is the composition of this isomorphism with the map induced by $W(k)[\epsilon] \xrightarrow{\epsilon \mapsto 0} W(k)$. $\square$

Remark 2.8. Both corollaries fail for $p = 2$. For a counterexample consider the algebra $\mathbb{F}_2$ over $\mathbb{Z}_2$: by Proposition 2.12 in [KN] the periodic cyclic homology $\mathrm{HP}_0(\mathbb{F}_2/\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[[y^{[\cdot]}]]/(y - 2)$(note that this ring is already 2-adically complete). The map $\mathbb{Z}_2[y] \xrightarrow{y \mapsto 2} \mathbb{Z}_2$ does not extend to this ring because $v_2(\frac{2^{2^n}}{2^n!}) = 1$ for every n and it is not hard to prove that the ring $\mathrm{HP}_0(\mathbb{F}_2/\mathbb{Z}_2)$ admits no homomorphism to $\mathbb{Z}_2$.

3. Crystalline periodic cyclic homology. Proof of the main theorem

We start by introducing a bit of notations. Given an algebra object A in a symmetric monoidal ∞-category $\mathcal{S}$, a right A-module M, and a left A-module N we denote by

$$(3.1) \qquad \text{``} M \otimes_A N \text{''} \in \mathrm{Fun}(\Delta^{op}, \mathcal{S})$$

the two-sided bar construction. This is a simplicial object of $\mathcal{S}$ that carries $[n] \in \Delta^{op}$ to $M \otimes A^{\otimes^n} \otimes N \in \mathcal{S}$.

For a DG category $\mathcal{C}$ over a perfect field k of odd characteristic we consider the simplicial object

$$(3.2) \qquad \text{``} \widehat{\overline{\mathrm{HH}}}(\mathcal{C}/W(k)) \otimes_{\widehat{\overline{\mathrm{HH}}}(k/W(k))} \widehat{\overline{\mathrm{HH}}}(W(k)/W(k)) \text{''} \in \mathrm{Fun}(\Delta^{op}, \widehat{\mathrm{Mod}}^{t}_{W(k)[S^1]}),$$

where $\widehat{\overline{\mathrm{HH}}}(k/W(k))$-module structure on $\widehat{\overline{\mathrm{HH}}}(W(k)/W(k))$ is given by the algebra homomorphism (2.5). The p-completed Tate invariants functor induces a functor between the corresponding categories of simplicial objects:

$$\mathrm{Fun}(\Delta^{op}, \widehat{\mathrm{Mod}}^{t}_{W(k)[S^1]}) \longrightarrow \mathrm{Fun}(\Delta^{op}, \widehat{\mathrm{Mod}}_{W(k)^{tS^1}}).$$

Definition 3.1. Define the crystalline periodic cyclic homology $\mathrm{HP}^{cris}(\mathcal{C}, W(k))$ to be the p-completion of

$$\mathrm{hocolim}_{\Delta^{op}}((\text{``} \widehat{\overline{\mathrm{HH}}}(\mathcal{C}/W(k)) \otimes_{\widehat{\overline{\mathrm{HH}}}(k/W(k))} \widehat{\overline{\mathrm{HH}}}(W(k)/W(k)) \text{''})^{tS^1}).$$

We also set

$$\mathrm{HP}^{cris}(\mathcal{C}, W_n(k)) = \mathrm{HP}^{cris}(\mathcal{C}, W(k)) \otimes_{W(k)} W_n(k).$$

Theorem 3.1. *Let k be a perfect field of characteristic $p > 2$. Then the following holds.*

(i) *For any DG category $\tilde{\mathcal{C}}$ over $W_n(k)$, we have a natural isomorphism*

$$\mathrm{HP}(\tilde{\mathcal{C}}/W_n(k)) \xrightarrow{\sim} \mathrm{HP}^{cris}(\mathcal{C}, W_n(k))$$

where $\mathcal{C} := \tilde{\mathcal{C}} \otimes_{W_n(k)} k$.

(ii) *For any DG category $\mathcal{C}$ over k and a lifting $\tilde{\mathcal{C}}$ of $\mathcal{C}$ over $W(k)$, we have a natural isomorphism*

$$\widehat{\mathrm{HP}}(\tilde{\mathcal{C}}/W(k)) \xrightarrow{\sim} \mathrm{HP}^{cris}(\mathcal{C}, W(k)).$$

(iii) *For any DG category $\mathcal{C}$ over k, there is a natural isomorphism of spectra*

$$\mathrm{HP}^{cris}(\mathcal{C}, W(k)) \xrightarrow{\sim} \widehat{\mathrm{TP}}(\mathcal{C}).$$

Proof. We first prove part (i) for $n = 1$. Since the tensor product commutes with colimits, we have that

$$(3.3) \quad \mathrm{HP}^{cris}(\mathcal{C}, W_1(k)) = \mathrm{hocolim}_{\Delta^{op}}((\text{``} \overline{\mathrm{HH}}(\mathcal{C}/W(k)) \otimes_{\overline{\mathrm{HH}}(k/W(k))} \overline{\mathrm{HH}}(k/k) \text{''})^{tS^1}).$$

Observe that $\overline{\mathrm{HH}}(k/W(k))$-module structures on $\overline{\mathrm{HH}}(k/k)$ and on $\overline{\mathrm{HH}}(\mathcal{C}/W(k))$ lift to the obvious $\mathrm{HH}(k/W(k))$-module structures on $\mathrm{HH}(k/k)$ and on $\mathrm{HH}(\mathcal{C}/W(k))$. It follows that the right-hand side of (3.3) can be rewritten as

$$\mathrm{hocolim}_{\Delta^{op}}((\text{``} \mathrm{HH}(\mathcal{C}/W(k)) \otimes_{\mathrm{HH}(k/W(k))} \mathrm{HH}(k/k) \text{''})^{tS^1}).$$

We have to construct an isomorphism between this colimit and $\mathrm{HP}(\mathcal{C}/k)$. Observe that

$$\mathrm{hocolim}_{\Delta^{op}}(\text{``}\mathrm{HH}(\mathcal{C}/W(k)) \otimes_{\mathrm{HH}(k/W(k))} \mathrm{HH}(k/k)\text{''}) \xrightarrow{\sim}$$

$$\mathrm{HH}(\mathcal{C}/W(k)) \otimes_{\mathrm{HH}(k/W(k))} \mathrm{HH}(k/k) \xrightarrow{\sim} \mathrm{HH}(\mathcal{C}/k).$$

In general, the Tate invariants functor does not commute with colimits. However, we get a morphism:

$$(3.4) \qquad \mathrm{hocolim}_{\Delta^{op}}((\text{``}\mathrm{HH}(\mathcal{C}/W(k)) \otimes_{\mathrm{HH}(k/W(k))} \mathrm{HH}(k/k)\text{''})^{tS^1}) \to \mathrm{HP}(\mathcal{C}/k).$$

We would like to show that (3.4) is an isomorphism. To do this we give another description of this map. The simplicial object "$\mathrm{HH}(\mathcal{C}/W(k)) \otimes_{\mathrm{HH}(k/W(k))} \mathrm{HH}(k/k)$" is given by

$$[n] \mapsto \mathrm{HH}(\mathcal{C} \otimes_{W(k)} k^{\otimes^n} \otimes_{W(k)} k/k),$$

where $k^{\otimes^n}$ stands for the n-fold tensor product over $W(k)$ Denote by

$$i_* : \text{DG categories over } k \to \text{DG categories over } W(k), \quad \mathcal{C} \mapsto \mathcal{C},$$

$$i^* : \text{DG categories over } W(k) \to \text{DG categories over } k, \quad \tilde{\mathcal{C}} \mapsto \tilde{\mathcal{C}} \otimes_{W(k)} k$$

the pair of adjoint functors. Setting

$$\Phi := i^* i_* : \text{DG categories over } k \to \text{DG categories over } k,$$

we identify the simplicial DG category

$$[n] \mapsto \mathcal{C} \otimes_{W(k)} k^{\otimes^n} \otimes_{W(k)} k$$

with the standard simplicial object

$$[n] \mapsto \Phi^{n+1}(\mathcal{C})$$

associated to the comonadic structure on Φ. By a general property of a pair of adjoint functors (Corollary 8.6.9 in [W]) the natural morphism

$$(3.5) \qquad\qquad\qquad \Phi^\bullet(\mathcal{C}) \to \mathcal{C}$$

to the constant simplicial DG category induces a homotopy equivalence

$$(3.6) \qquad\qquad\qquad i_* \Phi^\bullet(\mathcal{C}) \to i_* \mathcal{C}.$$

In particular, morphism (3.5) is a quasi-isomorphism:

$$\mathrm{hocolim}_{\Delta^{op}} \Phi^\bullet(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}.$$

Morphism (3.4) is the composition
(3.7)
$$\mathrm{HP}^{cris}(\mathcal{C}, W_1(k)) \cong \mathrm{hocolim}_{\Delta^{op}} \mathrm{HP}(\Phi^\bullet(\mathcal{C})/k) \to \mathrm{HP}(\mathrm{hocolim}_{\Delta^{op}} \Phi^\bullet(\mathcal{C})/k) \cong \mathrm{HP}(\mathcal{C}/k).$$

To show that (3.7) is an isomorphism we need the following lemma.

Lemma 3.2 ([AMN, Theorem 3.4]). *The functor*

$$\mathcal{F} \circ \mathrm{HP}(-/k) : \text{DG categories over } k \to \mathrm{Mod}_k \xrightarrow{\mathcal{F}} \mathrm{Sp},$$

where $\mathrm{Mod}_k \xrightarrow{\mathcal{F}} \mathrm{Sp}$ *is the forgetful functor from the category of k-vector spaces to the category of spectra, factors through* $i_* : \text{DG categories over } k \to \text{DG categories over } W(k)$. *In fact, for every DG category $\mathcal{C}$ over k, one has a functorial isomorphism in* Sp:

$$\mathrm{HP}(\mathcal{C}/k) \xrightarrow{\sim} \mathrm{TP}(\mathcal{C})/p.$$

We want to prove that (3.7) is an isomorphism. Since the forgetful functor $\mathcal{F}$ is conservative and commutes with colimits it suffices to check that

$$\text{(3.8)} \qquad \text{hocolim}_{\Delta^{op}} \, \mathcal{F} \circ \text{HP}(\Phi^{\bullet}(\mathcal{C})/k) \to \mathcal{F} \circ \text{HP}(\mathcal{C}/k)$$

is an isomorphism. Applying the functor $\text{TP}(-)/p$ to (3.6) and using the Lemma we derive a homotopy equivalence

$$\mathcal{F} \circ \text{HP}(\Phi^{\bullet}(\mathcal{C})/k) \to \mathcal{F} \circ \text{HP}(\mathcal{C}/k).$$

This proves that morphism (3.8) is an isomorphism.

Now we sketch a proof of (i) for any n. Consider the E_{∞}-algebras homomorphism

$$\text{(3.9)} \qquad \widehat{\widehat{\text{HH}}}(W_n(k)/W(k)) \longrightarrow \widehat{\widehat{\text{HH}}}(W(k)/W(k)),$$

defined as the composition

$$\widehat{\widehat{\text{HH}}}(W_n(k)/W(k)) \longrightarrow \widehat{\widehat{\text{HH}}}(k/W(k)) \longrightarrow \widehat{\widehat{\text{HH}}}(W(k)/W(k)),$$

where the first map is induced by the projection $W_n(k) \to k$ and the second map is (2.5). Given a DG category $\tilde{\mathcal{C}}$ over $W_n(k)$ we use (3.9) to define $\text{HP}^{cris}(\tilde{\mathcal{C}}, W(k))$ as the p-completion of

$$\text{hocolim}_{\Delta^{op}}((\text{``}\widehat{\widehat{\text{HH}}}(\tilde{\mathcal{C}}/W(k)) \otimes_{\widehat{\widehat{\text{HH}}}(W_n(k)/W(k))} \widehat{\widehat{\text{HH}}}(W(k)/W(k))\text{''})^{tS^1}).$$

The $W(k)$-linear functor $\tilde{\mathcal{C}} \to \mathcal{C}$ gives a morphsm

$$\text{(3.10)} \qquad \text{HP}^{cris}(\tilde{\mathcal{C}}, W(k)) \to \text{HP}^{cris}(\mathcal{C}, W(k)).$$

By looking at the reduction modulo p one checks that (3.10) is an isomorphism. Next, setting

$$\text{HP}^{cris}(\tilde{\mathcal{C}}, W_n(k)) = \text{HP}^{cris}(\tilde{\mathcal{C}}, W(k)) \otimes_{W(k)} W_n(k)$$

we shall construct a morphism

$$\text{(3.11)} \qquad \text{HP}^{cris}(\tilde{\mathcal{C}}, W_n(k)) \to \text{HP}(\tilde{\mathcal{C}}/W_n(k)).$$

The construction is based on the following property of the homomorphism (3.9): its reduction modulo p^n is given by

$$\text{HH}(W_n(k) \otimes_{W(k)} W_n(k)/W_n(k)) \longrightarrow \text{HH}(W_n(k)/W_n(k)).$$

Using this we identify $\text{HP}^{cris}(\tilde{\mathcal{C}}, W_n(k))$ with

$$\text{hocolim}_{\Delta^{op}}((\text{``}\text{HH}(\tilde{\mathcal{C}}/W(k)) \otimes_{\text{HH}(W_n(k)/W(k))} \text{HH}(W_n(k)/W_n(k))\text{''})^{tS^1}).$$

which admits a map to $\text{HP}(\tilde{\mathcal{C}}/W_n(k))$. Finally, we need to check that (3.11) is an isomorphism. We do this using induction on n and the following commutative diagram of $W_n(k)$-modules

$$\begin{array}{ccccc}
\text{HP}^{cris}(\mathcal{C}, W_1(k)) & \longrightarrow & \text{HP}^{cris}(\tilde{\mathcal{C}}, W_n(k)) & \longrightarrow & \text{HP}^{cris}(\tilde{\mathcal{C}} \otimes_{W_n(k)} W_{n-1}(k), W_{n-1}(k)) \\
\downarrow & & \downarrow & & \downarrow \\
\text{HP}(\mathcal{C}/k) & \longrightarrow & \text{HP}(\tilde{\mathcal{C}}/W_n(k)) & \longrightarrow & \text{HP}(\tilde{\mathcal{C}} \otimes_{W_n(k)} W_{n-1}(k)/W_{n-1}(k))
\end{array}$$

where the rows are distinguished triangles. We have already proven that the left vertical arrow is an isomorphism. The right vertical arrow is an isomorphism by the

induction assumption. We conclude that (3.11) is an isomorphism. This completes the proof of part (i).

The assertion of the part (ii) follows from (i). We, however, give a simpler proof which only uses (i) for $n = 1$. We have homomorphisms of E_∞-algebras

$$\widehat{\widehat{\mathrm{HH}}}(W(k)/W(k)) \longrightarrow \widehat{\widehat{\mathrm{HH}}}(k/W(k)) \longrightarrow \widehat{\widehat{\mathrm{HH}}}(W(k)/W(k)),$$

whose composition is the identity map. This gives morphisms

(3.12) $$\mathrm{HP}^{cris}(\mathcal{C}, W(k)) \longrightarrow \mathrm{HP}(\mathcal{C}/W(k)) \longrightarrow \mathrm{HP}^{cris}(\mathcal{C}, W(k))$$

whose composition is again the identity map. Now consider the morphisms

(3.13) $$\widehat{\mathrm{HP}}(\tilde{\mathcal{C}}/W(k)) \longrightarrow \widehat{\mathrm{HP}}(\mathcal{C}/W(k)) \longrightarrow \mathrm{HP}^{cris}(\mathcal{C}, W(k)),$$

where the first arrow comes from $W(k)$-linear functor $\tilde{\mathcal{C}} \to \mathcal{C}$ and the second arrow from (3.12). Modulo p the composition reduces to an isomorphism

$$\mathrm{HP}(\mathcal{C}/k) \xrightarrow{\sim} \mathrm{HP}^{cris}(\mathcal{C}, W_1(k))$$

from part (i). Hence, the composition (3.13) is an isomorphism.

For the assertion of part (iii) consider the arrows

$$\widehat{\mathrm{TP}}(\mathcal{C}) \longrightarrow \widehat{\mathrm{HP}}(\mathcal{C}/W(k)) \longrightarrow \mathrm{HP}^{cris}(\mathcal{C}, W(k))$$

where the second map is taken from (3.12). By Lemma 3.2 the composition reduces to an isomorphism modulo p. Hence, it is an isomorphism.

$\square$

Remark 3.3. Theorem 3.1 implies, in particular, that for any DG category $\tilde{\mathcal{C}}$ over $W_n(k)$ and an integer $m \le n$ the canonical map

$$\mathrm{HP}(\tilde{\mathcal{C}}/W_n(k)) \otimes_{W_n(k)} W_m(k) \to \mathrm{HP}(\tilde{\mathcal{C}} \otimes_{W_n(k)} W_m(k)/W_m(k))$$

is an isomorphism. This assertion is straightforward if $\tilde{\mathcal{C}}$ satisfies the following boundedness condition: $\mathrm{HH}_i(\tilde{\mathcal{C}} \otimes_{W_n(k)} k/k)$ vanish for $i \gg 0$ or if $\tilde{\mathcal{C}}$ admits a lift over $W(k)$ but we do not have a direct proof in general.

4. Explicit complex for HP^{cris}

Let A be a DG algebra over k such that the graded algebra $\bigoplus_{i \in \mathbb{Z}} A^i$ is a free associative graded algebra over k. In this section we construct an explicit complex representing $\mathrm{HP}^{cris}(A, W_2(k))$.

For a flat graded algebra C over a commutative ring S in this section we denote by $\mathrm{HH}(C/S)$ the graded module underlying the standard Hochschild complex of C over S and $b : \mathrm{HH}(C/S) \to \mathrm{HH}(C/S)[1]$, $B : \mathrm{HH}(C/S) \to \mathrm{HH}(C/S)[-1]$ are the Hochschild and Connes-Tsygan differentials respectively so that $(\mathrm{HH}(C/S)((u)), b + uB)$ is the totalization of the periodic cyclic complex of C. For a degree n derivation d of the algebra C we denote by L_d the induced degree n endomorphism of the Hochschild complex $\mathrm{HH}(C/S)$. In particular, if C is a DG algebra with the differential d then $(\mathrm{HH}(C/S)((u)), b + uB + L_d)$ is the periodic cyclic complex $\mathrm{HP}(C/S)$ of C.

The construction of the explicit complex is a formal consequence of the Cartan homotopy formula for periodic cyclic homology which we now recall:

Lemma 4.1. *To any degree n derivation D of the graded algebra underlying a DG algebra A over k we can associate a degree $n - 1$ endomorphism ι_D of the graded module $\mathrm{HH}(A, k)((u))$ such that the following relations are satisfied:*

$$[b + uB + L_d, \iota_D] = L_D + \iota_{[d,D]} \qquad \iota_{D_1+D_2} = \iota_{D_1} + \iota_{D_2}$$

Proof. Take ι_D to be $e_D + u^{-1} E_D$ from Proposition 4.1.8 in [L]. See also Definition 2.1 in [G]. $\square$

Pick a set I of free generators for the graded associative algebra $\bigoplus_i A^i = k\{x_i | i \in I\}$. Define a graded algebra $\bigoplus_i \widetilde{A}^i := W_2(k)\{x_i | i \in I\}$ freely generated by the same set of generators so that there is an obvious isomorphism $(\bigoplus_i \widetilde{A}^i) \otimes_{W_2(k)} k \simeq \bigoplus_i A^i$ of graded algebras. Let $\tilde{d} : \widetilde{A}^\bullet \to \widetilde{A}^{\bullet+1}$ be an arbitrary lift of the differential $d : A^\bullet \to A^{\bullet+1}$ to a derivation of the graded algebra $\widetilde{A}$. The operator $\tilde{d}$ need not satisfy the relation $\tilde{d}^2 = 0$ but there exists a unique k-linear map $D : A^\bullet \to A^{\bullet+2}$ such that $\tilde{d}^2 = pD$. Here pD refers to the operator on $\widetilde{A}$ induced by the composition $\widetilde{A}^\bullet \to A^\bullet \xrightarrow{D} A^{\bullet+2} \xrightarrow{p} \widetilde{A}^{\bullet+2}$.

Lemma 4.2. *The operator D is a derivation of the DG algebra A.*

Proof. D satisfies the Leibnitz rule because $\tilde{d}$ does so. It also commuted with the differential d because $pDd = \tilde{d}^2 d = \tilde{d}^3 = d\tilde{d}^2 = pdD$. $\square$

Define the following complex

$$(4.1) \qquad \mathrm{HP}^{cris}_{obj}(A, W_2(k)) := (\mathrm{HH}(\widetilde{A})((u)), b + uB + L_{\tilde{d}} + p\iota_D)$$

The differential squares to zero because $[b + uB, p\iota_D] = pL_D = L_{[\tilde{d},\tilde{d}]} = [L_{\tilde{d}}, L_{\tilde{d}}]$.

Lemma 4.3. *Suppose that $\tilde{C}$ is a flat graded algebra over $W_2(k)$ equipped with two degree 1 derivations $\tilde{d}_1, \tilde{d}_2$ satisfying $\tilde{d}_1 \equiv \tilde{d}_2 \mod p$ and $\tilde{d}_i^2 \equiv 0 \mod p$. Denote by D_1, D_2 the derivations of $C = \tilde{C} \otimes_{W_2(k)} k$ such that $pD_i = \tilde{d}_i^2$. There exists an isomorphism of complexes*

$$(\mathrm{HH}(\tilde{C})((u)), b + uB + L_{\tilde{d}_1} + p\iota_{D_1}) \simeq (\mathrm{HH}(\tilde{C})((u)), b + uB + L_{\tilde{d}_2} + p\iota_{D_2})$$

which reduces to the identity modulo p.

Proof. Let $D : C^\bullet \to C^{\bullet+1}$ be the degree 1 derivation such that $pD = \tilde{d}_1 - \tilde{d}_2$. The desired isomorphism is given by $\mathrm{id} + p\iota_D$. Indeed, $(\mathrm{id} + p\iota_D)(b + uB + L_{\tilde{d}_1} + p\iota_{D_1}) - (b + uB + L_{\tilde{d}_2} + p\iota_{D_2})(\mathrm{id} + p\iota_D) = [p\iota_D, b + uB] + L_{\tilde{d}_1} + p\iota_{D_1} + p\iota_D L_{\tilde{d}_1} - L_{\tilde{d}_2} - p\iota_{D_2} - pL_{\tilde{d}_2}\iota_D$ Since $\tilde{d}_1 \equiv \tilde{d}_2 \mod p$, we have $p\iota_D L_{\tilde{d}_1} - pL_{\tilde{d}_2}\iota_D = [p\iota_D, L_{\tilde{d}_1}]$ so the whole expression is equal to $[p\iota_D, b + uB + L_{\tilde{d}_1}] + L_{\tilde{d}_1} - L_{\tilde{d}_2} + p\iota_{D_1-D_2}$. This is equal to zero by Lemma 4.1 because $[\tilde{d}_1, pD] = \tilde{d}_1(\tilde{d}_1 - \tilde{d}_2) + (\tilde{d}_1 - \tilde{d}_2)\tilde{d}_2 = \tilde{d}_1^2 - \tilde{d}_2^2 = pD_1 - pD_2$. $\square$

Theorem 4.1. *There is a canonical quasi-isomorphism*

$$\mathrm{HP}^{cris}_{obj}(A, W_2(k)) \simeq \mathrm{HP}^{cris}(A, W_2(k))$$

where we regard the right-hand side as a DG module over $W_2(k)[u^{\pm 1}]$ using the equivalence of monoidal categories $\mathrm{Mod}_{W_2(k)^{tS^1}} \simeq \mathrm{Mod}_{W_2(k)((u))}$.

Proof. We will choose a particular flat resolution for the DG algebra A over $W(k)$ and use it to construct a morphism

$$\mathrm{HP}(A/W(k)) \to \mathrm{HP}^{cris}_{obj}(A, W_2(k))$$

that will induce the desired equivalence. Consider the free graded algebra $\breve{A} = W(k)\{x_i | i \in I\}$ over $W(k)$ and choose further a lift $\breve{d} : \breve{A} \to \breve{A}[1]$ of the derivation $\tilde{d}$. This yields a unique derivation $\widetilde{D} : \breve{A} \to \breve{A}[2]$ such that $p\widetilde{D} = \breve{d}^2$.

Consider the DG algebra $\breve{A}[\epsilon]$ whose underlying graded algebra is $\breve{A} \otimes_{W(k)} W(k)[\epsilon]$ with $\deg \epsilon = -1$ and the differential is given by $\breve{d} + p\partial_\epsilon + \epsilon\widetilde{D}$. Here ∂_ϵ is the derivation induced by $W(k)[\epsilon] \to W(k)[1]$. The map $\breve{A}[\epsilon] \to A$ induced by the reduction map $\breve{A} \to A$ and $\epsilon \mapsto 0$ is a quasi-isomorphism of DG algebras over $W(k)$ (it is surjective and $\frac{1}{p}\epsilon$ is a contracting homotopy for the kernel). Hence, the periodic cyclic homology $\mathrm{HP}(A/W(k))$ is quasi-isomorphic to the periodic cyclic complex of the (h-)flat algebra $\breve{A}[\epsilon]$:

$$\mathrm{HP}(\breve{A}[\epsilon]/W(k)) = (\mathrm{HH}(\breve{A}[\epsilon])((u)), b + B + \breve{d} + p\partial_\epsilon + \epsilon\widetilde{D})$$

The derivation $\widetilde{D}$ is equal to D modulo p, so the reduction of the above complex modulo p^2 is the periodic cyclic complex of a lift of the k-DG algebra $(A[\epsilon], d + \epsilon D)$. By Lemma 4.3 applied to the DG algebra $C = (A[\epsilon], d + \epsilon D)$ it follows that

$$\mathrm{HP}(\widetilde{A}[\epsilon]/W_2(k)) \simeq (\mathrm{HH}(\widetilde{A}[\epsilon])((u)), b + uB + L_{\tilde{d}} + \epsilon L_{\widetilde{D}} + p\iota_D)$$

Indeed, the derivation $\tilde{d} + \epsilon\widetilde{D}$ is a lift of the differential $d + \epsilon D$ and $(\tilde{d} + \epsilon\widetilde{D})^2 = \tilde{d}^2 = pD$. The latter complex now admits a map to $\mathrm{HP}^{cris}_{obj}(A, W_2(k))$ induced by $\epsilon \mapsto 0$.

We have constructed a map that fits into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{HP}(A/W(k)) & \longrightarrow & \mathrm{HP}^{cris}_{obj}(A, W_2(k)) \\
\downarrow & & \downarrow \\
\mathrm{HP}(A/k) & = \!\!=\!\!=\!\!= & \mathrm{HP}(A/k)
\end{array}
$$

Therefore, it induces a morphism $\mathrm{HP}^{cris}(A, W_2(k)) \to \mathrm{HP}^{cris}_{obj}(A, W_2(k))$ fitting into a morphism of distinguished triangles

$$
\begin{array}{ccccc}
\mathrm{HP}(A/k) & \longrightarrow & \mathrm{HP}^{cris}(A, W_2(k)) & \longrightarrow & \mathrm{HP}(A/k) \\
\| & & \downarrow & & \| \\
\mathrm{HP}(A/k) & \longrightarrow & \mathrm{HP}^{cris}_{obj}(A, W_2(k)) & \longrightarrow & \mathrm{HP}(A/k)
\end{array}
$$

Thus, this map is a quasi-isomorphsim and the theorem is proven. $\square$

Remark 4.4. By Lemma 13.5 in [D] any DG algebra A is quasi-isomorphic to to a DG algebra with free underlying graded algebra(furthermore, one can choose such semi-free model in a functorial way) so Theorem 4.1 provides an explicit complex computing $\mathrm{HP}^{cris}(A, W_2(k))$ for any DG algebra A over k.

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HARVARD UNIVERSITY, USA
Email address: apetrov@math.harvard.edu

NATIONAL RESEARCH UNIVERSITY "HIGHER SCHOOL OF ECONOMICS", RUSSIA
Email address: vologod@gmail.com