Algebraic Geometry I, Fall 2021

Problem Set 4

Due Friday, October 1, 2021 at 5 pm

1. Give an example of a locally ringed space $(X, \mathcal{O}_X)$ whose underlying topological space X consists of 1 point, but $(X, \mathcal{O}_X)$ is not a scheme.

2. Let X be a scheme whose underlying topological space has ≤ 2 points. Prove that X is an affine scheme.

3. Let $X = \{x, y, y'\}$ be a three point set. Topologize X by declaring the open sets to be $\varnothing, \{x\}, \{x, y\}, \{x, y'\}$, and X. Fix a prime number $p \in \mathbf{Z}$ and define a presheaf of rings $\mathcal{O}_X$ on X by setting

$$\mathcal{O}_X(X) = \mathcal{O}_X(\{x, y\}) = \mathcal{O}_X(\{x, y'\}) = \mathbf{Z}_{(p)}, \quad \mathcal{O}_X(\{x\}) = \mathbf{Q}, \quad \mathcal{O}_X(\varnothing) = 0,$$

where the restriction maps $\mathcal{O}_X(X) \to \mathcal{O}_X(\{x, y\})$ and $\mathcal{O}_X(X) \to \mathcal{O}_X(\{x, y'\})$ are the identity maps, and $\mathcal{O}_X(\{x, y\}) \to \mathcal{O}_X(\{x\})$ and $\mathcal{O}_X(\{x, y'\}) \to \mathcal{O}_X(\{x\})$ are the natural inclusions.

   (a) Check that the presheaf defined above is a sheaf, and that $(X, \mathcal{O}_X)$ is a scheme.

   (b) Prove that $(X, \mathcal{O}_X)$ is not an affine scheme.

4. Let X be a nonempty quasi-compact scheme. Prove that there exists a closed point of X.

5. Show by example that the intersection of two affine open subschemes of a scheme X is not necessarily an affine open subscheme.

6. Prove that for any ring A, there is an isomorphism $\Gamma(\mathbf{P}^n_A, \mathcal{O}_{\mathbf{P}^n_A}) \cong A$, where $\mathbf{P}^n_A$ is projective n-space over A, defined as in class via glueing.

7. (a) Let X and Y be locally ringed spaces. If $U \subset X$ is an open subset, then we regard U as a locally ringed space with structure sheaf $\mathcal{O}_U = \mathcal{O}_X|_U$. Show that if $X = \bigcup_{i \in I} U_i$ is an open cover and $f_i \colon U_i \to Y$ are morphisms of locally ringed spaces such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there is a unique morphism $f \colon X \to Y$ of locally ringed spaces such that $f|_{U_i} = f_i$ for all $i \in I$.
   More succinctly: the functor $U \mapsto \mathrm{Hom}_{\mathrm{LRS}}(U, Y)$ is a sheaf of sets on X.

   (b) Let X be a topological space and $X = \bigcup_{i \in I} U_i$ an open cover. Suppose we are given a sheaf (of say sets) $\mathcal{F}_i$ on U_i for each $i \in I$, together with isomorphisms

$$\phi_{ij} \colon \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j}$$

   for $i, j \in I$. Further, for every $i, j, k \in I$, suppose that on the open subset $U_i \cap U_j \cap U_k$ we have $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$, where by abuse of notation we have written ϕ_{ij} still for the isomorphism $\mathcal{F}_i|_{U_i \cap U_j \cap U_k} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j \cap U_k}$ given by the restriction of ϕ_{ij} (and similarly for ϕ_{jk}, ϕ_{ik}). Prove that there is a sheaf $\mathcal{F}$ on X together with isomorphisms $\psi_i \colon \mathcal{F}|_{U_i} \to \mathcal{F}_i$ such that $\phi_{ij} \circ \psi_i = \psi_j$ on $U_i \cap U_j$ for all $i, j \in I$.

   (c) Check the details of the general glueing lemma for schemes from class. For this part of the problem, you do not need to submit any written work.