Algebraic Geometry I, Fall 2021

Problem Set 3

Due Friday, September 24, 2021 at 5 pm

1. Read the statement of Yoneda's Lemma and work out the proof for yourself (see e.g. §1.3.10 of *Foundations of Algebraic Geometry* by Vakil). With this in mind, revisit the proof from class that $(g \circ f)^{-1}(\mathcal{H}) \cong f^{-1}g^{-1}(\mathcal{H})$ for continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$ of topological spaces and $\mathcal{H} \in \mathrm{Sh}(Z)$. For this problem, you do not need to submit any written work.

2. Let $f\colon X \to Y$ be a continuous map of topological spaces.

   (a) Prove that the inverse image functor $f^{-1}\colon \mathrm{Ab}(Y) \to \mathrm{Ab}(X)$ between categories of abelian sheaves is *exact*, i.e. if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence in $\mathrm{Ab}(Y)$ then
   $$0 \to f^{-1}\mathcal{F} \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{H} \to 0$$
   is an exact sequence in $\mathrm{Ab}(X)$.

   (b) Prove that the pushforward functor $f_*\colon \mathrm{Ab}(X) \to \mathrm{Ab}(Y)$ between categories of abelian sheaves is *left exact*, i.e. if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence in $\mathrm{Ab}(X)$ then
   $$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H}$$
   is an exact sequence in $\mathrm{Ab}(Y)$.

   (c) Show that the pushforward functor $f_*\colon \mathrm{Ab}(X) \to \mathrm{Ab}(Y)$ is *not* exact in general.

3. Let X be a topological space, and let $\mathcal{B}$ be a basis for the topology of X which is closed under intersections, i.e. for $U_1, U_2 \in \mathcal{B}$ we have $U_1 \cap U_2 \in \mathcal{B}$. In class, given a sheaf of sets $\mathcal{F}$ on $\mathcal{B}$, we defined a presheaf $\mathcal{F}_X$ on X by the formula
   $$\mathcal{F}_X(U) = \lim_{V \subset U, V \in \mathcal{B}} \mathcal{F}(V) \quad \text{for opens} \quad U \subset X,$$
   with the natural restriction maps.

   (a) Prove that $\mathcal{F}_X$ is a sheaf on X such that $\mathcal{F}_X(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$.

   (b) Prove that there is an equivalence between the category $\mathrm{Sh}(X)$ and the category of sheaves on $\mathcal{B}$. For this part of the problem, you do not need to submit any written work.

4. Let A be a ring and let $S \subset A$ be a multiplicative set. Prove that there is an isomorphism of rings
   $$S^{-1}A \cong \mathrm{colim}_{f \in S}\, A_f,$$
   where the maps appearing in the colimit diagram are, for every $f \in S$ dividing $f' \in S$, the canonical map $A_f \to A_{f'}$ (coming from the fact that f is invertible in $A_{f'}$).

5. Let A be a ring and let M be an A-module.

(a) By imitating our construction of $\mathcal{O}_{\mathrm{Spec}(A)}$ from class, construct a sheaf $\widetilde{M}$ on $\mathrm{Spec}(A)$ whose sections over any distinguished open $D(f)$ are given by the A_f-module M_f. Moreover, observe that $\widetilde{M}$ is an $\mathcal{O}_{\mathrm{Spec}(A)}$-module (see §2.2.13 of *Foundations of Algebraic Geometry* by Vakil for the general definition of an $\mathcal{O}_X$-module on a ringed space $(X, \mathcal{O}_X)$). For this part of the problem, you do not need to submit any written work.

(b) Describe the stalk of $\widetilde{M}$ at a point $\mathfrak{p} \in \mathrm{Spec}(A)$ as an $A_{\mathfrak{p}}$-module.

6. Let $X = \mathbf{C}$ with the Euclidean topology. Let $\mathcal{F}$ be the presheaf of abelian groups given on open subsets $U \subset X$ by

$$\mathcal{F}(U) = \begin{cases} \mathbf{Z} & \text{if } U = X, \\ 0 & \text{if } U \neq X, \end{cases}$$

with the natural restriction maps. Give a formula for the sections $\mathcal{F}^{\mathrm{sh}}(U)$ of the sheafification $\mathcal{F}^{\mathrm{sh}}$ over any open $U \subset X$.